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DERIVATIONS ON PRIME RINGS AND BANACH ALGEBRAS

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Abstract. In this paper we show that if D and G are continuous linear Jordan derivations on a Banach algebra A satisfying $[D(x), x]x - x[G(x), x] \in \text{rad}(A)$ for all $x \in A$, then both D and G map A into $\text{rad}(A)$.

1. Introduction

Throughout this paper R will represent an associative ring with center Z and A will represent an associative algebra over a complex field $\mathbb{C}$. The (Jacobson) radical of A is the intersection of all primitive ideals of A and will be denoted by $\text{rad}(A)$. $\mathbb{Z}$ will represent the set of all integers and $\mathbb{Z}^+$ will represent the set of all positive integers. A ring R is said to be n-torsion free if $nx = 0$, $x \in R$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$, and we make extensive use of the basic identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies that $a = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. An additive mapping D from R to R is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be centralizing on R if $[F(x), x] \in Z$ holds for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. Brešar showed that every Jordan derivation on a 2-torsion free semiprime ring

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is a derivation [1]. There has been considerable interest in commuting, centralizing, and related mappings in prime and semiprime rings. K. W. Jun and B. D. Kim [7] have obtained the algebraic condition that every derivation on a Banach algebra maps into its radical. In this paper we shall give the various algebraic conditions on prime ring that every derivation on the ring is zero and using these results, we show that every continuous linear Jordan derivation with some conditions on a Banach algebra maps into its radical.

2. Main results

We list a few more or less well-known results which will be needed in the sequel.

REMARK. R will represent a prime ring with center Z and extended centroid C.

1. Let c and ac be in the center of R. If c is not zero, then a is in the center of R.
2. If a derivation D of a prime ring R maps a nonzero left ideal of R into the center of R, then either $D = 0$ or R is commutative.
3. Suppose that the elements a_i, b_i in the central closure of R satisfy $\Sigma a_i y b_i = 0$. If $b_i \neq 0$ for some i then the a_i's are C-dependent.
4. The elements a, b in the central closure of R are C-dependent if and only if $ayb = bya$ holds for all $y \in R$.

The explanation of the notions of the extended centroid and the central closure of a prime ring, as well as the proof of Remark 3, can be found in [5, pp. 20–31].

E. Posner [9] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. We are going to generalize the Posner's theorem as follows. The following result is motivated by Brešar's result [2, Theorem 4.1].

THEOREM 2.1. *Let R be a noncommutative prime ring. Suppose that D and G are derivations of R such that $D(x)x - xG(x) \in Z$ for all $x \in R$. Then $D = G = 0$ on R.*

Proof. A linearization of $D(x)x - xG(x) \in Z$ gives

$$D(x)y + D(y)x - xG(y) - yG(x) \in Z \quad \text{for all } x, y \in R. \tag{2.1}$$

First assume that there exists a nonzero $c \in Z$. Taking $y = c$ in (2.1) we get

$$c(D(x) - G(x)) + D(c)x - xG(c) \in Z \text{ for all } x \in R. \tag{2.2}$$

Now let $y = c^2$ in (2.1). Then we obtain

$$c^2(D(x) - G(x)) + 2cD(c)x - 2cxG(c) \in Z.$$

That is,

$$c(D(c)x - xG(c)) + c(c(D(x) - G(x)) + D(c)x - xG(c)) \in Z$$

for all $x \in R$. Noting that the second summand is contained in Z by (2.2), we obtain $c(D(c)x - xG(c)) \in Z$ for every $x \in R$, hence by Remark 1, $D(c)x - xG(c) \in Z$. Thus (2.2) becomes $c(D(x) - G(x)) \in Z$ for all $x \in R$, and so, by Remark 1, $D(x) - G(x) \in Z$ for every $x \in R$. In view of Remark 2 we are forced to conclude that $D = G$. Thus in case $Z \neq 0$, we have $[D(x), x] \in Z$ by assumption, so $D = 0$ by the Posner's Theorem. Consequently, we arrive at $D = 0 = G$. In case $Z = 0$, by assumption, we get

$$D(x)x - xG(x) = 0 \qquad \text{for all } x \in R. \tag{2.3}$$

Linearizing this relation, we obtain

$$D(x)y + D(y)x = xG(y) + yG(x) \qquad \text{for all } x, y \in R. \tag{2.4}$$

Replace in (2.4) y by yx. The relation which we obtain can be written in the form $(D(x)y + D(y)x - xG(y))x + y(D(x)x - xG(x)) = xyG(x)$, hence it follows from (2.3) and (2.4) that

$$yG(x)x = xyG(x), \text{ i.e., } [yG(x), x] = 0 \qquad \text{for all } x, y \in R. \tag{2.5}$$

