Bull. Korean Math. Soc. **38** (2001), No. 4, pp. 689–699

PROJECTIVE SYSTEMS WHOSE SUPPORTS CONSIST OF THE UNION OF THREE LINEAR SUBSPACES

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ABSTRACT. We discuss the class of projective systems whose supports are the complement of the union of three linear subspaces in general position. We proves these codes are uniquely determined up to equivalence by their weight enumerators. Our result is a generalization of the result given in [1].

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements. Let $C \subset \mathbb{F}_q^n$ be a nondegenerate linear q-ary $[n,k,d]$-code with a generator matrix $G = (\mathbf{a}_1, \dots, \mathbf{a}_n)$, where $\mathbf{a}_i$ is a column vector of length k. Since C is nondegenerate, $\mathbf{a}_i \neq \mathbf{0}$ for any $i = 1, \dots, n$, whence we can associate $\mathbf{a}_i$ with the point $[\mathbf{a}_i]$ in the $k-1$ dimensional projective space $\mathbb{P}^{k-1}$ over $\mathbb{F}_q$. Then, G induces a positive 0-cycle $\sum_{i=1}^n [\mathbf{a}_i]$ on $\mathbb{P}^{k-1}$, whose support spans the whole space. Let $G' = (\mathbf{a}'_1, \dots, \mathbf{a}'_n)$ be another generator matrix of C. Then, 0-cycles $\sum_{i=1}^n [\mathbf{a}_i]$ and $\sum_{i=1}^n [\mathbf{a}'_i]$ are projectively equivalent. By definition, a projective system $\mathcal{X}_C$ associated with C is a representative of this equivalence class [2].

Conversely, a positive 0-cycle of length n in $\mathbb{P}^{k-1}$ whose support spans the whole space determines a nondegenerate $[n,k]_q$-code.

Let S be a subset of $\mathbb{P}^{k-1}$ which spans the whole space. Then, the 0-cycle $\sum_{P \in S} P$ induces a nondegenerate code C. Recently, Homma-Kim-Yoo [1] computed the weight enumerator $W_C(z)$ of C, in the case S (or S^c) is a union of linear subspaces $\mathcal{L}_i$ $(i = 1, \dots, r)$ of $\mathbb{P}^{k-1}$ in general position.

Received December 20, 2000.
2000 Mathematics Subject Classification: 94B05, 51E20, 05B25.
Key words and phrases: linear code, weight enumerator, projective system.
*Partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, #12640180.

In case $r = 2$, they also prove the converse, namely, if $W_C(z) = W_{C'}(z)$, then C' is equivalent to C.

In this paper, we shall prove the converse in case $r = 3$.

2. Known facts

In this section, we shall give several known facts concerning weight enumerator of codes corresponding to the projective systems and Homma Kim-Yoo's result.

LEMMA 2.1. *Let S be a subset of $\mathbb{P}^{k-1}$ such that $\dim\langle S\rangle = \dim\langle S^c\rangle = k-1$. Let $\mathcal{X}_1 = \sum_{P\in S} P$ and $\mathcal{X}_2 = \sum_{P\in S^c} P$ and let C_i be the code induced by $\mathcal{X}_i$, $i = 1, 2$. Then*

$$W_{C_2}(z) = 1 + \left(W_{C_1}\left(\frac{1}{z}\right) - 1\right) z^{q^{k-1}}.$$

Thus studying $W_{C_1}(z)$ is the same as studying $W_{C_2}(z)$. In case S is the complement of the union of linear subspaces, Homma-Kim-Yoo give the following;

THEOREM 2.2 (Homma-Kim-Yoo [1]). *Let $\mathcal{L}_i$, $i = 1, \ldots, r$ $(r \geq 2)$, be linear subspaces of dimension s_i in $\mathbb{P}^{k-1}$ in general position, where $l = r + \sum_{i=1}^{r} s_i \leq k$. Let $S = \mathbb{P}^{k-1} \setminus \bigcup_{i=1}^{r} \mathcal{L}_i$ and C be the code induced by the 0-cycle $\mathcal{X} = \sum_{P\in S} P$. Then*

$$W_C(z) = 1 + \left(q^{k-l} f\left(\frac{1}{z}\right) - 1\right) z^{q^{k-1}},$$

where

$$f(z) = \prod_{i=1}^{r} \left(1 + \left(q^{s_i+1} - 1\right) z^{q^{s_i}}\right).$$

