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TOPOLOGICAL CHARACTERIZATIONS OF CERTAIN LIMIT POINTS FOR MÖBIUS GROUPS

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Abstract. A limit point p of a Möbius group acting on B^m is called a concentration point if for every sufficiently small connected open neighborhood of p, the set of translates contains a local basis for the topology of p. For the case of two generator Schottky groups acting on B^2, we give characterizations for several different kinds of limit points.

1. Introduction

Let Γ be a discrete subgroup of hyperbolic isometries acting on the Poincaré disc $B^m, m \geq 2$. The discrete group Γ acts properly discontinuously in B^m, and acts on ∂B^m as a group of conformal homeomorphisms, but need not act properly discontinuously on ∂B^m. The action of Γ divides ∂B^m into two sets. The ordinary set $\Omega(\Gamma)$ is the largest open subset of ∂B^m on which Γ acts discontinuously. The complement of $\Omega(\Gamma)$ in ∂B^m is the limit set, denoted by $\Lambda(\Gamma)$ or simply Λ. The limit set $\Lambda(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(x)$ for one, hence for every, point $x \in B^m$. Equivalently, the limit set is the smallest nonempty closed set in ∂B^m on which Γ does not act discontinuously. If Λ contains two or fewer points, Γ is elementary, and contains a free abelian subgroup of finite index. Otherwise, Γ is nonelementary. In this paper, we always assume that Γ is nonelementary.

It is easy to see that $\Lambda(\Gamma) = \Lambda(\Gamma')$ for any nontrivial normal subgroup Γ' of Γ. Also, if x is any point of ∂B^m, then the accumulation points of any orbit of x under Γ lie in $\Lambda(\Gamma)$. For a nonelementary group Γ, define $CH(\Lambda)$ to be the smallest nonempty convex set in B^m which is invariant under the action of Γ; this is the convex hull of Γ. The

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boundary at infinity of $CH(\Lambda)$ is precisely Λ, and so $CH(\Lambda)$ contains every geodesic line in B^m both of whose endpoints at infinity are in Λ. By a neighborhood of p, we will always mean an open neighborhood of p in ∂B^m.

DEFINITION. One says that a neighborhood U of p *can be concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U)$ and $\gamma(U) \subset V$.

The limit point p is called a *concentration point* for Γ if there exists a neighborhood of W of p such that every neighborhood U of p with $U \subset W$ can be concentrated at p. If Γ is a Fuchsian group, a slight weaker concept than concentration point turns out to be important.

DEFINITION. A limit point p is called a *geodesic separation point* for the Fuchsian group Γ if for every sufficiently small connected neighborhood U of p, either U or $S^1 - U$ can be concentrated at p.

A *weak concentration point* which is the weakest reasonable concept of concentration is characterized in [5]. Namely, the limit point p is a weak concentration point for Γ if there exists a connected open set U that can be concentrated at p. The main property is that if the limit set approaches a point from two different tangential directions in ∂B^m then it is a weak concentration point. This implies every limit point of a geometrically finite group must be a weak concentration point. A limit point p is called a *controlled concentration point* if it has a neighborhood U such that for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ so that $p \in \gamma(U)$ and $\gamma(U) \subset V$. In the following sections, we will give characterizations for controlled concentration points and concentration points in the case of two generator Schottky groups.

DEFINITION. One says that a pair of open sets (U_1, U_2) in ∂B^m is an *admissible pair* if

(a) $\overline{U_2} \subset U_1$,
(b) $U_2 \cap \Lambda \neq \emptyset$ and
(c) $\Lambda \not\subset \overline{U_1}$.

One says that an admissible pair (U_1, U_2) *can be concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U_2) \subset \gamma(U_1) \subset V$.

DEFINITION. A geodesic λ is called a geodesic for Γ if both endpoints of λ are limit points of Γ. The limit point p is called a *Myrberg-Agard* *density point* for Γ if whenever μ is an oriented geodesic for Γ and α is a geodesic ray ending at p in $CH(\Lambda)$ (convex hull of Λ), there is a sequence of elements $\{\gamma_i\}$ such that $\{\gamma_i(\alpha)\}$ converge to μ in an oriented sense.

Using the characterization of Theorem 4.1 in [3], one can easily see that every Myrberg-Agard density point is a controlled concentration point. The next theorem will be useful to get a characterization of Myrberg-Agard density points for a two generator Schottky group.

THEOREM 1.1. *A limit point p is a Myrberg-Agard density point for Γ if and only if every admissible pair (U_1, U_2) can be concentrated at p.*

Proof. See Theorem 3.1 in [2]. □

2. Schottky groups and limit points

We will work with a 2-generator m-dimensional Schottky group Γ, although it will be apparent that the same phenomena occur for other examples (in particular, with more generators). The limit set of Γ is a Cantor set which can be understood quite explicitly using the sequence of crossings of a geodesic ray (ending at the limit point) with the translates of two fixed sides of a fundamental domain.