Replacing y by zy in (2.5), we then get $[z, x]yG(x) = 0$ for all $x, y, z \in R$. Since R is prime, for every $x \in R$ we have either $x \in Z$ or $G(x) = 0$. Using the fact that a group can not be the union of two proper subgroups, it follows that $G = 0$, since R is noncommutative. So (2.3) reduces to

$$D(x)x = 0 \qquad \text{for all } x \in R. \tag{2.6}$$

Linearizing (2.6), we get

$$D(x)y + D(y)x = 0 \qquad \text{for all } x, y \in R. \tag{2.7}$$

Replace y by $D(x)y$ to get $0 = D(x)^2y + D^2(x)yx + D(x)D(y)x = D^2(x)yx$ by (2.7). Using primeness of R, it follows that $D^2(x) = 0$ for all $x \in R$. So $D = 0$ by the Posner's first theorem [9]. The proof of the theorem is complete. □

By the above theorem, we obtain Posner's second theorem as a corollary.

COROLLARY 2.2. *Let R be a prime ring. If D is a nonzero derivation of R which is centralizing on R, then R is commutative.*

COROLLARY 2.3. *Let R be a noncommutative prime ring. Suppose there exist $a, b \in R$ and a derivation D of R such that the mapping $x \mapsto D(x) + ax + xb$ is centralizing on R. Then D is an inner derivation given by $D(x) = [x, a] = [b, x]$.*

Proof. Observe that the relation $[D(x) + ax + xb, x] \in Z$ can be written in the form $(D(x) - [x, a])x - x(D(x) - [b, x]) \in Z$. □

Taking a derivation D in Corollary 2.3 to be zero, we get

COROLLARY 2.4. *Let R be a prime ring. If $a, b \in R$ are such that the mapping $x \mapsto ax + xb$ (called a generalized inner derivation) is centralizing on R, then $a, b \in Z$.*

We give the well-known results which will be needed in the rest of this paper. The following lemma is due to L. O. Chung and J. Luh [4].

LEMMA 2.5 [4, Lemma 1]. *Let R be a $m!$-torsion free ring. Suppose that $t_1, t_2, \cdots, t_m \in R$ satisfy $kt_1 + k^2t_2 + \cdots + k^mt_m = 0$ for $k = 1, 2, \cdots, m$. Then $t_i = 0$ for all i.*

LEMMA 2.6 [8]. *Let R be a noncommutative prime ring of $(n+1)!$-torsion free and $D : R \to R$ a Jordan derivation such that $[D(x), x]x^n = 0$ or $x^n[D(x), x] = 0$ for all $x \in R$. Then $D = 0$.*

THEOREM 2.7. *Let R be a noncommutative prime ring of $6!$-torsion free. Suppose that there exists a Jordan derivation $D : R \to R$ such that $x[D(x), x]x = 0$ holds for all $x \in R$. Then we have $D = 0$ on R.*

Proof. Brešar showed that every Jordan derivation on a 2-torsion free semiprime ring is a derivation [1]. Thus D is a derivation. We introduce a symmetric biadditive mapping $F : R\times R \to R$ by the relation $F(x,y) = [D(x),y] + [D(y),x]$ for all $x,y \in R$. A routine calculation shows that the relation $F(xy,z) = F(x,z)y + xF(y,z) + D(x)[y,z] + [x,z]D(y)$ is fulfilled for all $x,y,z \in R$. Let us write $f(x)$ for $F(x,x)$. Thus $f(x) = 2[D(x),x]$ for all $x \in R$. The mappings f satisfies the relation $f(x+\lambda y) = f(x) + \lambda^2 f(y) + 2\lambda F(x,y)$ for all $x,y \in R$ and $\lambda \in \mathbb{Z}$. Now the assumption of the theorem can be written in the form

$$xf(x)x = 0, \qquad x \in R. \tag{2.8}$$

Replacing x by $x+\lambda y$ in (2.8), we get

$$\begin{aligned} 0 =& \lambda(xf(x)y + 2xF(x,y)x + yf(x)x) + \lambda^2(2xF(x,y)y + yf(x)y \\ &+ xf(y)x + 2yF(x,y)x) + \lambda^3(xf(y)y + 2yF(x,y)y + yf(y)x) \end{aligned}$$