They [1] also prove the converse of Theorem 2.2 in the following manner:

THEOREM 2.3 (Homma-Kim-Yoo [1]). *Let C be the code in* Theorem 2.2 *for $r = 2$ and some s_1, s_2. If C' is a code such that $W_{C'}(z) = W_C(z)$, then C' is equivalent to C.*

To prove this theorem they proves the following two lemmas:

LEMMA 2.4 (Homma-Kim-Yoo [1]). *Let C and C' be nondegenerate codes with the same weight enumerators. Then, $\mathcal{X}_C = \sum_{P \in \mathrm{Supp}(\mathcal{X}_C)} P$ if and only if $\mathcal{X}_{C'} = \sum_{P \in \mathrm{Supp}(\mathcal{X}'_C)} P$.*

To state the next lemma, we introduce a notation

$$N(m) = \frac{q^{m+1}-1}{q-1} = q^m + q^{m-1} + \cdots + 1,$$

which is the number of points in $\mathbb{P}^m$.

LEMMA 2.5 (Homma-Kim-Yoo [1]). *Let n, s, t be non-negative integers satisfying $n = s + t + 1$. Let S be a set of points in $\mathbb{P}^n$ such that $^{\#}S = N(s) + N(t)$. Let $\mathcal{H}'$.'s be the sets of hyperplanes in $\mathbb{P}^n$ as follows:*

$$\begin{aligned} \mathcal{H}'_s &= \{H \in \mathbb{P}^{n*} | ^{\#}(H \cap S) = N(s-1) + N(t)\}, \\ \mathcal{H}'_t &= \{H \in \mathbb{P}^{n*} | ^{\#}(H \cap S) = N(s) + N(t-1)\}, \\ \mathcal{H}'_{st} &= \{H \in \mathbb{P}^{n*} | ^{\#}(H \cap S) = N(s-1) + N(t-1)\}. \end{aligned}$$

Assume $^{\#}\mathcal{H}'_s = N(s)$, $^{\#}\mathcal{H}'_t = N(t)$ and $\mathbb{P}^{n*} = \mathcal{H}'_s \cup \mathcal{H}'_t \cup \mathcal{H}'_{st}$. Then, there are subspaces $\mathcal{L}_1$, $\mathcal{L}_2$ in $\mathbb{P}^n$ such that $\dim \mathcal{L}_1 = s$, $\dim \mathcal{L}_2 = t$, $\mathcal{L}_1 \cap \mathcal{L}_2 = \phi$ and $S = \mathcal{L}_1 \cup \mathcal{L}_2$.

3. Uniqueness theorem for $r = 3$

In this section, we shall prove the converse of Theorem 2.2 in case $r = 3$.

THEOREM 3.1. *Let C be the code in Theorem 2.2 for $r = 3$ and some s_1, s_2, s_3. Further, assume $q \geq 5$. If C' is a code such that $W_{C'}(z) = W_C(z)$, then C' is equivalant to C.*

Using the same argument as in the proof of Theorem 2.2 (cf. [1]), we can prove this theorem if we can prove, Theorem 3.2 below:

THEOREM 3.2. *Assume that $q \geq 5$. Let n, s, t, u be non-negative integers satisfying $n = s+t+u+2$. Let S be a set of points in $\mathbb{P}^n$ such that ${}^{\#}S = N(s) + N(t) + N(u)$. Let $\mathcal{H}.$'s be the sets of hyperplanes in $\mathbb{P}^n$ as follows:*

$$\begin{aligned}
\mathcal{H}_s &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s-1) + N(t) + N(u)\}, \\
\mathcal{H}_t &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s) + N(t-1) + N(u)\}, \\
\mathcal{H}_u &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s) + N(t) + N(u-1)\}, \\
\mathcal{H}_{st} &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s-1) + N(t-1) + N(u)\}, \\
\mathcal{H}_{su} &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s-1) + N(t) + N(u-1)\}, \\
\mathcal{H}_{tu} &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s) + N(t-1) + N(u-1)\}, \\
\mathcal{H}_{stu} &= \{H \in \mathbb{P}^{n*} | {}^{\#}(H \cap S) = N(s-1) + N(t-1) + N(u-1)\}.
\end{aligned}$$

