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POSITIVE COEXISTENCE OF STEADY STATES FOR COMPETITIVE INTERACTING SYSTEM WITH SELF-DIFFUSION PRESSURES

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ABSTRACT. We discuss the existence of positive solutions to a certain nonlinear elliptic system representing a competition interaction with self-diffusion. The method used here is a fixed point index theory in a positive cone. We give a sufficient condition for the existence of positive solutions.

1. Introduction and existence theorem

In this paper, we consider the following coupled elliptic system representing a competitive interaction between two different species with self-diffusion :

$$(1.1) \qquad \begin{cases} -\Delta\varphi(u)u = uf(u,v) & \\ -\Delta\psi(v)v = vg(u,v) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian operator, Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and the functions φ, ψ, f, g satisfy certain conditions.

These equations can be thought of as the steady-state equations for a system of generalized Lotka-Volterra equations under suitable conditions on f and g. u and v may represent the densities of two species of interacting populations in problems arising in many applications, namely biology, ecology, etc. The functions f and g are called the relative growth rates of those populations. In mathematical model, two species compete each other if these two relative growth rates are decreasing in the other

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opposer, respectively. For example, if u and v are in competition, then $f_v < 0$ and $g_u < 0$. See Chapter 14 in [11] for details.

We say that the system (1.1) has a *positive solution* (u, v) if $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$. The existence of a positive solution (u, v) to system (1.1) is also called a *positive coexistence.*

In [5], L. Li and R. Logan investigated the existence of positive solutions of the elliptic system under Dirichlet boundary conditions using the fixed point index theory for the competition model:

$$\begin{cases} -\Delta u = u[f(u) - g(v)] & \\ -\Delta v = v[b(v) - a(u)] & \text{in } \Omega. \end{cases}$$

In [7], A. Leung and G. Fan, by using the Schauder fixed point theorem, found a positive solution for the following elliptic systems between appropriate upper and lower solutions under some conditions of f and g.

$$\begin{cases} -\Delta\varphi(u) = f(x, u, v) & \\ -\Delta\varphi(v) = g(x, u, v) & \text{in } \Omega, \\ (u, v) = (0, 0) & \text{on } \partial\Omega. \end{cases}$$

In [8], W. Ruan considered the coupled competition elliptic system with the linear diffusion and growth rates under homogeneous Dirichlet boundary condition :

$$(1.2) \qquad \begin{cases} -\Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)]u = u(a_1 - b_{11}u - b_{12}v) & \\ -\Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)]v = v(a_2 - b_{21}u - b_{22}v) & \text{in } \Omega, \end{cases}$$

where α_i, β_{ij}, a_i, b_{ij} are nonnegative constants with $\alpha_i > 0$, $b_{ii} > 0$ for $i = 1, 2$. The author applied the index theory to show the existence of positive solutions to the system (1.2). This system (1.2) was proposed first by Shigesada *et al.* in [10].

If the diffusion rate in reaction-diffusion equations only depends on the density of one species itself, it is called self-diffusion pressure. On the other hand, if the diffusion is affected by the density of the other species, we call it cross-diffusion pressure. One can also refer [2], [6] for systems with cross-diffusion pressures which are linear with respect to the densities.

In this paper, we give a sufficient condition for the existence of positive solutions of system (1.1) which has a self-diffusion pressure by using the method of the fixed point index of compact operators in a positive cone.

Consider the system (1.1) satisfying the following hypotheses:

(H1.1) $\varphi(0), \psi(0) > 0$ and $\varphi(u), \psi(v)$ are C^1 in u, v with $\varphi_u(u), \psi_v(v) \geq 0$ for all $u, v \geq 0$, respectively.

(H1.2) f and g are C^1 in u and v.

(H1.3) f_u, f_v, g_u and $g_v < 0$ for $(u, v) \in [0, \infty) \times [0, \infty)$.

(H1.4) There exist constants $C_1 > 0$ and $C_2 > 0$ such that $f(C_1, 0)$, $g(0, C_2) < 0$.

In the above assumptions, (H1.3) represents the competing interactions between two species and (H1.4) implies the logistic property of growth rates for each species.

