Bull. Korean Math. Soc. **38** (2001), No. 4, pp. 679–687

REMARKS ON THE STABILITY OF ADDITIVE FUNCTIONAL EQUATION

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Abstract. In this paper, using an idea from the direct method of Hyers, we give the conditions in order for a linear mapping near an approximately additive mapping to exist.

1. Introduction

In 1940, S. M. Ulam [11] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive functions was solved by D. H. Hyers [4] under the assumption that G_1 and G_2 are Banach spaces. Let E_1 be a real normed space and E_2 a real Banach space. In 1941 D. H. Hyers [4] considered approximately additive mappings $f : E_1 \to E_2$ satisfying $\|f(x+y) - f(x) - f(y)\| \le \epsilon$ for all $x, y \in E_1$. He proved that the limit $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E_1$ and that $T : E_1 \to E_2$ is the unique additive mapping satisfying $\|f(x) - T(x)\| \le \epsilon$. No continuity condition is required

Received April 2, 2001. Revised June 22, 2001.
2000 Mathematics Subject Classification: 39B52, 39B72.
Key words and phrases: Hyers-Ulam-Rassias stability, functional equation.
This work was supported by grant No. 981-0102-012-1 from the KOSEF.

for this result, but if $f(tx)$ is continuous in the real t for each fixed x, then the mapping T is linear. In 1978, a generalized solution to Ulam's problem for approximately linear mappings was given by Th. M. Rassias [8]. He considered a mapping $f : E_1 \to E_2$ satisfying the condition of continuity of $f(tx)$ in t for each fixed x and assumed the weaker condition $\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$, for all $x, y \in E_1$, where $\theta \geq 0$ and $0 \leq p < 1$. He proved that the above function $T : E_1 \to E_2$ is the unique linear mapping satisfying $\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$. The proof given in [8] works also when $p < 0$. In 1990 Th. M. Rassias asked the question whether such a theorem can also be proved for $p \geq 1$. In [2] Z. Gajda followed a similar approach as in [8] and obtained a solution of this problem for $p > 1$. His result states that the mapping $T : E_1 \to E_2$ defined by $T(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ is the unique additive mapping satisfying $\|f(x) - T(x)\| \leq \frac{2\theta}{2^p-2}\|x\|^p$. The problem when $p = 1$ is not true. Counterexamples for the corresponding assertion in the case $p = 1$ were constructed by Gajda [2] and Rassias and Šemrl [9]. Y. H. Lee and K. W. Jun [7] have improved the stability problem for approximately additive mappings. This leads to the problem of proving the similar results replacing the right-hand side with $H(\|x\|, \|y\|)$, where H is a two variable real function on $\mathbb{R}_+ \times \mathbb{R}_+$. Some answers to this question were given recently by Rassias and Šemrl [10] and Isac and Rassias [5]. Therefore the general question is to find weaker conditions under which the direct method works. The stability problems of several functional equations have been extensively investigated by a number of authors. In this paper, using an idea from the direct method of Hyers, we shall give conditions in order for a linear mapping near an approximately additive mapping to exist.

2. Stability of additive functional equation

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers. Recall that a mapping $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is homogeneous of degree $p \geq 0$ if it satisfies $H(tu, tv) = t^p H(u, v)$ for all $t, u, v \in \mathbb{R}^+$. The following theorem is due to Th. M. Rassias and P. Šemrl [10].

THEOREM 2.1 [10, Theorem 1]. *Let E_1 be a real normed space, E_2 a Banach space. Assume that $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically increasing symmetric homogeneous function of degree p, where*

$p \geq 0,\ p \neq 1$ and define $H(1,1) = \theta$. Let $f : E_1 \to E_2$ satisfy

$$\|f(x+y) - f(x) - f(y)\| \leq H(\|x\|, \|y\|)$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{|2 - 2^p|} H(\|x\|, \|x\|) = \frac{\theta}{|2 - 2^p|} \|x\|^p$$

for all $x \in E_1$. Moreover, if for every fixed $x \in E_1$, there exists a real number $\delta_x > 0$ such that the function $t \mapsto \|f(tx)\|$ is bounded on $[0, \delta_x]$, then T is linear.

We will show that Theorem 2.1 is still valid if the condition of monotonically increasing symmetric homogeneous function of degree p is changed to the weaker condition as follows.