Assume ${}^{\#}\mathcal{H}_s = N(s)$, ${}^{\#}\mathcal{H}_t = N(t)$, ${}^{\#}\mathcal{H}_u = N(u)$, ${}^{\#}\mathcal{H}_{st} = N(n-u-1) - N(s) - N(t)$, ${}^{\#}\mathcal{H}_{su} = N(n-t-1) - N(s) - N(u)$, ${}^{\#}\mathcal{H}_{tu} = N(n-s-1) - N(t) - N(u)$ and $\mathbb{P}^{n*} = \mathcal{H}_s \cup \mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{st} \cup \mathcal{H}_{su} \cup \mathcal{H}_{tu} \cup \mathcal{H}_{stu}$. Then, there are subspaces $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ in $\mathbb{P}^n$ such that $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are mutually disjoint, $\dim \mathcal{L}_1 = s$, $\dim \mathcal{L}_2 = t$, $\dim \mathcal{L}_3 = u$ and $S = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.

REMARK. In case at least two of s, t, u coincide, the corresponding $\mathcal{H}.$'s coincide. For example, if $s = t \neq u$, then $\mathcal{H}_s = \mathcal{H}_t$ and $\mathcal{H}_{su} = \mathcal{H}_{tu}$. In this case, one should read the assumption for ${}^{\#}\mathcal{H}.$'s as ${}^{\#}\mathcal{H}_s = 2N(s)$ and ${}^{\#}\mathcal{H}_{su} = 2\,(N(n-s-1) - N(s) - N(u))$. One should also read the assumption in a similar manner for other cases.

To prove this theorem, we need several lemmas. To describe these lemmas and their proofs, we introduce a notation. For a linear subspace (or a point) $\mathcal{L}$ in $\mathbb{P}^n$, let $\check{\mathcal{L}}$ be the set of hyperplanes which contain $\mathcal{L}$. Evidently, $\dim \check{\mathcal{L}} = n - 1 - \dim \mathcal{L}$.

LEMMA 3.3. *Let S be a subset of $\mathbb{P}^n$ such that ${}^{\#}(H \cap S) \geq N(s-1) + N(t-1) + N(u-1)$ for every hyperplane H. Let L be a line in $\mathbb{P}^n$. Assume there is a point P in L which is not contained in S. Then ${}^{\#}(L \cap S) \geq 3$. Furthermore, equality occurs only if ${}^{\#}(H \cap S) = N(s-1) + N(t-1) + N(u-1)$ for every $H \in \check{P} \setminus \check{L}$.*

The proof is just the same procedure as in the proof of Theorem 3.1, Claim 1 in [1]. Thus, we omit it.

LEMMA 3.4. *Let S be as in Lemma 3.3. Let M be a plane in $\mathbb{P}^n$ and L be a line which is contained in M. Let $\alpha = {}^{\#}(L \cap S)$ and $\beta = {}^{\#}(M \cap S)$. Then $\beta \geq \alpha q + 3$.*

In particular, if L is not contained in S, then $\beta \geq 3q + 3$ and if there exists $H \in \check{L} \setminus \check{M}$ so that ${}^{\#}(H \cap S) > N(s-1) + N(t-1) + N(u-1)$, then $\beta \geq 3q + 2$.

Proof. Since ${}^{\#}(\check{L} \setminus \check{M}) = N(n-2) - N(n-3) = q^{n-2}$, we have

$$(1) \qquad \sum_{H \in \check{L} \setminus \check{M}} {}^{\#}(H \cap S) \geq q^{n-2} \left(N(s-1) + N(t-1) + N(u-1) \right).$$

On the other hand, we have

$$\begin{aligned} {}^{\#}(H \cap S) &= {}^{\#}(H \cap (S \setminus M)) + {}^{\#}(H \cap S \cap L) \\ &= {}^{\#}(H \cap (S \setminus M)) + \alpha \end{aligned}$$

for every $H \in \check{L} \setminus \check{M}$.

