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A NOTE ON CLARKSON'S INEQUALITIES

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Abstract. It is proved that if for each n, $1 \le p_n \le 2$ and the (p_n, p_n') Clarkson inequality holds in each Banach space X_n then the (t, t') Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$, the ℓ_r-sum of X_n's, where $1 \le r < \infty$, $t = \min\{p, r, r'\}$ and $p = \inf\{p_n\}$. The (p, p') Clarkson inequality is preserved by quotient maps and a new proof of a Takahashi-Kato theorem stating that the (p, p') Clarkson inequality holds in a Banach space X if and only if it holds in its dual space X^* is given.

1. Introduction

In 1936, while proving the uniform convexity of ℓ_p and L_p $(1 < p < \infty)$ Clarkson [2] proved that if X is either ℓ_p or L_p and $x, y \in X$ then for $p \ge 2$ $(\frac{1}{p} + \frac{1}{p'} = 1)$

$$\|x+y\|^p + \|x-y\|^p \le 2(\|x\|^{p'} + \|y\|^{p'})^{p-1}, \tag{1}$$

$$2(\|x\|^p + \|p\|^p)^{p'-1} \le \|x+y\|^{p'} + \|x-y\|^{p'}, \tag{2}$$

$$2(\|x\|^p + \|p\|^p) \le \|x+y\|^p + \|x-y\|^p \le 2^{p-1}(\|x\|^p + \|y\|^p). \tag{3}$$

For $1 < p \le 2$ these inequalities hold in the reverse sense.

Setting $x + y = \xi$, $x - y = \eta$, we can see that (1) is equivalent to (2), and the right side of (3) is equivalent to the left side of (3). Moreover, (3) follows from (1) and inequality $2(a^{p'} + b^{p'})^{p-1} \le 2^{p-1}(a^p + b^p)$ for positive real numbers a and b. Therefore, any Banach space X satisfying inequality (1) satisfies the rest inequalities and the uniform convexity of X follows. For $1 < p \le 2$, the inequality corresponding to (2) is

$$\|x+y\|^{p'} + \|x-y\|^{p'} \le 2(\|x\|^p + \|y\|^p)^{p'-1}, \tag{4}$$

which is equivalent to (1).

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Later, Clarkson's inequalities have been studied in various Banach spaces and the inequalities themselves have been generalized in various ways(Ch XVIII in [10], [3], [4], [5], [7], [8], [12], [13]). Extensive studies of Clarkson's inequalities were done by Kato and Miyazaki([4], [5], [6], [11]), and also by Kato and Takahashi([6], [7], [12]). Following Takahashi and Kato[12], let us say that for $1 \le p \le 2$ the (p, p') Clarkson inequality holds in a Banach space X if

$$(5) \qquad (\|x+y\|^{p'} + \|x-y\|^{p'})^{1/p'} \le 2^{1/p'}(\|x\|^p + \|y\|^p)^{1/p}$$

holds for all x and y in X. For $p = 1$, inequality (5) should be understood to be

$$\max\{\|x+y\|, \|x-y\|\} \le \|x\| + \|y\|.$$

Therefore, the $(1, \infty)$ Clarkson inequality holds in every Banach space. Setting $x + y = \xi$ and $x - y = \eta$, we can see that (5) is equivalent to :

$$(6) \qquad (\|x\|^{p'} + \|y\|^{p'})^{1/p'} \le 2^{-1/p}(\|x+y\|^p + \|x-y\|^p)^{1/p}.$$

In 1997, Takahashi and Kato [12] proved that if for $1 \le p \le 2$, the (p, p') Clarkson inequality holds in a Banach space X, then the (t, t') Clarkson inequality holds in Lebesgue-Bochner space $L_r(X)$ $(1 \le r < \infty)$, where $t = \min\{p, r, r'\}$.

