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PRIME RADICALS OF FORMAL POWER SERIES RINGS

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ABSTRACT. In this note we study the prime radicals of formal power series rings, and the shapes of them under the condition that the prime radical is nilpotent. Furthermore we observe the condition structurally, adding related examples to the situations that occur naturally in the process.

1. Formal power series rings

Throughout this paper all rings are associative with identity. Given a ring R, the prime radical is denoted by $P(R)$; the polynomial ring and the formal power series ring over R are denoted by $R[x]$ and $R[[x]]$ with x the indeterminate, respectively; also $R[X]$ and $R[[X]]$ denote the polynomial ring and the formal power series ring over R with X a set of commuting indeterminates (possibly infinite) over R, respectively.

In this section we obtain some informations for the prime radicals of formal power series rings.

THEOREM 1.1.. *Given a ring R, the following conditions are equivalent:*

(1) *R is semiprime.*
(2) *$R[X]$ is semiprime.*
(3) *$R[[X]]$ is semiprime.*

Proof. (1)⇔(2): By the well-known fact that $P(R[X]) = P(R)[X]$.

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(1)⇒(3): First we introduce some notations and terminologies for simpler computations. Suppose $X = \{x_\alpha \mid \alpha \in A\}$, and we may assume that A is a well-ordered set. Let $\mathbb{N}$ be the set of positive integers, and for any nonempty finite subset $I = \{(\alpha_1, m_1), (\alpha_2, m_2), \ldots, (\alpha_r, m_r)\}$ of $A \times \mathbb{N}$, define the monomial X^I by

$$X^I = x_{\alpha_1}^{m_1} x_{\alpha_2}^{m_2} \cdots x_{\alpha_r}^{m_r} (\alpha_1 < \alpha_2 < \cdots < \alpha_r) \text{ and } X^{\emptyset} = 1_R.$$

We give terminologies to I as follows:

$\deg(I) = m_1 + m_2 + \cdots + m_r$: the degree of X^I, and

$|I| = r$: the number of elements of I.

Next for two nonempty finite subsets $I = \{(\alpha_1, m_1), (\alpha_2, m_2), \ldots, (\alpha_r, m_r)\}$ and $J = \{(\beta_1, n_1), (\beta_2, n_2), \ldots, (\beta_s, n_s)\}$, we write $I < J$ if one of the following four conditions holds:

(I) $\deg(I) < \deg(J)$,

(II) $\deg(I) = \deg(J)$ and $|I| < |J|$,

(III) $\deg(I) = \deg(J), |I| = |J|$, and there exists a positive integer k such that $\alpha_i = \beta_i, m_i = n_i$ for all $i \leq k$, and $\alpha_{k+1} < \beta_{k+1}$, and

(IV) $\deg(I) = \deg(J), |I| = |J|$, and there exists a positive integer k such that $\alpha_i = \beta_i, m_i = n_i$ for all $i \leq k$, and $\alpha_{k+1} = \beta_{k+1}$ with $m_{k+1} > n_{k+1}$.

For any finite subsets I and J of $A \times \mathbb{N}$, set $I + J$ be the finite subset with $X^{I+J} = X^I X^J$. Then it is clear that for any nonempty finite subsets I, J and K of $A \times \mathbb{N}$,

(I) if $I < J$ and $J < K$ then $I < K$, and

(II) if $I < J$ then $I + K < J + K$, in particular $2I \overset{\text{let}}{=} I + I < I + J$.

Notice that every nonzero power series f in $R[[X]]$ is of the form

$$f = a_0 + \sum_{n=1}^{\infty} a_{I_n} X^{I_n},$$

where $a_0, a_{I_n} \in R$ and each I_n is a finite subset of $A \times \mathbb{N}$ with $I_n < I_{n+1}$ for all $n \geq 1$. Now we claim $fRf \neq 0$. This is true if $a_0 \neq 0$ since R is semiprime, so we assume $a_0 = 0$. Without loss of generality, we may let $a_{I_1} \neq 0$. Note that

$$2I_1 = I_1 + I_1 < I_p + I_q$$

for all positive integers p, q with $p + q > 2$, hence the coefficient of the term X^{2I_1} in fuf, with $u \in R$, is $a_{I_1} u a_{I_1}$. Therefore we have $fRf \neq 0$, this completes the claim.