For any point $Q \in S \setminus M$, we have

$$\begin{aligned} {}^{\#}\left\{ H \in \check{L} \setminus \check{M} \mid Q \in H \right\} &= {}^{\#}(\check{L} \cap \check{Q}) - {}^{\#}(\check{M} \cap \check{Q}) \\ &= N(n-3) - N(n-4) \\ &= q^{n-3}. \end{aligned}$$

Thus,

$$\begin{aligned} (2) \quad \sum_{H \in \check{L} \setminus \check{M}} {}^{\#}(H \cap S) &= {}^{\#}(S \setminus M) q^{n-3} + \alpha\, {}^{\#}(\check{L} \setminus \check{M}) \\ &= \left(N(s) + N(t) + N(u) - \beta \right) q^{n-3} + \alpha q^{n-2}. \end{aligned}$$

By (1) and (2), we have $\beta \leq \alpha q + 3$. If there exists $H \in \check{L} \setminus \check{M}$ so that ${}^{\#}(H \cap S) > N(s-1) + N(t-1) + N(u-1)$, then equality does not hold in (1), whence we have $\beta < \alpha q + 3$. This completes the proof. □

Henceforth, we assume $q \geq 5$.

LEMMA 3.5. *Let S be a subset of $\mathbb{P}^n$ and let H be a hyperplane in $\mathbb{P}^n$ such that ${}^{\#}(H \cap S) > N(s-1) + N(t-1) + N(u-1)$ and ${}^{\#}(H^c \cap S) \geq 3$. Assume M is a plane which contains at least 3 points in $H^c \cap S$ and $M = \langle M \cap H^c \cap S \rangle$. Then, either*

i) *M is contained in S*

or

ii) *There exist a line L_0 and a point P_0 such that $M \cap S \cap H^c = (L_0 \cap H^c) \cup \{P_0\}$.*

Proof. Assume M is not contained in S. Let $L = M \cap H$, $M' = M \setminus L$ and $S_0 = M' \cap S$.

Case 1. Assume $L \subset S$. Since $^{\#}S_0 \geq 3$, we have $^{\#}(M \cap S) = {}^{\#}S_0 + {}^{\#}(L \cap S) \geq q + 4$. By the assumption, there exists a line L_1 in M which is not contained in S. Hence, by Lemma 3.4, we have $^{\#}(M \cap S) \leq 3q + 3$. Thus $^{\#}S_0 \leq 2q + 2$.

By Lemma 3.4, there is no line L_1 in M such that $^{\#}(L_1 \cap S) = 1$. Hence, $^{\#}(L' \cap S_0) \geq 1$ for every line L' in M.

Subcase 1-1. Assume there exists a line L_1 in M which is contained in $M \cap S$ and is different from L. Since every line in M passing through the point $L_1 \cap L$ contains at least one point in S_0. Hence,

$$^{\#}M \cap S \geq {}^{\#}L_1 + {}^{\#}L - 1 + (q - 1) = 3q. \tag{3}$$

If there would exist a line L_2 in M such that $^{\#}L_2 \cap S = 2$, then by Lemma 3.4, we had $^{\#}(M \cap S) \leq 2q + 3 < 3q$. This is a contradiction. Hence, every line in M passing through $L_1 \cap L$ contains at least two points in S_0. Thus, we have

$$^{\#}M \cap S \geq {}^{\#}L_1 + {}^{\#}L - 1 + 2(q - 1) = 4q - 1 > 3q + 3.$$

This contradicts Lemma 3.4.