To define Γ, we work in the Poincaré unit disc B^m. Let a and a' be the geodesic hyperplanes in B^m which lie in the spheres in $\mathbf{R}^m$ with centers at the points (1.1,0,...,0) and (-1.1,0,...,0), say. Similarly, let b and b' lie in the spheres with centers at the points (0,...,0,1.1) and (0,...,0,-1.1). Choose a, a', b, b' so that they are mutually disjoint. As the generators of Γ, select two orientation-preserving hyperbolic isometries: one carrying a to a' and one carrying b to b'. Fix one of the direction normal to a as the positive direction. It determines a positive normal direction for each translate of a. Similarly, we label b and its translates. A crossing of an oriented geodesic of geodesic ray in B^m with a translate of a or b will be called a *positive* crossing when it agrees with the selected direction; otherwise it will be called a *negative* crossing.

Suppose α is a geodesic ray in B^m, which does not lie in a translate of a and b. Then α crosses a sequence (finite or infinite, possibly of length 0) of translates of a and b. (When a geodesic ray starts in a translate, we count that intersection as a crossing.) To α, we associate a sequence $S(\alpha) = x_1x_2x_3$...of elements in the set $\{a, \overline{a}, b, \overline{b}\}$ in the following way.

If the nth crossing of α with the union of the translates of a and b is a positive crossing with a translate of a, then $x_n = a$. If the nth crossing is a negative crossing with a translate of a, then $x_n = \bar{a}$. For crossings with translates of b, the elements b and $\bar{b}$ are assigned similarly. Note that $S(\alpha)$ is an infinite sequence if and only if α ends at a limit point of Γ, and note that, for each sequence $S = x_1x_2x_3...$ of elements of the set $\{a, \bar{a}, b, \bar{b}\}$ (with the property that for no n is x_nx_{n+1} in the set $\{a\bar{a}, \bar{a}a, b\bar{b}, \bar{b}b\}$), there exists a geodesic ray α with $S(\alpha) = S$.

Using these sequences, the controlled concentration points of Γ can be characterized. The following characterization appears in [1], but we reproduce its proof here for the convenience of the reader.

PROPOSITION 2.1. *Let p be a limit point of Γ which is the endpoint of a geodesic ray α with $S(\alpha) = x_1x_2x_3\cdots$. Then p is a controlled concentration point for Γ if and only if $S(\alpha)$ has the following property. There exists N such that for all $n \geq N$, for all positive k, and for all M, there exists $m \geq M$ such that $x_{n+i} = x_{m+i}$ for all i with $0 \leq i \leq k$.*

Proof. Denote by λ_n the translate of a or b whose crossing with α determines x_n, and by U_n the neighborhood of p bounded by the endpoints of λ_n. Suppose the condition in the Proposition holds. By truncating α, we may assume that every subsequence reappears infinitely often. Let m_n be an integer so that $x_{m_n+i} = x_i$ for $1 \leq i \leq n$. Let γ_{m_n} be the element of Γ that translates λ_1 to λ_{m_n+1}. Note that this element translates λ_{k+1} onto λ_{m_n+k+1} for all $1 \leq k < n$. Given a neighborhood V of p, choose n so large that λ_n has endpoints in V. Then $\gamma_{m_n}(U_1) \subseteq V$ and $p \in \gamma_{m_n}(V)$, showing that U_1 can be concentrated with control. Conversely, suppose p is a controlled concentration point and choose N large enough so that U_N, and hence every neighborhood of p inside U_N, can be concentrated with control. For any $n, k > N$ and any M, there exists γ so that $\gamma(U_n) \subseteq U_M$ and $\gamma^{-1}(p) \in U_{n+k}$. This γ must move $\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}$ onto a sequence of translates of a and b crossed by α, with endpoints in U_M. Thus the condition of Proposition 2.1 holds. □

The next theorem which is originally stated in [2] (Theorem 3.3) is revised. The author would like to thank Darryl McCullough for improvement of the original statement.

We say a sequence $y_1y_2\cdots y_n$ is *admissible* if no pair y_iy_{i+1} is in $\{a\bar{a}, \bar{a}a, b\bar{b}, \bar{b}b\}$.

THEOREM 2.2. *The limit point p is a Myrberg-Agard density point if and only if for every (equivalently, for some) ray α ending at p, every admissible sequence appears as a subsequence of $S(\alpha)$.*

Proof. Denote by W_n the neighborhood of p determined by the half space bounded by translates of a or b whose crossing with α determines x_n of the sequence $S(\alpha) = x_1x_2x_3\cdots$.

Let p be a Myrberg-Agard density point and let α be a geodesic ray ending at p. For each admissible sequence $y_1y_2\cdots y_n$, we have an admissible pair (U_1, U_2) determined by the half spaces bounded by translates of a or b such that $U_1 = W_{y_1}$ and $U_2 = W_{y_n}$. For a neighborhood V of p, we choose an x_j with $W_{x_j} \subset V$. Since the admissible pair (U_1, U_2) can be concentrated at p from Theorem 1.1, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U_2) \subset \gamma(U_1) \subset W_{x_j}$. Therefore every admissible sequence $y_1y_2\cdots y_n$ appears as a subsequence of $S(\alpha)$.