Let $f \in R[[X]]$ with $fR[[X]]f = 0$, then clearly $fRf = 0$. By the claim we have $f = 0$, and so $R[[X]]$ is semiprime.

(3)⇒(1): Let $a \in R$ with $aRa = 0$. Then clearly $aR[[X]]a = 0$, but $R[[X]]$ is semiprime by the condition and we get $a = 0$; hence R is semiprime. □

COROLLARY 1.2. *Given a ring R, $P(R[[X]]) \subseteq P(R)[[X]]$.*

Proof. Note that $(R/P(R))[[X]]$ is semiprime by Theorem 1.1. Since $R/P(R)[[X]] \cong R[[X]]/P(R)[[X]]$ and the prime radical is the smallest such one, we have $P(R[[X]]) \subseteq P(R)[[X]]$. □

The inclusion in Corollary 1.2 may be strict by the following.

EXAMPLE 1.3. Let F be a field and let V be a infinite dimensional left vector space over F with $\{v_1, v_2, \dots\}$ a basis. For the endomorphism ring $A = \mathrm{End}_F(V)$, define

$$J = \{f \in A \mid \mathrm{rank}(f) < \infty \text{ and } f(v_i) \in \sum_{j<i} Fv_j\}.$$

Let R be the F-subalgebra of A generated by J and 1_A. This is just the ring mentioned in [7, Example 2.7.38]. Note that every element in J is strongly nilpotent in R and $R/J \cong F$, then $P(R) = J$, that is, $P(R)$ contains all nilpotent elements in R. Moreover every element in J is also strongly nilpotent in $R[[x]]$, hence $P(R)[x] \subseteq P(R[[x]])$. Let e_{ij} be the infinite matrix over F with (i,j)-entry 1 and elsewhere 0. Take

$$f(x) = e_{12} + e_{34}x + \cdots + e_{(2n+1)(2n+2)}x^n + \cdots$$

and

$$g(x) = e_{23} + e_{45}x + \cdots + e_{(2n+2)(2n+3)}x^n + \cdots$$

in $P(R)[[x]]$. Then $f(x)^2 = 0 = g(x)^2$; but the coefficients of $(f(x) + g(x))^k$ are

$$e_{1(k+1)}, e_{2(k+2)}, \dots, e_{n(k+n)}, \dots \text{ for } k = 2, 3, \dots,$$

and so it is not nilpotent. Therefore $f(x) \notin P(R[[x]])$ or $g(x) \notin P(R[[x]])$, it then follows $P(R)[[x]] \nsubseteq P(R[[x]])$.

We obtain a condition for which the converse of Corollary 1.2 may be true.

COROLLARY 1.4. *Given a ring R, $P(R)$ is nilpotent if and only if so is $P(R[[X]])$ with $P(R[[X]]) = P(R)[[X]]$.*

Proof. $\Leftarrow$: Obvious. $\Rightarrow$: Since $P(R)$ is nilpotent, $P(R)[[X]]$ is also nilpotent and so $P(R)[[X]] \subseteq P(R[[X]])$. By Corollary 1.2, we have $P(R[[X]]) = P(R)[[X]]$ and then it is nilpotent. □

Based on Corollary 1.4, we observe the condition, that $P(R)$ is nilpotent, further in the next section.

2. Nilpotent prime radicals

Given a ring R, consider the condition: (*) $P(R)$ is nilpotent. This is a Morita invariant property as we see later in this section, and we study the prime radicals of formal power series rings with this condition. A ring with finite right Krull dimension satisfies (*) by [5]; and a ring, which is right Goldie or satisfies ascending chain condition on both right and left annihilators, satisfies (*) by [4] and [1, Theorem 1.34]. First we may obtain the following directly by [3, Corollary 3.2.2].