Subcase 1-2. Assume every line $L_1 (\neq L)$ in M is not contained in S. Since $L_1 \cap L \in S$, by Lemma 3.3, we have $^{\#}(L_1 \cap S_0) = 1$ or 2. Put $\gamma = {}^{\#}S_0$. For a point Q in S_0, every line passing through Q contains at most one point in $S_0 \setminus \{Q\}$. Since there are exactly $q + 1$ such lines, we have $\gamma \leq q + 2$. For a point $P \in L$, every line passing through P contains at least one point in S_0. Hence, $\gamma \geq q$.

On the other hand,

$$^{\#}\{L_1 | \text{ line in } M \text{ such that } {}^{\#}(L_1 \cap S_0) = 2\} = \frac{\gamma(\gamma - 1)}{2}, \tag{4}$$

$$^{\#}\{L_1 | \text{ line in } M \text{ such that } {}^{\#}(L_1 \cap S_0) = 1\} = \gamma(q + 1 - (\gamma - 1)). \tag{5}$$

Since there is no line which does not meet S_0, we have

$$\begin{aligned} \frac{\gamma(\gamma - 1)}{2} + \gamma(q + 2 - \gamma) &= q^2 + q \\ &= {}^{\#}\{\text{lines in } M \text{ different from } L\}. \end{aligned}$$

However, we do not have an integral solution γ of this equality.

Case 2. Assume L is not contained in S. By Lemma 3.3, we have $\alpha = {}^{\#}(L \cap S) \leq 3$. Let P be a point in $L \setminus S$. Since $H \in \check{P}$, by the latter part of Lemma 3.3, we have $^{\#}(L_1 \cap S_0) \leq 2$ for every line L_1 in M passing through P. Hence, we have $\beta = {}^{\#}(M \cap S) \leq 2q + \alpha$.

If $\beta < 2q + \alpha$, then there is a line L_1 in M such that ${}^{\#}(L_1 \cap S) \leq 1$. Hence, we have $\beta \leq q + 3$ by Lemma 3.4.

Subcase 2-1. Assume $\beta = 2q + \alpha$. There exist a line L_1 in M such that ${}^{\#}(L_1 \cap S_0) \geq 3$, otherwise by the same argument as in Subcase 1-2, we have $\gamma = \beta - \alpha \leq q + 2$ which is a contradiction. Since $H \in \check{P}_1$ for $P_1 = L_1 \cap L$, by Lemma 3.3 we have $L_1 \subset S$ and ${}^{\#}(S_0 \setminus L_1) = q$.

Assume these q points lie on a line L_2. Then, there is a line L_3 passing through P_1 such that ${}^{\#}(L_3 \cap S) = 1$. Hence, by Lemma 3.4 we have $\beta \leq q + 3$ which is a contradiction.

Therefore, there are points Q_1, Q_2 in $S_0 \setminus L_1$ such that the line L_2 containing Q_1, Q_2 does not pass through P_1. Hence, $L_1 \cap L_2 \in S_0$. Thus, by Lemma 3.3, L_2 is contained in S. Since ${}^{\#}(S_0 \cap (L_1 \cup L_2)) = 2q - 1$, there is a point $Q \in S_0 \setminus (L_1 \cup L_2)$. Thus, there exists a line L_3 containing Q and two points in $S_0 \cap (L_1 \cup L_2)$, whence L_3 is contained in S by Lemma 3.3. Then, ${}^{\#}(S_0 \cap (L_1 \cup L_2 \cup L_3)) = 3q - 2$ which contradicts $\gamma = 2q$.

Subcase 2-2. Assume $\beta \leq q + 3$. If there were a line L_1 in M such that ${}^{\#}(L_1 \cap S) = 0$, then by Lemma 3.4, $\beta \leq 3$. Since $\gamma = \beta - \alpha \geq 3$, we had $\alpha = 0$ and $\beta = 3$. Again, by Lemma 3.4, $\beta \leq 2$ since $H \in \check{L}$. This is a contradiction.