for all $x,y \in R$, $\lambda \in \mathbb{Z}$. Applying Lemma 2.5, we have

$$0 = xf(x)y + 2xF(x,y)x + yf(x)x, \quad x,y \in R. \tag{2.9}$$

Let us replace in (2.9) y by yx. Then, by (2.8) and (2.9), we get

$$\begin{aligned} 0 =& xf(x)yx + 2xF(x,y)x^2 + 2xyf(x)x + 2x[y,x]D(x)x \\ =& -yf(x)x^2 + 2xyf(x)x + 2x[y,x]D(x)x \end{aligned} \tag{2.10}$$

for all $x,y \in R$. The substitution yx for y in the above relation leads to $0 = x[y,x]xD(x)x$, which can be written in the form

$$xyx^2D(x)x = x^2yxD(x)x, \qquad x,\ y \in R. \tag{2.11}$$

Putting in (2.10) $y = D(x)xy$, we obtain

$$xD(x)xyx^2D(x)x = x^2D(x)xyxD(x)x, \qquad x,\ y \in R. \tag{2.12}$$

If $xD(x)x \neq 0$, then it follows from (2.12) and Remark 4 that $x^2D(x)x = \alpha xD(x)x$ for some $\alpha \in C$. Applying the last relation to (2.11), we get $0 = (\alpha x - x^2)yxD(x)x$, $y \in R$, which yields $x^2 = \alpha x$ since R is prime. Since R is of characteristic not two, we can rewrite (2.10) as follows.

$$\begin{aligned} 0 =& xy(2D(x)x^2 - xD(x)x) - x^2yD(x)x \\ &+ y(xD(x)x^2 - D(x)x^3), \qquad y \in R. \end{aligned} \tag{2.13}$$

We set conveniently

$$a = 2D(x)x^2 - xD(x)x, b = D(x)x, c = xD(x)x^2 - D(x)x^3, a_i = x^i, i = 1, 2$$

Thus we have by (2.13)

$$(2.14) \qquad 0 = a_1ya + a_2yb + yc, \qquad y \in R.$$

Substituting za_1y for y in (2.14), we obtain $0 = a_1za_1ya + a_2za_1yb + za_1yc$ for all $y, z \in R$. But on the other hand we see from (2.14) that $a_1za_1ya = -a_1za_2yb - a_1zyc$. Comparing the last two relations, we arrive at $0 = (a_2za_1 - a_1za_2)yb + (za_1 - a_1z)yc$ for all $y, z \in R$, which gives

$$(2.15) \qquad 0 = (za_1 - a_1z)yc, \qquad y, z \in R,$$

since a_1, a_2 are C-dependent. Now it follows from (2.15) that we have either $a_1 = x \in Z$ or $c = -[D(x), x]x^2 = 0$ by primeness of R. Of course, in both cases $[D(x), x]x^2 = 0$. We have therefore proved that $[D(x), x]x^2 = 0$ in case $xD(x)x \neq 0$. So R is the union of its subsets $P = \{x \in R : xD(x)x = 0\}$ and $Q = \{x \in R : [D(x), x]x^2 = 0\}$. Suppose $D \neq 0$. The well known results [3] and Lemma 2.6 then tell us that $P \neq R$ and $Q \neq R$. Thus there exist $x, y \in R$ such that $x \notin Q$ and $y \notin P$, hence $x \in P$ and $y \in Q$. If we consider $x + \lambda y$, $\lambda \in \mathbb{Z}$, then we see that either $x + \lambda y \in P$ or $x + \lambda y \in Q$. If this element lies in P, then we have

$$(2.16) \qquad \begin{aligned} &\lambda\{xD(x)y + xD(y)x + yD(x)x\} + \lambda^2\{xD(y)y \\ &+ yD(x)y + yD(y)x\} + \lambda^3 yD(y)y = 0. \end{aligned}$$

If it lies in Q, then
(2.17)

$$\begin{aligned} &[D(x), x]x^2 \\ &+ \lambda\{[D(x), x](xy + yx) + ([D(x), y] + [D(y), x])x^2\} \\ &+ \lambda^2\{[D(x), x]y^2 + ([D(x), y] + [D(y), x])(xy + yx) + [D(y), y]x^2\} \\ &+ \lambda^3\{([D(x), y] + [D(y), x])y^2 + [D(y), y](xy + yx)\} = 0. \end{aligned}$$