Let $A(x) > 0$ in $C^2(\overline{\Omega})$ and $B(x) \in L^\infty(\Omega)$. Then it is easy to observe that the following eigenvalue problem:

$$\begin{cases} A(x)\Delta\widetilde{\phi} + B(x)\widetilde{\phi} = \lambda\widetilde{\phi} & \text{in } \Omega, \\ \widetilde{\phi}(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is equivalent to the problem

$$\begin{cases} \Delta[A(x)\phi] + B(x)\phi = \lambda\phi & \text{in } \Omega, \\ \phi(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

if we let $\widetilde{\phi} := A(x)\phi$ in (1.3). Thus the well-known variational property (see Chapter 11 in [11]) of eigenvalues of (1.3) can be applied to the problem (1.4). In fact, the following observation is useful throughout in this article.

OBSERVATION 1.1. *The eigenvalue problem* (1.4) *has a positive solution* $\phi \in C^1(\overline{\Omega})$ *and a unique* λ. *Moreover,* λ *is decreasing in* $A(x)$ *and increasing in the ratio* $B(x)/A(x)$.

Throughout this paper, let $\lambda_1(\Delta A(x) + B(x))$ denote the unique eigenvalue λ of the eigenvalue problem (1.4) corresponding to the unique positive eigenfunction $\phi(x)$.

Now we give the existence theorem of positive solution for the system (1.1).

Denote the nonnegative nonzero solution of our system, if they exist, when one of the species is absent by $(u_0, 0)$ and $(0, v_0)$. In fact, such semi-trivial solutions exist if $\lambda_1(-\Delta) < \frac{f(0,0)}{\varphi(0)}$ and $\lambda_1(-\Delta) < \frac{g(0,0)}{\psi(0)}$ (See Theorem 2.9 in section 2).

THEOREM 1.2. *Consider the system* (1.1) *with hypotheses* (H1.1)–(H1.4). *Suppose that* $\lambda_1(-\Delta) < \frac{f(0,0)}{\varphi(0)}$ *and* $\lambda_1(-\Delta) < \frac{g(0,0)}{\psi(0)}$. *Then*

(i) The nonnegative solution (u, v) *of* (1.1) *has a priori bounds;* $u(x) \le C_1$, $v(x) \le C_2$ *where* C_1 *and* C_2 *are the constants in* (H1.4).

(ii) *If* $\lambda_1(\Delta\varphi(0) + f(0, v_0)) > 0$ *and* $\lambda_1(\Delta\psi(0) + g(u_0, 0)) > 0$, *then* (1.1) *has a positive solution.*

2. Preliminaries

In this section, we state the some known lemmas and give the existence theorem for the scalar equation.

We consider the following scalar equation:

$$\begin{cases} -\Delta\varphi(u)u = uf(u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The functions $\varphi : [0, \infty) \to [0, \infty)$ and $f : [0, \infty) \to \mathbb{R}$ are assumed to satisfy the following hypotheses:

(H2.1) $\varphi(0) > 0$ and $\varphi(u)$ is C^1 in u with $\varphi_u(u) \geq 0$ for all $u \geq 0$.

(H2.2) $f(u)$ is C^1 in u with $f_u(u) < 0$ for all $u \geq 0$.

(H2.3) $f(0) > 0$ and $f(u) \leq 0$ on $[C_0, \infty)$ for some constant $C_0 > 0$.

REMARK 2.1. Define a mapping $G : [0, \infty) \to [0, \infty)$ by $G(u) := \varphi(u)u$ for all $x \in \overline{\Omega}$. Then by the hypothesis (H2.1), we have

$$\frac{\partial G}{\partial u} = \varphi_u(u)u + \varphi(u) > 0 \text{ for all } (x, u) \in \overline{\Omega} \times [0, \infty).$$

Thus the map $G(u)$ has a continuous inverse and denote it by $G^{-1}(u)$. Also we can see that $\frac{\partial}{\partial u}(G^{-1}(u)) > 0$ for all $u \geq 0$ by the inverse function theorem.

DEFINITION 2.2. A function $u \in C(\overline{\Omega})$ is called a solution of (2.1) if $\varphi(u)u \in C^{2,\alpha}(\Omega)$ and $u(x)$ satisfies (2.1).