THEOREM 2.2. *Let E_1 be a real normed space, E_2 a Banach space. Let $p \geq 0,\ p \neq 1$ and let $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping satisfying $H(tx, ty) \leq t^p H(x, y)$ for all $t, x, y \in \mathbb{R}^+$. Suppose that a function $f : E_1 \to E_2$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq H(\|x\|, \|y\|) \tag{2.1}$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{|2 - 2^p|} H(\|x\|, \|x\|) \leq \frac{H(1,1)}{|2 - 2^p|} \|x\|^p \tag{2.2}$$

for all $x \in E_1$. Moreover, if for every fixed $x \in E_1$, there exists a real number $\delta_x > 0$ such that the function $t \mapsto \|f(tx)\|$ is bounded on $[0, \delta_x]$, then T is linear.

Proof. Case $p < 1$.

Let x be any fixed element in E_1. The relation (2.1) for $y = x$ yields $\|f(2x) - 2f(x)\| \leq H(\|x\|, \|x\|)$, which implies

$$\|2^{-1} f(2x) - f(x)\| \leq 2^{-1} H(\|x\|, \|x\|). \tag{2.3}$$

Claim that

$$\|2^{-n}f(2^n x) - f(x)\| \le \frac{1 - 2^{n(p-1)}}{2 - 2^p} H(\|x\|, \|x\|) \tag{2.4}$$

for any positive integer n. The verification of (2.4) follows by induction on n. Note that (2.4) reduces to (2.3) for $n = 1$. Assume now that (2.4) holds and we want to prove it for the case $n+1$. We write the inequality (2.4) for $2x$ instead of x and divide by 2. We obtain

$$\begin{aligned} \|2^{-n-1}f(2^{n+1}x) - 2^{-1}f(2x)\| &\le \frac{1 - 2^{n(p-1)}}{2(2 - 2^p)} H(2\|x\|, 2\|x\|) \\ &\le \frac{1 - 2^{n(p-1)}}{2(2 - 2^p)} 2^p H(\|x\|, \|x\|). \end{aligned}$$

By the triangle inequality, together with the inequality (2.3), we get

$$\begin{aligned} \|2^{-n-1}f(2^{n+1}x) - f(x)\| &\le \|2^{-n-1}f(2^{n+1}x) - 2^{-1}f(2x)\| \\ &\quad + \|2^{-1}f(2x) - f(x)\| \\ &\le \frac{1 - 2^{(n+1)(p-1)}}{2 - 2^p} H(\|x\|, \|x\|). \end{aligned}$$

Thus (2.4) is valid for any positive integer n, and it follows that

$$\|2^{-n}f(2^n x) - f(x)\| \le \frac{1}{2 - 2^p} H(\|x\|, \|x\|) \tag{2.5}$$

because $\{2^{n(p-1)}\}$ converges to zero, as $p < 1$. However, for $m > n > 0$,

$$\begin{aligned} \|2^{-m}f(2^m x) - 2^{-n}f(2^n x)\| &= 2^{-n}\|2^{-(m-n)}f(2^{m-n}2^n x) - f(2^n x)\| \\ &\le 2^{-n}\frac{1 - 2^{(m-n)(p-1)}}{2 - 2^p} H(2^n\|x\|, 2^n\|x\|) \\ &\le 2^{n(p-1)}\frac{1 - 2^{(m-n)(p-1)}}{2 - 2^p} H(\|x\|, \|x\|). \end{aligned}$$

Because the right-hand side of the above sequence of inequalities tends to zero if n tends to infinity. Therefore $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence. But E_2, as a Banach space, is complete, and thus the sequence converges. Define $T(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$ for all $x \in E_1$. Inequality (2.4) implies

$$\|T(x) - f(x)\| \le \frac{1}{2 - 2^p} H(\|x\|, \|x\|) \le \frac{H(1,1)}{2 - 2^p}\|x\|^p.$$

It follows from (2.1) that

$$\begin{aligned}2^{-n}\|f[2^n(x+y)]-f[2^nx]-f[2^ny]\| &\leq 2^{-n}H(2^n\|x\|,2^n\|y\|)\\ &\leq 2^{n(p-1)}H(\|x\|,\|y\|),\end{aligned}$$

which implies $T(x+y)=T(x)+T(y)$, since the sequence $\{2^{n(p-1)}\}$ converges to zero when n tends to infinity. This condition implies that $T(2^nx)=2^nT(x)$ for any $x\in E_1$. We want to prove that T is the unique such function. Suppose that there exists another one V such that $\|V(x)-f(x)\|\leq \frac{1}{2-2^q}H_1(\|x\|,\|x\|)$ for a certain function H_1 with the corresponding $q<1$. Then we have

$$\begin{aligned}\|T(x)-V(x)\| &= 2^{-n}\|T(2^nx)-V(2^nx)\|\\ &\leq 2^{-n}\frac{H(2^n\|x\|,2^n\|x\|)}{2-2^p}+2^{-n}\frac{H_1(2^n\|x\|,2^n\|x\|)}{2-2^q}\\ &\leq 2^{n(p-1)}\frac{H(\|x\|,\|x\|)}{2-2^p}+2^{n(q-1)}\frac{H_1(\|x\|,\|x\|)}{2-2^q}.\end{aligned}$$

Since both terms of the right-hand side in the above inequalities tend to zero for n tending to infinity, T coincides with V.