Thus, for every $\lambda \in \mathbb{Z}$, one of these two possibilities holds. But either (2.16) has more than three solutions or (2.17) has more than four solutions. In view of Lemma 2.5, this contradicts the choice of x and y

such that $yD(y)y \neq 0$ and $[D(x), x]x^2 \neq 0$. The proof of the theorem is complete. $\square$

Brešar [1] showed that every Jordan derivation on a 2-torsion free semiprime ring is a derivation. In recent paper [11] Vukman has proved that in case there exists a nonzero derivation $D : R \to R$, where R is a prime ring of characteristic different from 2 and 3, such that the mapping $x \mapsto [D(x), x]$ is centralizing on R, R is commutative. We are going to generalize this theorem mentioned above as follows.

THEOREM 2.8. *Let R be a noncommutative prime ring of 6!-torsion free. Suppose that D and G are Jordan derivations of R such that $[D(x), x]x - x[G(x), x] \in Z$ for all $x \in R$. Then we have $D = 0 = G$ on R.*

Proof. By the Brešar' result [1] D, G are derivations. We use the notations f, F in Theorem 2.7 and h, H similarly, defined by $H(x, y) = [G(x), y] + [G(y), x]$, $h(x) = H(x, x)$, respectively. The assumption of the theorem can now be written in the form

$$f(x)x - xh(x) \in Z \quad \text{for all } x \in R. \tag{2.18}$$

First assume there exists a nonzero $c \in Z$. The linearization of (2.18) gives

$$\begin{aligned} f(x)y + 2F(x, y)x + 2F(x, y)y + f(y)x - 2xH(x, y) \\ - xh(y) - 2yH(x, y) - yh(x) \in Z \end{aligned} \tag{2.19}$$

for all $x, y \in R$. The substitution $-x$ for x in the above relation leads to

$$\begin{aligned} f(x)y + 2F(x, y)x - 2F(x, y)y - f(y)x - 2xH(x, y) \\ + xh(y) + 2yH(x, y) - yh(x) \in Z \end{aligned} \tag{2.20}$$

for all $x, y \in R$. Now from (2.19) and (2.20) we obtain $f(x)y + 2F(x, y)x - yh(x) - 2xH(x, y) \in Z$, which can be written in the form

$$\begin{aligned} [D(x), x]y + [D(x, y]x + [D(y), x]x \\ - y[G(x), x] - x[G(x), y] - x[G(y), x] \in Z \end{aligned} \tag{2.21}$$

for all $x, y \in R$. Taking $y = c$ in (2.21), we get

$$([D(x), x] - [G(x), x])c + [D(c), x]x - x[G(c), x] \in Z \tag{2.22}$$

for all $x \in R$. Now let $y = c^2$ in (2.21), then we obtain

$$([D(x), x] - [G(x), x])c^2 + 2[cD(c), x]x - 2x[cG(c), x] \in Z$$

for all $x \in R$. That is, the above relation can now be rewritten in the form

$$\begin{aligned} c\{([D(x), x] - [G(x), x])c + [D(c), x]x - x[G(c), x]\} \\ + c([D(c), x]x - x[G(c), x]) \in Z \end{aligned}$$

for all $x \in R$. Noting that the first summand is contained in Z by (2.22), we obtain $c([D(c), x]x - x[G(c), x]) \in Z$ for every $x \in R$, and hence by Remark 1 $[D(c), x]x - x[G(c), x] \in Z$. Thus (2.22) becomes $([D(x), x] - [G(x), x])c \in Z$ for all $x \in R$. And so, by Remark 1, $[D(x), x] - [G(x), x] \in Z$ for all $x \in R$. In view of [9, Theorem 2], we are forced to conclude that $D - G = 0$.