Thus for a point P in $L \setminus S$, ${}^{\#}(L_1 \cap S) \geq 1$ for every line L_1 passing through P. This implies that $\gamma = \beta - \alpha \geq q$. Since $H \in \check{L}$, by Lemma 3.4 we have $\beta \leq \alpha q + 2$. Hence, there are the following 5 possibilities

i) $\beta = q + 1$, $\alpha = 1$
ii) $\beta = q + 2$, $\alpha = 2$
iii) $\beta = q + 3$, $\alpha = 3$
iv) $\beta = q + 2$, $\alpha = 1$
v) $\beta = q + 3$, $\alpha = 2$.

For cases 3, 3, 3, since $M = \langle S_0 \rangle$ and $\gamma = {}^{\#}S_0 = q$, these q points do not lie on a line. Hence, any triple of points in S_0 does not lie on a line.

Applying (4), (5) for $\gamma = q$, we have

$$ {}^{\#}\{L_1 | \text{line in } M \text{ such that } L_1 \cap S_0 = \phi\} = q^2 + q - \frac{q(q-1)}{2} - 2q = \frac{q(q-1)}{2}. $$

Since

$$ {}^{\#}\{L_1 | \text{line in } M \text{ such that } L_1 \cap L \cap S \neq \phi, L_1 \cap S_0 \neq \phi\} \geq \frac{q}{2}\alpha, $$

we have

$$ {}^{\#}\{L_1 | \text{line in } M \text{ such that } L_1 \neq L, L_1 \cap L \cap S \neq \phi, L_1 \cap S_0 = \phi\} \leq \frac{q}{2}\alpha. $$

Hence,

$$^{\#}\{L_1|\text{line in } M \text{ such that } L_1 \cap S = \phi\} \geq \frac{q(q-1)}{2} - \frac{q}{2}\alpha > 0.$$

Therefore, there exists a line L_1 in M such that $L_1 \cap S = \phi$. This is a contradiction.

For cases 3, 3, assuming any q points of S_0 lie on a line, as in the previous case, we have

$$^{\#}\{L_1|\text{line in } M \text{ such that } L_1 \cap S = \phi\} \geq \frac{(q-2)(q+1)}{2} - \alpha\frac{q}{2} > 0.$$

Hence, there exists a line L_1 which is contained in S. This completes the proof. □

LEMMA 3.6. *Let S, H be as in Lemma 3.5. Further, assume $^{\#}(H^c \cap S) = q^s$. Then, $\mathcal{L} = \langle H^c \cap S\rangle$ is of dimension s and contained in S.*

Proof. Let $r = \dim \mathcal{L}$. Since $q^r = {}^{\#}(\mathcal{L} \cap H^c) \geq q^s$, it is sufficient to prove that $\mathcal{L} \cap H^c$ is contained in S. In case $s = 1$, if $r \geq 2$, then one could choose non-collinear 3 points P_1, P_2, P_3 in $H^c \cap S$. Let $M = \langle P_1, P_2, P_3\rangle$. Then, by Lemma 3.5, $^{\#}(H^c \cap S) \geq {}^{\#}(H^c \cap M) \geq q+1$. This is a contradiction. Hence, $r = 1$.

Therefore, we may assume $s \geq 2$. It is sufficient to show that for every non-collinear 3 points P_1, P_2, P_3 in $H^c \cap S$, $\langle P_1, P_2, P_3\rangle \cap H^c$ is contained in $H^c \cap S$.

Assume there exists a plane M such that $H^c \cap M$ contains at least non-collinear 3 points in S and is not contained in S. Then, by Lemma 3.5, there is a line L_1 on M which is contained in S and a point $P \in L_1$ on $H^c \cap M$.

Take and fix a point R on $L_1 \cap H^c$. Let Q be a point in $(H^c \cap S) \setminus M$. Then, one can apply Lemma 3.5 for the plane $M' = \langle P, Q, R\rangle$. Since the line $\langle P, R\rangle$ is not contained in S, M' is not contained in S. Then, either the line $\langle P, Q\rangle$ or $\langle R, Q\rangle$ is contained in S. Since M' is not contained in S, not both of $\langle P, Q\rangle$ and $\langle R, Q\rangle$ are contained in S. If $\langle R, Q\rangle$ is contained in S, then the plane spanned by Q and L_1 is contained in S. Thus for each point Q in $(H^c \cap S) \setminus M$, one of the following possibilities occurs.

i) the plane spanned by Q and L_1 is contained in S.
ii) the line $\langle P, Q\rangle$ is contained in S.