Case $1<p$.

Putting $\frac{x}{2}$ in place of x and y in inequality (2.1), we obtain

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|\leq H\left(\left\|\frac{x}{2}\right\|,\left\|\frac{x}{2}\right\|\right)\leq\frac{1}{2^p}H(\|x\|,\|x\|) \tag{2.6}$$

for all $x\in E_1$. Hence for each $n\in\mathbb{N}$ and every $x\in E_1$, we have by (2.6)

$$\begin{aligned}&\left\|f(x)-2^nf\left(\frac{x}{2^n}\right)\right\|\\ &\leq \left\|f(x)-2f\left(\frac{x}{2}\right)\right\|+2\left\|f\left(\frac{x}{2}\right)-2f\left(\frac{x}{2^2}\right)\right\|\\ &\quad+\cdots+2^{n-1}\left\|f\left(\frac{x}{2^{n-1}}\right)-2f\left(\frac{x}{2^n}\right)\right\|\\ &\leq 2^{-p}H(\|x\|,\|x\|)+2^{1-p}H\left(\frac{\|x\|}{2},\frac{\|x\|}{2}\right)\\ &\quad+\cdots+2^{n-1-p}H\left(\frac{\|x\|}{2^{n-1}},\frac{\|x\|}{2^{n-1}}\right)\\ &\leq 2^{-p}H(\|x\|,\|x\|)+2^{1-2p}H(\|x\|,\|x\|)+\cdots\\ &\quad+2^{n-1-np}H(\|x\|,\|x\|)\\ &=2^{-p}(1+2^{1-p}+2^{2(1-p)}+\cdots+2^{(n-1)(1-p)})H(\|x\|,\|x\|)\\ &\leq\frac{1}{2^p-2}H(\|x\|,\|x\|).\end{aligned} \tag{2.7}$$

Now, fix an $x \in E_1$ and choose arbitrary $m, n \in \mathbb{N}$ such that $m > n$. Then by (2.7)

$$\begin{aligned}\Big\|2^m f\Big(\frac{x}{2^m}\Big) - 2^n f\Big(\frac{x}{2^n}\Big)\Big\| &= 2^n \Big\|f\Big(\frac{x}{2^n}\Big) - 2^{m-n} f\Big(\frac{1}{2^{m-n}}\frac{x}{2^n}\Big)\Big\| \\ &\leq \frac{2^n}{2^p-2} H\Big(\frac{\|x\|}{2^n}, \frac{\|x\|}{2^n}\Big) \\ &\leq \frac{2^n}{(2^p-2)2^{np}} H(\|x\|, \|x\|),\end{aligned}$$

which becomes arbitrarily small as $n \to \infty$. On account of the completeness of the space E_2, this implies that the sequence $\{2^n f(\frac{x}{2^n})\}$ is convergent for each $x \in E_1$. Thus T is correctly defined by $T(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. Moreover it satisfies condition

$$\|f(x) - T(x)\| \leq \frac{1}{2^p-2} H(\|x\|, \|x\|) \leq \frac{H(1,1)}{2^p-2}\|x\|^p,$$

which results on letting $n \to \infty$ in (2.7). Finally, replacing x by $\frac{x}{2^n}$ and y by $\frac{y}{2^n}$ in (2.1) and then multiplying both sides of the resulting inequality by 2^n, we get

$$\Big\|2^n f\Big(\frac{x+y}{2^n}\Big) - 2^n f\Big(\frac{x}{2^n}\Big) - 2^n f\Big(\frac{y}{2^n}\Big)\Big\| \leq 2^{n(1-p)} H(\|x\|, \|y\|)$$

for $x, y \in E_1$. Since the right-hand side of this inequality tends to zero as $n \to \infty$, it becomes apparent that the mapping T is additive. It is also clear what has to be changed in the proof of the uniqueness of T. The remaining assertion in the theorem is proved by the same argument as that of [10]. Assume that for every fixed $x \in E_1$ there exists a positive real δ_x such that the function $t \mapsto \|f(tx)\|$ is bounded on $[0, \delta_x]$. Fix $z \in E_1$ and $\varphi \in E_2^*$ (the dual space of E_2). Let us denote

$$M_z = \sup\{\|f(tz)\| : t \in [0, \delta_z]\}.$$

Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = \varphi(T(tz))$. It is obvious that ϕ is additive. For any real number $t \in [0, \delta_z]$, we have

$$\begin{aligned}|\phi(t)| = |\varphi(T(tz))| \leq \|\varphi\|\|T(tz)\| &\leq \|\varphi\|(\|T(tz) - f(tz)\| + \|f(tz)\|) \\ &\leq \|\varphi\| \left(\frac{1}{|2^p-2|} H(\|tz\|, \|tz\|) + M_z\right) \\ &\leq \|\varphi\| \left(\frac{\delta_z^p}{|2^p-2|} H(\|z\|, \|z\|) + M_z\right).\end{aligned}$$