DEFINITION 2.3. A function $\hat{u}(x) \in C(\overline{\Omega})$ is called an upper solution of (2.1) if $\hat{u}$ satisfies the following conditions:

$$-\Delta\varphi(\hat{u})\hat{u} \geq \hat{u}f(\hat{u}) \quad \text{in } \Omega, \quad \hat{u}(x) \geq 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

Similarly, we define a lower solution $\overline{u}(x)$ of (2.1) by reversing the inequalities in (2.2).

DEFINITION 2.4. Let X be a nonempty subset of some ordered set Y. A fixed point x of a map $f : X \to Y$ is called maximal(minimal) if every fixed point y of f in X satisfies $x \geq y (y \geq x)$.

LEMMA 2.5. *Let $A(x) > 0$ in $C^2(\overline{\Omega})$, $B(x) \in L^\infty(\Omega)$ and $u \geq 0$, $u \not\equiv 0$ in Ω with $u = 0$ on $\partial\Omega$.*
(i) *If $0 \not\equiv (\Delta A(x) + B(x))u \geq 0$, then $\lambda_1(\Delta A(x) + B(x)) > 0$.*
(ii) *If $0 \not\equiv (\Delta A(x) + B(x))u \leq 0$, then $\lambda_1(\Delta A(x) + B(x)) < 0$.*
(iii) *If $(\Delta A(x) + B(x))u \equiv 0$, then $\lambda_1(\Delta A(x) + B(x)) = 0$.*

Proof. We only prove (i). Let $\phi(x) > 0$ be the eigenfunction corresponding to the principal eigenvalue $\lambda_1(\Delta A(x) + B(x))$. Then we have $0 < \int_\Omega \phi(\Delta A(x) + B(x))u = \lambda_1(\Delta A(x) + B(x)) \int_\Omega \phi u$. Since $u \not\equiv 0$, $\lambda_1(\Delta A(x) + B(x)) > 0$. □

Let $T : E \to E$ be a linear operator on a Banach space. Denote the spectral radius of T by $r(T)$.

LEMMA 2.6. *Assume that T is a compact positive linear operator on an ordered Banach space. Let $u > 0$ be a positive element. Then*
(i) *If $Tu > u$, then $r(T) > 1$.*
(ii) *If $Tu < u$, then $r(T) < 1$.*
(iii) *If $Tu = u$, then $r(T) = 1$.*

Proof. See Lemma 2.3 in [4]. □

LEMMA 2.7. *The nonnegative solution $u(x)$ of* (2.1) *with hypotheses* (H2.1)–(H2.3) *has a priori bound; $u(x) \leq C_0$ for all $x \in \overline{\Omega}$.*

Proof. Assume that $u(x) > C_0$ for some $x \in \Omega$. Let $\Omega_1 = \{x \in \Omega : u(x) > C_0\}$. Then we have $-\Delta\varphi(u)u = uf(u) < uf(C_0) \leq 0$ in Ω_1. By the strong maximum principle, we can see that $\varphi(u)u \leq \varphi(C_0)C_0$ for all $x \in \Omega_1$. By the monotonicity of $\varphi(u)u$, we must have $u(x) \leq C_0$ for all $x \in \Omega_1$, which is a contradiction to the definition of Ω_1. □

COROLLARY 2.8. *The nonnegative solution $u(x)$ of* (2.1) *with hypotheses* (H2.1)–(H2.3) *satisfies $f(u(x)) \geq 0$.*

Proof. In the proof of Lemma 2.7, take a constant C_0 as the unique root of $f(u) = 0$ for u. □

REMARK 2.9. By virtue of Corollary 2.8, we can easily see that $\frac{f(u)}{\varphi(u)}$ is monotone decreasing on $0 \le u \le C_0$ where C_0 is the unique root of $f(u) = 0$ for u.

THEOREM 2.10. *Let hypotheses* (H2.1)–(H2.3) *be hold.*
(i) *If* $\lambda_1(-\Delta) \ge \frac{f(0)}{\varphi(0)}$, *then* (2.1) *has no positive solution.*
(ii) *If* $\lambda_1(-\Delta) < \frac{f(0)}{\varphi(0)}$, *then* (2.1) *has a unique positive solution.*

Proof. (i) If there exists a positive solution $u(x)$ of (2.1), then we have $\lambda_1(\Delta\varphi(u) + f(u)) = 0$ by Lemma 2.5 (iii). But since $\lambda_1(-\Delta) \ge \frac{f(0)}{\varphi(0)}$ and $\frac{f(u)}{\varphi(u)}$ is monotone decreasing on $0 \le u \le C_0$, we must have $\lambda_1(\Delta\varphi(u) + f(u)) < 0$ by Observation 1.1, which is a contradiction.