In this paper we will obtain a result analogous to that of Takahashi and Kato [12]. In Theorem 3, we will prove that if for each n, $1 \le p_n \le 2$ and the (p_n, p_n') Clarkson inequality holds in each Banach space X_n then the (t, t') Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$, where $t = \min\{p, r, r'\}$, $p = \inf\{p_n\}$ and $1 \le r < \infty$. In Theorem 4, we will prove that the (p, p') Clarkson inequality is preserved by quotient maps.

2. Clarkson's inequality

We begin with reviewing a few definitions. If $\{X_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces X_n's, for $1 \le p < \infty$ the ℓ_p-sum $(\sum_{n=1}^{\infty} X_n)_p$ of X_n's is the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X_n$ with $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. This space is a Banach space under the norm defined by $\|x\| = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$. The dual space of $(\sum_{n=1}^{\infty} X_n)_p$ is $(\sum_{n=1}^{\infty} X_n^*)_{p'}$, where X_n^* is the dual space of X_n and p' is the conjugate exponent of p. The ℓ_p-sum of two Banach spaces X and Y is defined in an obvious manner and will be denoted by $X \oplus_p Y$. The dual space of $X \oplus_p Y$ is $X^* \oplus_{p'} Y^*$.

The Rademacher functions r_n's are defined by $r_n(t) = \mathrm{sgn}(\sin(2^n \pi t))$, $0 \le t \le 1$. Observe that if X is a Banach space and $x, y \in X$ then

$$\int_0^1 \|x + r_1(t)y\| \, dt = \frac{1}{2}\{\|x+y\| + \|x-y\|\}.$$

We need two useful theorems due to Kato and Takahashi ([7], [12]).

THEOREM 1 [12]. *Suppose $1 \le r < p \le 2$. If the (p,p') Clarkson inequality holds in a Banach space X, then the (r,r') Clarkson inequality holds in X.*

Kato and Takahashi [7] gave an elegant proof of the following theorem. However, we will give a new proof of the theorem for its own interest.

THEOREM 2 [7]. *For $1 \le p \le 2$, the (p,p') Clarkson inequality holds in a Banach space X if and only if it holds in X^*.*

New proof. Suppose the (p,p') Clarkson inequality holds in X, and $x^*, y^* \in X^*$. We will prove the inequality (6) for x^* and y^*. Let $\delta > 0$. We choose $(x,y) \in X \oplus_p X$ such that $\|x\|^p + \|y\|^p = 1$ and $(\|x^*\|^{p'} + \|y^*\|^{p'})^{1/p'} \le (1+\delta)(x^*(x) + y^*(y))$. Then we have

$$\begin{aligned}
& \frac{1}{1+\delta}(\|x^*\|^{p'} + \|y^*\|^{p'})^{1/p'} \\
\le\; & x^*(x) + y^*(y) \\
=\; & \int_0^1 (x^* + r_1(t)y^*)(x + r_1(t)y) \, dt \\
\le\; & \Big(\int_0^1 \|x^* + r_1(t)y^*\|^p \, dt\Big)^{1/p} \Big(\int_0^1 \|x + r_1(t)y\|^{p'} \, dt\Big)^{1/p'} \\
=\; & \{\frac{1}{2}(\|x^*+y^*\|^p + \|x^*-y^*\|^p)\}^{1/p} \{\frac{1}{2}(\|x+y\|^{p'} + \|x-y\|^{p'})\}^{1/p'} \\
\le\; & 2^{-\frac{1}{p}}(\|x^*+y^*\|^p + \|x^*-y^*\|^p)^{1/p}(\|x\|^p + \|y\|^p)^{1/p}.
\end{aligned}$$

Since $\delta > 0$ is arbitrary and $\|x\|^p + \|y\|^p = 1$, inequality (6) holds in X^*, and hence the (p,p') Clarkson inequality holds in X^*.