Let

$$\begin{aligned}\delta &= {}^{\#}\{M' | \text{plane such that } L_1 \subset M' \subset S \cap \mathcal{L}\} \\ \varepsilon &= {}^{\#}\{L' | \text{line such that } P \in L' \subset S \cap \mathcal{L}\}.\end{aligned}$$

Since ${}^{\#}((M' \cap H^c) \setminus L_1) = q^2 - q$ and ${}^{\#}((L' \cap H^c) \setminus \{P\}) = q - 1$, we have

$$\begin{aligned}q^s &= {}^{\#}(H^c \cap S) \\ &= \delta(q^2 - q) + \varepsilon(q-1) + {}^{\#}(L' \cap H^c) + 1 \\ &= (\delta q + \varepsilon + 1)(q-1) + 2.\end{aligned}$$

Since $q \geq 5$, this equation does not hold. This completes the proof. □

Proof of Theorem 3.2. Without lose of generality, we may assume that $s \leq t \leq u$. First, assume $s > t$. let H_1 be hyperplane in $\mathcal{H}_s$, and let $\mathcal{L}_1 = \langle H_1^c \cap S \rangle$. Then, by Lemma 3.6, $\dim \mathcal{L}_1 = s$ and $\mathcal{L}_1 \subset S$. Let H_1' be another hyperplane in $\mathcal{H}_s$ and let $\mathcal{L'}_1 = \langle H'^c_1 \cap S \rangle$. Then, we shall show $\mathcal{L}_1 = \mathcal{L}_1'$.

For a hyperplane H, if $\mathcal{L}_1 \subset H$, then ${}^{\#}(H \cap S) \geq N(s) > N(s-1) + N(t) + N(u)$, whence $H \in (\mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{tu})$. Similarly if $\mathcal{L}_1' \subset H$, then $H \in (\mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{tu})$. On the other hand,

$$\begin{aligned}N(n-s-1) &= {}^{\#}\{H \in \mathbb{P}^{n*} | \mathcal{L}_1 \subset H\} \\ &\leq {}^{\#}(\mathcal{L}_t \cup \mathcal{L}_u \cup \mathcal{L}_{tu}) \\ &= N(n-s-1).\end{aligned}$$

Hence

$$\begin{aligned}\{H \in \mathbb{P}^{n*} | \mathcal{L}_1 \subset H\} &= \{H \in \mathbb{P}^{n*} | \mathcal{L}_1' \subset H\} \\ &= \mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{tu}.\end{aligned}$$

Thus, we have $\mathcal{L}_1 = \mathcal{L}_1'$.

Let $S' = S \setminus \mathcal{L}_1$. For $H \in \mathcal{H}_s$, since $\mathcal{L}_1 = \langle H^c \cap S \rangle$, we have $S' \subset H$. Thus, $\langle S' \rangle \subset \bigcap\{H | H \in \mathcal{H}_s\}$. Let $r = \dim\langle S' \rangle$. Then,

$$\begin{aligned}N(n-r-1) &= {}^{\#}\{H \in \mathbb{P}^{n*} | \langle S' \rangle \subset H\} \\ &\geq {}^{\#}\mathcal{H}_s \\ &= N(s).\end{aligned}$$

Hence, $r \leq n - s - 1 = t + u + 1$. Since, $\mathbb{P}^n = \langle S \rangle \subset \langle \mathcal{L}_1 \cup \langle S' \rangle \rangle$, we have

$$\begin{aligned}n = \dim\langle \mathcal{L}_1 \cup \langle S' \rangle \rangle &= \dim \mathcal{L}_1 + \dim\langle S' \rangle - \dim(\mathcal{L}_1 \cap \langle S' \rangle) \\ &\leq s + t + u + 1 - \dim(\mathcal{L}_1 \cap \langle S' \rangle),\end{aligned}$$

whence $\mathcal{L}_1 \cap \langle S' \rangle = \phi$.