(ii) To prove the existence and uniqueness of a positive solution of (2.1), we divide the proof into three steps.

Step 1 : Construction of upper and lower solution of (2.1).
Let $\hat{u}(x) = C_0$ where C_0 is a constant in (H2.3). Then we can easily check that $\hat{u}(x)$ is a upper solution of (2.1), i.e.,

$$(2.3) \qquad \begin{cases} -\Delta\varphi(\hat{u})\hat{u} \ge \hat{u}f(\hat{u}) & \text{in } \Omega, \\ \hat{u} \ge 0 & \text{on } \partial\Omega. \end{cases}$$

To construct a lower solution $\overline{u}$, let $\phi > 0$ be the eigenfunction corresponding to $\lambda_1(-\Delta)$. Then we have $-\Delta\phi = \lambda_1(-\Delta)\phi < \frac{f(0)}{\varphi(0)}\phi$ in Ω. The last inequality follows from the assumption. Equivalently we can have $-\Delta\varphi(0)\phi < \phi f(0)$ in Ω, and so there is a $\epsilon > 0$ such that $-\Delta\varphi(\epsilon\phi)\phi \le \phi f(\epsilon\phi)$ in Ω by the continuity. If we take $\overline{u} = \epsilon\phi$, then we can see that

$$(2.4) \qquad \begin{cases} -\Delta\varphi(\overline{u})\overline{u} \le \overline{u}f(\overline{u}) & \text{in } \Omega, \\ \overline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

So $\overline{u} = \epsilon\phi$ is a positive lower solution of (2.1).

Step 2 : Existence of a positive solution of (2.1).
In Step 1, we can ensure that $\overline{u} \le \hat{u}$ for suitable ϵ.
Define a compact operator $F : [[\overline{u}, \hat{u}]] \to C(\overline{\Omega})$ by $F := G^{-1} \circ H$ where

$[[\overline{u}, \hat{u}]]$ denotes the ordered interval in $C(\Omega)$. Here G^{-1} is the inverse of the map $G(u) = \varphi(u)u$ in Remark 2.1 and H is given by

$$Hu := (-\Delta + M)^{-1}[(f(u) + M\varphi(u))u]$$

where $M > 0$ is a sufficiently large constant such that $(f(u) + M\varphi(u))u$ is monotone increasing with respect to u. Such constant M exists by (H2.1). Observe that u is a solution of (2.1) if and only if u is a fixed point of F. Adding $M\varphi(\hat{u})\hat{u}$, $M\varphi(\overline{u})\overline{u}$ and applying $G^{-1} \circ (-\Delta + M)^{-1}$ to both sides of (2.3) and (2.4), respectively, we can have $F(\hat{u}) \leq \hat{u}$ and $F(\overline{u}) \geq \overline{u}$. By Corollary 6.2 in [1], we can conclude that F has a minimal fixed point u_m and a maximal fixed point u_M in $[[\overline{u}, \hat{u}]]$.

Step 3 : Uniqueness of positive solution of (2.1).
Let u_m and u_M be minimal and maximal fixed point of F obtained in Step 2. If u is a positive solution of (2.1), then we have $u_m \leq u \leq u_M$ since u is also a fixed point of F. So it suffices to show that $u_m = u_M$.

If we assume that $u_m < u_M$, then by Remark 2.9 and Observation 1.1, we have $\lambda_1(\Delta\varphi(u_m) + f(u_m)) > \lambda_1(\Delta\varphi(u_M) + f(u_M))$. But Lemma 2.5 (iii) implies $\lambda_1(\Delta\varphi(u_m) + f(u_m)) = \lambda_1(\Delta\varphi(u_M) + f(u_M)) = 0$. So we can derive a contradiction. This completes the proof. □

Let X be a Banach space and let F be a Fréchet differentiable compact operator in X which maps a closed convex set E into itself. In [1], a fixed point index $ind_E(F, U)$ can be defined on each open subset U of E where boundary contains no fixed points of F. The index is defined through the Leray-Schauder degree

$$ind_E(F, U) = deg_X(I - F \circ \gamma, \gamma^{-1}(u), 0),$$

where I is the identity map in X and $\gamma : X \to E$ is a retraction of E.