In case $Z = 0$, by our assumption, we get

$$[D(x), x]x - x[G(x), x] = 0 \quad \text{for all} \quad x \in R. \tag{2.23}$$

Linearizing (2.23), we obtain

$$f(x)y + 2B(x, y)x - yh(x) - 2xH(x, y) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.24}$$

Replacing in (2.24) y by x^2, we obtain the relation by (2.23)

$$\begin{aligned} 0 =& f(x)x^2 + 2B(x, x^2)x - x^2h(x) - 2xH(x, x^2) \\ =& f(x)x^2 + 2(f(x)x + xf(x))x - x^2h(x) - 2x(h(x)x + xh(x)) \\ =& 3f(x)x^2 + 2xf(x)x - 3x^2h(x) - 2xh(x)x \\ =& f(x)x^2 - xf(x)x \\ =& [f(x), x]x \qquad \text{for all} \quad x \in R. \end{aligned}$$

Observe that the relation $0 = f(x)x - xh(x) = [f(x), x] + x(f(x) - h(x))$ holds for all $x \in R$. Right multiplication of this relation by x leads to

$$0 = x[D(x) - G(x), x]x \quad \text{for all} \quad x \in R. \tag{2.25}$$

Applying Theorem 2.7 to the relation (2.25), we have $D - G = 0$. Therefore in any case $D = G$ and the assumption can be written by

$[[D(x), x], x] \in Z$, so $D = 0$ by the Vukman's theorem [11]. Consequently we arrive at $D = 0 = G$. The proof of the theorem is complete.□

Neglecting the fact that in our result we have an additional assumption concerning the torsion of the ring, we can say that the above theorem generalizes the Vukman's theorem.

COROLLARY 2.9. *Let R be a prime ring of $6!$-torsion free. If D is a nonzero derivation of R such that $[[D(x), x], x] \in Z$ for all $x \in R$, then R is commutative.*

By virtue of Theorem 2.8, we can characterize a derivation with some specific property.

COROLLARY 2.10. *Let R be a noncommutative prime ring of $6!$-torsion free. Suppose there exist $a, b \in R$ and a derivation D of R such that the mapping $x \mapsto [D(x), x] + [a, x]x + x[b, x]$ is centralizing on R. Then D is an inner derivation given by $D(x) = [x, a] = [b, x]$.*

Proof. Observe that the relation $[[D(x), x] + [a, x]x + x[b, x], x] \in Z$ can be written in the form $([D(x)-[x, a], x])x - x([D(x)-[b, x], x]) \in Z$.□

Taking a derivation D in Corollary 2.10 to be zero, we get

COROLLARY 2.11. *Let R be a prime ring of $6!$-torsion free. If $a, b \in R$ are such that the mapping $x \mapsto [a, x]x + x[b, x]$ is centralizing on R, then $a, b \in Z$.*

Using the above algebraic results, we show that every continuous linear Jordan derivation on a Banach algebra maps into its radical.

THEOREM 2.12. *Let A be a Banach algebra, and let D and G be continuous linear Jordan derivations on A. If $[D(x), x]x - x[G(x), x] \in$ $\mathrm{rad}(A)$ for all $x \in A$, then both D and G map A into* $\mathrm{rad}(A)$.

Proof. Let P be a primitive ideal of A. Since D and G are continuous, by [10, Lemma 3.2] we have $D(P) \subseteq P$ and $G(P) \subseteq P$. Then we can define Jordan derivations D_P and G_P on A/P by

$$D_P(\hat{x}) = D(x) + P,\ G_P(\hat{x}) = G(x) + P,\ \hat{x} = x + P$$

for all $x \in A$. The factor algebra A/P is prime and semisimple, since P is a primitive ideal. Thus both D_P and G_P are derivations by the

Brešar's result [1]. Johnson [6] has proved that every derivation on a semisimple Banach algebra is continuous. Combining this result with the Singer-Wermer theorem, we obtain that there are no nonzero derivations on a commutative semisimple Banach algebra. Hence in case A/P is commutative, we have $D_P = 0$ and $G_P = 0$. It remains to show that $D_P = 0$ and $G_P = 0$ in the case when A/P is noncommutative. The assumption of the theorem gives $[D_P(\hat{x}), \hat{x}]\hat{x} - \hat{x}[G_P(\hat{x}), \hat{x}] = 0$, $\hat{x} \in A/P$. All the assumption of Theorem 2.8 is fulfilled. Thus we have both $D_P = 0$ and $G_P = 0$. In any case $D_P = 0$ and $G_P = 0$. Hence we see that $D(A) \subseteq P$ and $G(A) \subseteq P$. Since P is any primitive ideal, the result follows. This completes the proof. □

The following corollary is the special case of Theorem 2.12.

COROLLARY 2.13. *Let A be a Banach algebra, and let D be a continuous linear Jordan derivation on A. If $[[D(x), x], x] \in \operatorname{rad}(A)$ for all $x \in A$, then D maps A into* $\operatorname{rad}(A)$.

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