It is a well known fact that if an additive function $\phi : \mathbb{R} \to \mathbb{R}$ is bounded on an interval of positive length, then it is of the form $\phi(t) = \phi(1)t$ for all real values of t [1, Corollary 2.5]. Therefore $\varphi(T(tz)) = \varphi(tT(z))$ for any $t \in \mathbb{R}$, and consequently T is a linear mapping. □

REMARK 2.3. In Theorem 2.2, the mapping T is also linear if for each $x \in E_1$ the transformation $t \mapsto f(tx)$ is continuous [8]. The condition (2.2) is still true for all $x \in E_1 - \{0\}$ when $p < 0$. Furthermore in case $p < 1$, the condition $H(2x, 2y) \leq 2^p H(x, y)$ has been only used and in case $p > 1$, we have used the condition $H(\frac{1}{2}x, \frac{1}{2}y) \leq \frac{1}{2^p} H(x, y)$. Thus it is easy for someone to see that the proof of Theorem 2.2 as given above shows that the condition

$$\|f(x) - T(x)\| \leq \frac{1}{|2 - 2^p|} H(\|x\|, \|x\|)$$

is still true under the condition $H(tx, ty) \leq t^p H(x, y)$ for all $x, y \in \mathbb{R}^+$, where $t = 2$, $\frac{1}{2}$ and $p \in \mathbb{R} - \{1\}$. Hence we obtain the following corollaries, which were the results of Th. M. Rassias [8], Z. Gajda [2], G. Isac and Th. M. Rassias [6]. In particular, in case $p < 1$, the conclusion of Theorem 2.2 coincides with the result of Găvruta [3].

The following corollary can be found in [2].

COROLLARY 2.4. *Let E_1 be a real normed space, E_2 a Banach space. Let $f : E_1 \to E_2$ be a mapping for which there exist two constant $\epsilon \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$ ($E_1 - \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2 - 2^p|} \|x\|^p$$

for all $x \in E_1$ ($E_1 - \{0\}$ if $p < 0$). Moreover, if for every fixed $x \in E_1$ the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then the mapping T is linear.

The following corollary can be referred in [6].

COROLLARY 2.5. *Let E_1 be a real normed space, E_2 a Banach space. Let $f : E_1 \to E_2$ be a mapping for which there exist three constant $\epsilon \geq 0$ and $p_1,\ p_2 \in \mathbb{R} - \{1\}$ such that $p_2 \leq p_1 < 1$ or $1 < p_2 \leq p_1$ and*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^{p_1} + \|y\|^{p_2})$$

for all $x, y \in E_1$ ($E_1 - \{0\}$ if $p_i < 0$ for some i). Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \begin{cases} \frac{\epsilon(\|x\|^{p_1} + \|x\|^{p_2})}{2-2^{p_1}}, & (p_2 \leq p_1 < 1), \\ \frac{\epsilon(\|x\|^{p_1} + \|x\|^{p_2})}{2^{p_2}-2}, & (1 < p_2 \leq p_1) \end{cases}$$

for all $x \in E_1$ ($E_1 - \{0\}$ if $p_i < 0$ for some i). Moreover, if for every fixed $x \in E_1$ the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then the mapping T is linear.

Proof. To apply Theorem 2.2, we consider $H(t,s) = \epsilon(t^{p_1} + s^{p_2})$. In case $p_2 \leq p_1 < 1$, we have

$$H(2t, 2s) = \epsilon(2^{p_1}t^{p_1} + 2^{p_2}s^{p_2}) \leq 2^{p_1}\epsilon(t^{p_1} + s^{p_2}) = 2^{p_1}H(t,s).$$

In case $1 < p_2 \leq p_1$, we get

$$H\Big(\frac{t}{2}, \frac{s}{2}\Big) = \epsilon\Big(\frac{t^{p_1}}{2^{p_1}} + \frac{s^{p_2}}{2^{p_2}}\Big) \leq \frac{1}{2^{p_2}}\epsilon(t^{p_1} + s^{p_2}) = \frac{1}{2^{p_2}}H(t,s).$$

Applying Theorem 2.2 and using Remark 2.3, we obtain the results. □

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