For $y \in E$, define a wedge W_y by

$$W_y = cl\{x \in X : y + rx \in E \text{ for some } r > 0\},$$

where "cl" means the closure of the set. Let X_y be the maximal subspace of X contained in W_y. If X has the decomposition $X = X_y \oplus Y_y$ where Y_y is a closed linear subspace, then the index of F at y can be found by analyzing certain eigenvalue problems in Y_y and X_y. Let $T : X \to Y_y$ be the projection operator of Y_y along X_y.

The following theorem can be obtained from Theorem 2.1 and Theorem 2.2 in [9].

THEOREM 2.11. *Suppose $y \in X$ is a fixed point of F such that W_y is generating, i.e., $X = cl\{W_y - W_y\}$ and $X =: X_y \oplus Y_y$ for some closed linear subspace Y_y. If $F'(y)$ has no nonzero fixed point in W_y, then $ind_E(F,U)$ exists and*

(i) *$ind_E(F,y) = 0$ if $T \circ F'(y)$ has an eigenvalue $\lambda > 1$.*

(ii) *$ind_E(F,y) = ind_{X_y}(F'(y),0)$ if $T \circ F'(y)$ has no eigenvalue greater than 1.*

3. Proof of Theorem 1.2

We can easily prove Theorem 1.2 (i) by the similar argument in Lemma 2.7. Also we can obtain the following corollary by the slight modification of Corollary 2.8.

COROLLARY 3.1. *Assume that the semi-trivial solutions u_0 and v_0 of (1.1) are exist. Then $f(u_0(x),0)$, $g(0,v_0(x)) \geq 0$ for all $x \in \Omega$.*

We introduce the following notations.

$$D := \{(u,v) \in C_0(\Omega) \oplus C_0(\Omega) \ : \ u \leq Q+1, \ v \leq R+1\},$$
$$K := \{u \in C_0(\Omega) \ : \ 0 \leq u(x), \ x \in \overline{\Omega}\},$$
$$W := K \oplus K,$$
$$P_\rho := \{(u,v) \in W \ : \ u \leq \rho, \ v \leq \rho\}, \ \ \rho > 0,$$
$$D' := (intD) \cap W.$$

Set $\rho = \max\{C_1, C_2\} + 1$. Define a compact operator $F : P_\rho \to W$ by $F(u,v) := (R(u,v), S(u,v))$ where $R(u,v) = G_1^{-1} \circ (-\Delta + M)^{-1}[(f(u,v) + M\varphi(u))u]$ and $S(u,v) = G_2^{-1} \circ (-\Delta + M)^{-1}[(g(u,v) + M\psi(v))v]$. Here $G_1^{-1}(u)$ and $G_2^{-1}(v)$ are the inverse of the map $G_1(u) = \varphi(u)u$ and $G_2(v) = \psi(v)v$, respectively, and $M > 0$ is a constant sufficiently large so that $(f(u,v) + M\varphi(u))u$ and $(g(u,v) + M\psi(v))v$ are increasing with respect to u and v, respectively, for all $(u,v) \in P_\rho$. Note that D' is open in W and every positive solution of (1.1) is a fixed point of the compact operator F in D' of Banach space $X \equiv C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$.

LEMMA 3.2. *Let the hypotheses* (H1.1)–(H1.4) *be hold. If $\lambda_1(\Delta\varphi(0) + f(0,0)) > 0$ and $\lambda_1(\Delta\psi(0) + g(0,0)) > 0$, then $ind_W(F,(0,0)) = 0$.*

Proof. For the point $y = (0,0)$, observe that $W_y = K \oplus K$ and $X_y = \{0\}$, so $Y_y = X$ and $T = I$ where I is the identity operator in X. To use theorem 2.11, consider the eigenvalues of $L := F'(0,0)$. By the calculation, we have

$$L\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\varphi(0)}(-\Delta + M)^{-1}[(f(0,0) + M\varphi(0))\xi] \\ \dfrac{1}{\psi(0)}(-\Delta + M)^{-1}[(g(0,0) + M\psi(0))\eta] \end{pmatrix}$$