Conversely, let (U_1, U_2) be an admissible pair at p and let V be a neighborhood of p. Choose a pair (W_1, W_2) which are half spaces bounded by translates of a or b such that $W_2 \subset U_2$ and $W_1 \supset U_1$. If there is an element $\gamma \in \Gamma$ that concentrates (W_1, W_2) will concentrate (U_1, U_2).

For a geodesic ray α ending at p and meeting W_1, form a sequence $y_{i_1}y_{i_2}\cdots y_{i_m}$ of $S(\alpha)$ by crossing of α with translates of a or b so that

$$W_1 = W_{y_{i_1}} \supset \ldots \supset W_{y_{i_m}} = W_2.$$

Then the sequence $y_{i_1}y_{i_2}\cdots y_{i_m}$ of $S(\alpha)$ is admissible.

Now choose an x_j with $W_{x_j} \subset V$, and past x_j there must have an appearance $x_kx_{k+1}\cdots x_{k+m-1} = y_{i_1}y_{i_2}\cdots y_{i_m}$, so there exists an element $\gamma \in \Gamma$ that moves $W_1 - W_2$ onto $W_{x_k} - W_{x_{k+m-1}}$. Therefore (U_1, U_2) can be concentrated at p. By using Theorem 1.1, p is a Myrberg-Agard density point.

If (U_1, U_2) is not an admissible pair at p, then because the orbit of any limit point is dense in the limit set, there is $\tau \in \Gamma$ so that $\tau^{-1}(p) \in U_2$. Therefore $p \in \tau(U_2) \subset \tau(U_1)$ hence $(\tau(U_1), \tau(U_2))$ is an admissible pair at p. Now we apply the same argument as in the above to show that $(\tau(U_1), \tau(U_2))$ can be concentrated at p. This also implies the pair (U_1, U_2) can be concentrated at p. Again by using Theorem 1.1, p is a Myrberg-Agard density point. This completes the proof of the theorem 2.2. □

The next theorems work only for Schottky groups acting on B^2. Let Γ be a Schottky group with 2 generators as is described in the beginning of section 2 but we need to choose the geodesics in B^2 which lie in the spheres in $\mathbf{R}^2$ with centers at the points (1.1,0), (-1.1,0), (0,1.1) and (0,-1.1).

Denote by a_n a sequence of n a's, and by $\overline{a_n}$ a sequence of n $\overline{a}$'s. The following theorem which is a slight modification of Theorem 4.2 in [4] gives examples of concentration points but not Myberg-Agard density points for a two generator Schottky group. Other interesting phenomena of concentration points and related properties can be found in [4].

THEOREM 2.3. *For each increasing sequence of positive integers* $1 \leq i_1 < j_1 < i_2 < j_2 < i_3 \cdots$, *if* p *is a limit point which is the endpoint of a geodesic ray whose crossing is*

$$ba_{i_1}b\overline{a_{j_1}}ba_{i_2}b\overline{a_{j_2}}ba_{i_3}b\overline{a_{j_3}}\cdots$$

then p *is a concentration point but not a Myrberg-Agard density point.*

The next theorem characterizes limit points for finitely generated Fuchsian groups.

THEOREM 2.4. (Theorem 3.2 in [4]) *Let* Γ *be a Fuchsian group. If* Γ *is finitely generated, then every limit point of* γ *is either a parabolic fixed point or a geodesic separation point.*

Now we will give an example which shows that in Theorem 2.4 the hypothesis that Γ is finitely generated is necessary.

EXAMPLE 2.5. *A conical limit point for an infinitely generated 2-dimensional Schottky group which is not a geodesic separation point.*

For each positive integer k, let $z_k = \exp\left(\frac{i\pi}{2}(1-\frac{1}{k})\right) \in S^1 \subset \overline{B^2}$. Denote by ℓ_n the geodesic in B^2 through the origin with one end limiting to z_n. Choose geodesics λ_n perpendicular to ℓ_n, near z_n, with diameters small and limiting to 0 sufficiently fast so that each λ_n separates z_n from all other z_i, and the λ_n are pairwise disjoint. Let λ'_n be the image of λ_{n+1} under the reflection across the line through the origin perpendicular to ℓ_{n+1}. For $n \geq 1$ let γ_n be the hyperbolic isometry which moves λ_n onto λ'_n, carries the complementary region of λ_n containing z_n to the complementary region of λ'_n containing z_n, and carries ℓ_n to ℓ_{n+1}. The group generated by the γ_n is a Schottky group of infinite rank. For

$n \geq 1$ let $\tau_n = \gamma_1^{-1}\gamma_2^{-1}\cdots\gamma_n^{-1}$ and $\mu_1 = \lambda_1$ and $\mu_n = \tau_{n-1}(\lambda_n)$ for $n \geq 2$. Then the μ_n form a nested sequence limiting toward z_1, and each μ_n is the only translate of λ_n that crosses ℓ_1. Therefore z_1 is not a geodesic separation point. But $\tau_n^{-1}(\ell_1) = \ell_{n+1}$, and the images of the origin under τ_n form a sequence of points on ℓ_1 limiting to z_1. Therefore z_1 is a conical limit point for Γ.

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