LEMMA 2.1. *Given a ring R the following statements are equivalent:*
(1) *R satisfies (*).*
(2) *$P(R)$ is the largest nilpotent one-sided ideal in R.*
(3) *$P(R)$ is the only nilpotent semiprime ideal in R.*
(4) *R contains a nilpotent semiprime ideal.*

PROPOSITION 2.2. *Suppose that a ring R satisfies (*). Then we have the following assertions:*
(1) *eRe satisfies (*) for each nonzero idempotent $e \in R$.*
(2) *The n by n full matrix ring $\mathrm{Mat}_n(R)$ over R satisfies (*) for any positive integer n.*

Proof. (1) Let S be a nilpotent semiprime ideal of R. Then clearly $eSe \subsetneq eRe$. Letting $eaeReae \subseteq eSe$ with $a \in R$, we have $eae \in S$ and so $eae \in eSe$. Since eSe is a nilpotent semiprime ideal of eRe, eRe satisfies (*) by Lemma 2.1.

(2) Note that $P(\mathrm{mat}_n(R)) = \mathrm{mat}_n(P(R))$, so $\mathrm{Mat}_n(R)$ satisfies (*).□

COROLLARY 2.3. *Suppose that a ring R satisfies (*). Then for every finitely generated projective right R-module P, $\mathrm{End}_R(P)$ satisfies*

(*); *especially the condition* (*) *is a Morita invariant property, where* $\mathrm{End}_R(P)$ *is the endomorphism ring of* P *over* R.

Proof. Note that $\mathrm{End}_R(P) \cong e\mathrm{Mat}_n(R)e$ for some $e^2 = e \in \mathrm{Mat}_n(R)$ and some positive integer n. So we obtain the results from Proposition 2.2. □

LEMMA 2.4. *Let* $R_1, R_2, \ldots, R_n$ *be rings and* $R = \oplus_{i=1}^n R_i$. *Then* R *satisfies* (*) *if and only if* R_i *satisfies* (*) *for all* i.

Proof. By the fact that $P(R) = \oplus_{i=1}^n P(R_i)$. □

Given a ring R, $\mathrm{Spec}(R)$ represents the set of all prime ideals in R.

LEMMA 2.5. *Let* R *be a ring, and* $e = e^2 \in R$ *with* $e \neq 0$ *and* $e \neq 1$. *Then we have the following assertions:*
(1) $P(eRe) = eP(R)e$.
(2) *If* R *is semiprime then so is* eRe.

Proof. For any prime ideal P of R, note that either $ePe = eRe$ or ePe is a prime ideal of eRe. So we have $P(eRe) \subseteq eP(R)e$ since $eP(R)e = e(\cap_{P\in\mathrm{Spec}(R)} P)e = \cap_{P\in\mathrm{Spec}(R)} ePe$. Next let $a \in P(R)$, then eae is strongly nilpotent in R and also is in eRe; hence $eae \in P(eRe)$ and $eP(R)e \subseteq P(eRe)$. So we prove (1), and (2) follows immediately. □

PROPOSITION 2.6. *Let* R *be a ring, and* $e = e^2 \in R$ *with* $e \neq 0$ *and* $e \neq 1$. *Then* R *satisfies* (*) *if and only if both* eRe *and* $(1-e)R(1-e)$ *satisfy* (*).

Proof. $\Rightarrow$: By Lemma 2.5(1). $\Leftarrow$: Let $Q = P(R)$. Then $eQe = P(eRe)$ and $(1-e)Q(1-e) = P((1-e)R(1-e))$ by Lemma 2.5(1); hence eQe and $(1-e)Q(1-e)$ are nilpotent by the conditions. It then follows that eQ and $(1-e)Q$ are also nilpotent, say $(eQ)^n = 0 = ((1-e)Q)^n$ for some positive integer n. Notice that $Q^{2n+2} = (eQ + (1-e)Q)^{2n+2}$ is the sum of terms of the form $(eQ)^{i_1}((1-e)Q)^{j_1} \cdots (eQ)^{i_m}((1-e)Q)^{j_m}$ with $\sum_{k=1}^m (i_k + j_k) = 2n+2$, and that each of this term is contained in either $(eQ)^{n+1}$ of $((i-e)Q)^{n+1}$. Consequently $Q = eQ + (1-e)Q$ is nilpotent. □

COROLLARY 2.7. *For a ring* R *and a positive integer* n, *the following statements are equivalent:*
(1) R *satisfies* (*).