for each $\binom{\xi}{\eta} \in X$. Let ϕ be the positive eigenfunction corresponding to the eigenvalue $\mu := \lambda_1(\Delta\varphi(0) + f(0,0)) > 0$. Then $\Delta\varphi(0)\phi + f(0,0)\phi = \mu\phi > 0$, and so $-\Delta\varphi(0)\phi + M\varphi(0)\phi < (f(0,0) + M\varphi(0))\phi$. Thus it follows that $T\phi := \frac{1}{\varphi(0)}(-\Delta + M)^{-1}[(f(0,0) + M\varphi(0))\phi] > \phi$. So by Lemma 2.6 (i), $r(T) > 1$. Using the Krein-Rutman theorem, we have that $r(T)$ is an eigenvalue of T with positive eigenfunction ϕ_1. Thus if we consider the pair $\binom{\phi_1}{0}$ and $\lambda = r(T) > 1$, we have an eigenvalue greater than one with a positive eigenfunction, and so we can conclude that $ind_W(F,(0,0)) = 0$ by Theorem 2.11 (i). $\square$

LEMMA 3.3. *Let the hypotheses* (H1.1)–(H1.4) *be hold and assume that* $\lambda_1(\Delta\varphi(0)+f(0,0)) > 0$ *and* $\lambda_1(\Delta\psi(0)+g(0,0)) > 0$. *If* $\lambda_1(\Delta\psi(0)+g(u_0,0)) > 0$ *and* $\lambda_1\ (\Delta\varphi(0) + f(0,v_0)) > 0$, *then* $ind_W(F,(u_0,0)) = ind_W(F,(0,v_0)) = 0$.

Proof. We only calculate the index for the point $y = (u_0, 0)$ since the calculation of $ind_W(F,(0,v_0))$ is virtually the same.

For a point $y = (u_0, 0)$, observe that $W_y = C_0(\Omega) \oplus K$, $X_y = C_0(\Omega) \oplus \{0\}$ and set an operator $L := F'(u_0, 0)$. If we let $Y_y = \{0\} \oplus C_0(\Omega)$, then $X = X_y \oplus Y_y$ with projection $T(u,v) = (0,v)$. By the calculation, we have

$$L\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\varphi_u(u_0)u_0 + \varphi(u_0)}(-\Delta + M)^{-1}h(\xi,\eta) \\ \dfrac{1}{\psi(0)}(-\Delta + M)^{-1}k(\xi,\eta) \end{pmatrix}$$

for each $\binom{\xi}{\eta} \in X$ where $h(\xi,\eta) = [f(u_0,0) + M\varphi(u_0) + u_0(f_u(u_0,0) + M\varphi_u(u_0))]\xi + u_0 f_v(u_0,0)\eta$ and $k(\xi,\eta) = (g(u_0,0) + M\psi(0))\eta$.

Step 1 : Existence of $ind_W(F,(u_0,0))$.

Let $\binom{\xi}{\eta} \in W_y$ be a fixed point of L. Then we have

$$(3.1) \qquad \begin{aligned} & -\Delta(\varphi_u(u_0)u_0 + \varphi(u_0))\xi \\ = \; & (f(u_0,0) + u_0 f_u(u_0,0))\xi + u_0 f_v(u_0,0)\eta, \end{aligned}$$

$$(3.2) \qquad -\Delta\psi(0)\eta = g(u_0,0)\eta.$$

If $\eta \not\equiv 0$ in W_y, then $\eta > 0$, and so we must have λ_1 $(\Delta\psi(0) + g(u_0,0)) = 0$ by Lemma 2.5 (iii), which is a contradiction to the assumption. Now since $\eta \equiv 0$ in W_y, (3.1) can be expressed by $-\Delta$ $(\varphi_u(u_0)u_0 + \varphi(u_0))$ $\xi = (f(u_0,0) + u_0 f_u(u_0,0))\xi$. If $\xi \not\equiv 0$ in W_y, then we can see that 0 is an eigenvalue of Δ $(\varphi_u(u_0)u_0 + \varphi(u_0)) + (f(u_0,0) + u_0 f_u(u_0,0))I$, and so λ_1 $(\Delta$ $(\varphi_u(u_0)u_0 + \varphi(u_0)) + f(u_0,0) + u_0 f_u(u_0,0))$ ≥ 0 by Observation 1.1. But since $f(u_0,0) \geq 0$ by Corollary 3.1, we have $\frac{f(u_0,0)}{\varphi(u_0)} > \frac{f(u_0,0)+u_0 f_u(u_0,0)}{\varphi_u(u_0)u_0+\varphi(u_0)}$, and so $\lambda_1(\Delta(\varphi_u(u_0)u_0 + \varphi(u_0)) + f(u_0,0) + u_0 f_u(u_0,0)) < \lambda_1(\Delta\varphi(u_0) + f(u_0,0)) = 0$, which is also a contradiction. The last equality follows from Lemma 2.5 (iii). Consequently we can conclude that L has no nonzero fixed point in W_y, and thus $ind_W(F,(u_0,0))$ exists by Theorem 2.11.