Conversely, if the (p,p') Clarkson inequality holds in X^* then it holds in X^{**} and hence in $X (\subseteq X^{**})$. □

Combining the Clarkson's original proof of a Clarkson's inequality for ℓ_p [2] and Takahashi-Kato proof of the (t,t') Clarkson inequality for $L_r(X)$ [12], we have the following :

THEOREM 3. *Suppose $1 \leq r < \infty$, $1 \leq p_n \leq 2$ for each n and $p = \inf\{p_n\}$. If for each n, the (p_n, p'_n) Clarkson inequality holds in X_n, then the (t, t') Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$, where*

$$t = \begin{cases} r & \text{if} \quad 1 \leq r \leq p \\ p & \text{if} \quad p \leq r \leq p' \\ r' & \text{if} \quad p' \leq r < \infty \end{cases} .$$

Proof. Since the $(1, \infty)$ Clarkson inequality for $(\sum_{n=1}^{\infty} X_n)_r$ is trivial, we assume that $r > 1$ and $p > 1$.

Let $p \leq r \leq p'$, and let $x = (x_n), y = (y_n) \in (\sum_{n=1}^{\infty} X_n)_r$ with $x_n, y_n \in X_n$. Then

$$\begin{aligned}
& \|x+y\|^{p'} + \|x-y\|^{p'} \\
= \; & (\sum_{n=1}^{\infty} \|x_n + y_n\|^r)^{p'/r} + (\sum_{n=1}^{\infty} \|x_n - y_n\|^r)^{p'/r} \\
= \; & (\sum_{n=1}^{\infty} (\|x_n + y_n\|^{p'})^{r/p'})^{p'/r} + (\sum_{n=1}^{\infty} (\|x_n - y_n\|^{p'})^{r/p'})^{p'/r} \\
\leq \; & (\sum_{n=1}^{\infty} (\|x_n + y_n\|^{p'} + \|x_n - y_n\|^{p'})^{r/p'})^{p'/r} \\
& \text{(by Minkowski's inequality for } r/p' \leq 1) \\
\leq \; & 2(\sum_{n=1}^{\infty} (\|x_n\|^p + y_n\|^p)^{r/p})^{p'/r} \\
& \text{(by the } (p_n, p'_n) \text{ Clarkson inequalities in } X_n\text{'s)} \\
\leq \; & 2((\sum_{n=1}^{\infty} \|x_n\|^r)^{p/r} + (\sum_{n=1}^{\infty} \|y_n\|^r)^{p/r})^{p'/p} \\
& \text{(by Minkowski's inequality for } r/p \geq 1) \\
= \; & 2(\|x\|^p + \|y\|^p)^{p'/p}.
\end{aligned}$$

Therefore, the (p, p') Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$.

If $1 < r \leq p$, then by Theorem 1 the (r, r') Clarkson inequality holds in every X_n and hence in $(\sum_{n=1}^{\infty} X_n)_r$ by the preceding part. If $p' < r < \infty$, then $1 < r' < p$ and the (r', r) Clarkson inequality holds in every X_n^* and hence in $(\sum_{n=1}^{\infty} X_n^*)_{r'} = ((\sum_{n=1}^{\infty} X_n)_r)^*$. Therefore, the (r', r) Clarkson inequality holds in $(\sum_{n=1}^{\infty} X_n)_r$. □

Recall that a linear map T from a Banach space X to a Banach space Y is called a quotient map if T carries the open unit ball of X onto the open unit ball of Y. The (p, p') Clarkson inequality is preserved by quotient maps. More specifically we have :

THEOREM 4. *Suppose X and Y are Banach spaces, and the (p, p') Clarkson inequality holds in X. If there exists a quotient map $T : X \to Y$, then the (p, p') Clarkson inequality holds in Y. In particular, if Z is a closed subspace of X, then the (p, p') Clarkson inequality holds in X/Z.*

Proof. Suppose the (p, p') Clarkson inequality holds in X and $T : X \to Y$ is a quotient map. Then $T^* : Y^* \to X^*$ is an isometry into X^*. Since the (p, p') Clarkson inequality holds in X^*. It holds in $T^*(Y^*)$ and hence in Y^*. By Theorem 2, the (p, p') Clarkson inequality holds in Y. The (p, p') Clarkson inequality for X/Z is obvious. □

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