(2) *The n by n full matrix ring over R satisfies* (*).
(3) *The n by n upper triangular matrix ring over R satisfies* (*).
(4) *The n by n lower triangular matrix ring over R satisfies* (*).

Proof. (1)$\Leftrightarrow$(2): By Proposition 2.2(2) and the fact $P(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(P(R))$.

(1)$\Leftrightarrow$(3): Let U be the n by n upper triangular matrix ring over R. Consider an ideal

$$I=\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{ii} \in P(R) \text{ and } a_{ij} \in R \text{ with } i \neq j \right\}$$

in U. Say $P(R)^k = 0$, then $I^{k+n} = 0$ and $U/I \cong \oplus_{i=1}^{n} R_i$ with $R_i = R/P(R)$ for all i. So U contains a nilpotent semiprime ideal, hence satisfies (*) by Lemma 2.1. For the converse, consider $e = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then $R \cong eUe$ and so R satisfies (*) by Proposition 2.2(1).

(1)$\Leftrightarrow$(4): Similar to the proof of (1)$\Leftrightarrow$(3). □

By these results one may conjecture that subrings of given a ring R also satisfy (*) if R satisfies (*). However the following erases the possibility.

EXAMPLE 2.8. Let F be a field. For each positive integer n, let R_n be the n by n full matrix ring over F. Set R be the F-subalgebra of the direct product $\prod_{n=1}^{\infty} R_n$ generated by the direct sum $\oplus_{n=1}^{\infty} R_n$ and $1_{\prod_{n=1}^{\infty} R_n}$. Then $P(R) = \oplus_{n=1}^{\infty} P(R_n) = 0$. Next let S be the subring of R generated by the direct sum of n by n upper triangular matrix rings, say S_n, for all n and $1_{\prod_{n=1}^{\infty} R_n}$. Then $P(S) = \oplus_{n=1}^{\infty} P(S_n)$ is not nilpotent since

$$P(S_n) = \begin{pmatrix} 0 & F & F & \cdots & F & F \\ 0 & 0 & F & \cdots & F & F \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & F \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \text{ for all } n.$$

As a byproduct of $\prod_{n=1}^{\infty} S_n$ in Example 2.8, the condition (*) is not closed for direct products.

In the following process, we may show that the condition (*) is somewhat meaningful to study. Indeed, we obtain the converse process of the previous one, from Theorem 1.1 to Corollary 1.4, by the following Theorem 2.9 and Corollary 2.10.

THEOREM 2.9. *Given a ring R we have the following assertions:*
(1) *R satisfies (*) if and only if $R[X]$ satisfies (*).*
(2) *R satisfies (*) if and only if $R[[X]]$ satisfies (*) with $P(R[[X]]) = P(R)\,[[X]]$.*

Proof. (1) By the well-known fact $P(R[X]) = P(R)[X]$.

(2) First consider the case of $R[[x]]$, i.e., $|X| = 1$. Since $P(R)$ is nilpotent, $P(R)[[x]]$ is also nilpotent and so $P(R)[[x]] \subseteq P(R[[x]])$. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots \in P(R[[x]])$. Consider the epimorphism $\pi : R[[x]] \to R$ with $\pi(g(x)) = g(0)$. For each prime ideal P of R, $\pi^{-1}(P)$ is a prime ideal of $R[[x]]$ and so $\pi^{-1}(P)$ contains $f(x)$. Since $\pi(f(x)) = a_0 \in P$, we have $a_0 \in P(R) \subseteq P(R)[[x]] \subseteq P(R[[x]])$; hence $a_1x + a_2x^2 + \cdots \in P(R[[x]])$, and $a_1 + a_2x + \cdots \in P(R[[x]])$. In the same manner, we also obtain that $a_1 \in P(R)$ and $a_2 + a_3x + \cdots \in P(R[[x]])$. Inductively every a_i is in $P(R)$, i.e., $P(R[[x]]) \subseteq P(R)[[x]]$; hence $P(R[[x]]) = P(R)[[x]]$. So $R[[x]]$ satisfies (*). By this result, when $|X|$ is finite we have that $P(R[[X]]) = P(R)[[X]]$ and $R[[X]]$ satisfies (*), inductively.