<u>Step 2</u> : Calculation of $ind_W(F,(u_0,0))$.

To use Theorem 2.11, consider the eigenvalues of $T \circ L$. Observe that $T \circ L$ has an eigenfunction of the form $\binom{0}{\phi}$. Let ϕ be the positive eigenfunction corresponding to the positive eigenvalue $\mu := \lambda_1$ $(\Delta\psi(0)$ $+g(u_0,0))$. As in the proof of Lemma 3.3, we can have $\frac{1}{\psi(0)}$ $(-\Delta+M)^{-1}$ $[(g(u_0,0) + M\psi(0))\phi] > \phi$, and so $r(T \circ L) > 1$ by Lemma 2.6 (i). Thus we can conclude that ind_W $(F,(u_0,0)) = 0$ by Theorem 2.11 (i). □

Recall that $D' = (intD) \cap W$ where $D := \{(u,v) \in C_0(\Omega) \oplus C_0(\Omega) :$ $u \leq Q+1,\ v \leq R+1\}$. Then we can observe that D' contains the trivial and semi-trivial solutions $(0,0)$, $(u_0,0)$, $(0,v_0)$ of (1.1) by Theorem 1.2 (i).

LEMMA 3.4. *Let D' be the bounded open set in W defined as above. Then $ind_W(F,D') = 1$.*

Proof. Clearly, $\partial D'$ contains no fixed point of F, and so $ind_W(F,D')$ exists.

Define an operator F_μ by $F_\mu(u,v) = (R_\mu(u,v), S_\mu(u,v))$ for $\mu \in [0,1]$,

$$R_\mu(u,v) = G_1^{-1} \circ (-\Delta + M)^{-1}[(\mu f(u,v) + M\varphi(u))u],$$
$$S_\mu(u,v) = G_2^{-1} \circ (-\Delta + M)^{-1}[(\mu g(u,v) + M\psi(v))v],$$

then clearly $F = F_1$ and, for each μ, a fixed point of F_μ is a solution of the problem

$$\left\{ \begin{array}{ll} -\Delta\varphi(u)u = \mu u f(u,v) & \\ -\Delta\psi(v)v = \mu v g(u,v) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega. \end{array} \right. \tag{3.3}$$

The converse is also true. By Theorem 1.2 (i), we can see that every fixed point of F_μ satisfies $u(x) \le C_1$ and $v(x) \le C_2$ in Ω for each $\mu \in [0,1]$, and so every fixed point of F_μ is in D but not on ∂D. Thus the homotopy invariance property of index shows that $ind_W(F_\mu, D')$ is independent of μ. So $ind_W(F, D') = ind_W(F_1, D') = ind_W(F_0, D')$. Also since if $\mu = 0$, then (3.3) has only the trivial solution $(0,0)$, we get $ind_W(F_0, D') = ind_W(F_0, (0,0))$.

Let $y = (0,0)$. Then we have $\overline{W}_y = K \oplus K$ and

$$L \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\varphi(0)}(-\Delta + M)^{-1}(M\varphi(0)\xi) \\ \dfrac{1}{\psi(0)}(-\Delta + M)^{-1}(M\psi(0)\eta) \end{pmatrix}$$

for each $\binom{\xi}{\eta} \in X$ where $L := F_0'(0,0)$.