Since, there is a bijection of $\mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{tu}$ to $\mathbb{P}^{(t+u+1)*}$, we can apply Theorem 2.3 to conclude the existence of the subspaces $\mathcal{L}_2$, $\mathcal{L}_3$ such that $\dim \mathcal{L}_2 = t$, $\dim \mathcal{L}_3 = u$ and $\mathcal{L}_2 \cup \mathcal{L}_3 = S'$. Thus, we get the conclusion, in case $s > t$.

Second, we assume $s = t > u$. In this case, we cannot get $\langle H_1^c \cap S\rangle = \langle H_2^c \cap S\rangle$ for H_1, $H_2 \in \mathcal{H}_s$. However, taking the third hyperplane $H_3 \in \mathcal{H}_s$, we can show that at least two of $\mathcal{L}_j = \langle H_j^c \cap S\rangle$ $(j = 1, 2, 3)$ coincide.

Assume $\mathcal{L}_i \neq \mathcal{L}_j$ $(1 \leq i < j \leq 3)$. Let $\mathcal{H}_j = \{H \in \mathbb{P}^{n*} | \mathcal{L}_j \subset H\}$ and $\mathcal{H}_{ij} = \{H \in \mathbb{P}^{n*} | \mathcal{L}_i \cup \mathcal{L}_j \subset H\}$. Then,

$$\begin{aligned} & {}^{\#}\{H \in \mathbb{P}^{n*} | \mathcal{L}_j \subset H \text{ for some } j\} \\ \geq\ & {}^{\#}\mathcal{H}_1 +{}^{\#} \mathcal{H}_2 +{}^{\#} \mathcal{H}_3 -{}^{\#} \mathcal{H}_{12} -{}^{\#} \mathcal{H}_{13} -{}^{\#} \mathcal{H}_{23}. \end{aligned}$$

Since $\dim\langle \mathcal{L}_i \cup \mathcal{L}_j\rangle \geq s + 1$, we have ${}^{\#}\mathcal{H}_{ij} \leq N(n - s - 2)$. As in the case of $s > t$, we have $H \in \mathcal{H}_s \cup \mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{su} \cup \mathcal{H}_{tu}$ if $\mathcal{L}_j \subset H$. Thus

$$\begin{aligned} 2N(n-s-1) - N(u) &= {}^{\#}(\mathcal{H}_s \cup \mathcal{H}_t \cup \mathcal{H}_u \cup \mathcal{H}_{su} \cup \mathcal{H}_{tu}) \\ &\geq 3N(n-s-1) - 3N(n-s-2). \end{aligned}$$

This is a contradiction.

Assume $\mathcal{L}_1 \neq \mathcal{L}_2$. Let $S_j' = S \setminus \mathcal{L}_j$ $(j = 1, 2)$. Then, for $H \in \mathcal{H}_s$, if $\mathcal{L}_j = \langle H^c \cap S\rangle$, then $S_j' \subset H$.

Consider 2 subspaces $\mathcal{L}_j' = \bigcap\{H \in \mathcal{H}_s | S_j' \subset H\} \supset \langle S_j'\rangle$. Let $r_j = \dim\langle S_j'\rangle$. As in the previous case, we have $N(n-r_1-1)+N(n-r_2-1) \geq 2N(s)$, whence $r_1 \leq s + u + 1$ or $r_2 \leq s + u + 1$. We assume, say, $r_1 \leq s+u+1$. Then $\mathcal{L}_1 \cap \langle S_1'\rangle = \phi$ and $\dim\langle S_1'\rangle = s+u+1$. Then, we can apply Theorem 2.3 again, to obtain the conclusion.

Finally, we consider the case $s = t = u$. In this case, we show that at least two of $\mathcal{L}_j = \langle H_j^c \cap S\rangle$, $H_j \in \mathcal{H}_s$ $(j = 1, \ldots, 4)$ coincide. Then, we have $\dim\langle S \setminus \mathcal{L}_1\rangle = 2s + 1$ and apply Theorem 2.3. The procedure is similar to the above case, so we omit the proof. This completes the proof of the theorem. □

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