Let $\binom{\xi}{\eta}$ be an eigenfunction of L with corresponding eigenvalue λ. Then we can have

$$\left\{ \begin{array}{ll} -\Delta\varphi(0)\xi = \dfrac{1-\lambda}{\lambda} M\varphi(0)\xi & \\ -\Delta\psi(0)\eta = \dfrac{1-\lambda}{\lambda} M\psi(0)\eta & \text{in } \Omega, \\ (\xi,\eta) = (0,0) & \text{on } \partial\Omega, \end{array} \right.$$

and the above problem has no nonzero solution for $\lambda \ge 1$. Therefore $\lambda < 1$. So we can see that every eigenvalue of L is less than 1. Thus we may conclude $ind_W(F_0, (0,0)) = 1$. □

By virtue of Lemma 3.2 - 3.4, we can prove the Theorem 1.2 (ii).

Proof of Theorem 1.2 (ii). Suppose that F has no positive fixed point in D'. Then by Lemma 3.4 and the additivity of index, we have

$$ind_W(F,(0,0))+ind_W(F,(u_0,0))+ind_W(F,(0,v_0)) = ind_W(F,D') = 1. \tag{3.4}$$

If $\lambda_1(\Delta\varphi(0)+f(0,v_0)) > 0$ and $\lambda_1(\Delta\psi(0)+g(u_0,0)) > 0$, then by Lemma 3.2 and 3.3,

$$ind_W(F,(0,0)) + ind_W(F,(u_0,0)) + ind_W(F,(0,v_0)) = 0,$$

which is a contradiction to (3.4). Therefore our problem must have a positive solution in D'. □

4. Example

Finally we give a simple model with self-diffusion pressure which the diffusion and growth rates are nonlinear with respect to the densities of species.

Consider the following system of a competition interacting with self-diffusion:

$$\begin{cases} -\Delta[(\alpha_1+\beta_1 u)^m u] = (a_1 - b_{11}u^k - b_{12}v)u & \\ -\Delta[(\alpha_2+\beta_2 v)^n v] = (a_2 - b_{21}u - b_{22}v^l)v & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where α_i, a_i, β_i, b_{ij} are nonnegative constants with $\alpha_i > 0$, $b_{ii} > 0$ for $i = 1,2$ and $m,n,k,l > 0$. One can observe that if $\beta_1 = \beta_2 = 0$ and $k = l = 1$, then the model (4.1) is a classical Lotka-Volterra system with competing interaction between any two species.

COROLLARY 4.1. *Assume that $a_i > \alpha_i \ \lambda_1\ (-\Delta)$ for $i = 1,2$. If $\lambda_1(-\Delta) < \min\left\{ \frac{a_1 b_{22}^{\frac{1}{l}} - a_2^{\frac{1}{l}} b_{12}}{\alpha_1^m b_{22}^{\frac{1}{l}}}, \frac{a_2 b_{11}^{\frac{1}{k}} - a_1^{\frac{1}{k}} b_{21}}{\alpha_2^n b_{11}^{\frac{1}{k}}} \right\}$, then (4.1) has a positive solution.*

Proof. The assumption $a_i > \alpha_i\lambda_1(-\Delta)$ for $i = 1,2$ ensure that the semi-trivial solutions $u_0 > 0$, $v_0 > 0$ and we can see that $u_0 \le (\frac{a_1}{b_{11}})^{\frac{1}{k}}$ and $v_0 \le (\frac{a_2}{b_{22}})^{\frac{1}{l}}$ by Lemma 2.7. Note that $\frac{a_1 - b_{12}v}{\alpha_1^m}$, $\frac{a_2 - b_{21}u}{\alpha_2^n}$ are monotone decreasing for $v \ge 0$ and $u \ge 0$, respectively. So by using these facts and Observation 1.1, we have $\lambda_1(\Delta\alpha_1^m + a_1 - b_{12}v_0) \ge \lambda_1(\Delta\alpha_1^m + a_1 - b_{12}(\frac{a_2}{b_{22}})^{\frac{1}{l}}) > 0$ and $\lambda_1(\Delta\alpha_2^n + a_2 - b_{21}u_0) \ge \lambda_1(\Delta\alpha_2^n + a_2 - b_{21}(\frac{a_1}{b_{11}})^{\frac{1}{k}}) > 0$.

The last inequalities follows from the assumptions. Hence (4.1) has a positive solution by Theorem 1.2 (ii). □

REMARK 4.2. Besides the above example, one can use Theorem 1.2(ii) to have a positive coexistence for more general model as long as the system satisfies (H1.1)–(H1.4).

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