Next consider the general case. Since $P(R)$ is nilpotent, $P(R)[[X]]$ is also nilpotent and so $P(R)[[X]] \subseteq P(R[[X]])$. Let $f(X) = a_0 + a_1x_{11}x_{12}\cdots x_{1n_1} + a_2x_{21}x_{22}\cdots x_{2n_2} + \cdots \in P(R[[X]])$ with $a_i \in R$. Consider the epimorphism $p : R[[X]] \to R$ with $p(g(X))$ the constant term of $g(X)$. For each prime ideal S of R, $p^{-1}(S)$ is a prime ideal of $R[[X]]$ and so $p^{-1}(S)$ contains $f(X)$. Since $p(f(X)) = a_0 \in S$, we have $a_0 \in P(R) \subseteq P(R)[[X]] \subseteq P(R[[X]])$; hence

$$a_1x_{11}x_{12}\cdots x_{1n_1} + a_2x_{21}x_{22}\cdots x_{2n_2} + \cdots \in P(R[[X]]).$$

Let $Y = X - \{x_{11}, x_{12}, \cdots, x_{1n_1}\}$. Then

$$R[[X]] = R[[x_{11}, x_{12}, \cdots, x_{1n_1}]][[Y]].$$

Define another epimorphism $\Pi : R[[X]] \to R[[x_{11}, x_{12}, \cdots, x_{1n_1}]]$ by $\Pi(b_0 + b_1y_{11}y_{12}\cdots y_{1m_1} + b_2y_{21}y_{22}\cdots y_{2m_2} + \cdots) = b_0$ for $y_{ij} \in Y$ and $b_k \in R[[x_{11}, x_{12}, \cdots, x_{1n_1}]]$.

For each prime ideal Q of $R[[x_{11}, x_{12}, \cdots, x_{1n_1}]]$, $\Pi^{-1}(Q)$ is a prime ideal of $R[[X]]$ and so $\Pi^{-1}(Q)$ contains $a_1\, x_{11}\, x_{12} \cdots x_{1n_1} + a_2\, x_{21}\, x_{22} \cdots x_{2n_2} + \cdots$. Notice

$$P(R[[x_{11}, x_{12}, \cdots, x_{1n_1}]]) = P(R)[[x_{11}, x_{12}, \cdots, x_{1n_1}]]$$

by the previous argument. Since

$$\Pi(a_1 x_{11} x_{12} \cdots x_{1n_1} + a_2 x_{21} x_{22} \cdots x_{2n_2} + \cdots) = a_1 x_{11} x_{12} \cdots x_{1n_1} \in Q,$$

we have

$$a_1 x_{11} x_{12} \cdots x_{1n_1} \in P(R)[[x_{11}, x_{12}, \cdots, x_{1n_1}]];$$

hence a_1 is also contained in $P(R)$. Next consider $R[[x_{21}, x_{22}, \cdots, x_{2n_2}]]$, $R[[x_{31}, x_{32}, \cdots, x_{3n_3}]]$, and so on. Proceeding in this method, we inductively obtain that every a_i is contained in $P(R)$ and $f(X) \in P(R)[[X]]$; hence $P(R[[X]]) = P(R)[[X]]$. Therefore $R[[X]]$ satisfies (*). The converse is obvious. □

COROLLARY 2.10. *Given a ring R, we have the following assertions:*

(1) *R is semiprime if and only if $R[X]$ is semiprime if and only if $R[[X]]$ is semiprime.*
(2) $P(R[[X]]) \subseteq P(R)[[X]]$.

Proof. (1) Theorem 2.9 applies with the condition of $P(R) = 0$. From (1), we have the result (2) by the same proof as Corollary 1.2. □

A subset I of a ring R is called *left* (*right*) *T-nilpotent* provided that for every sequence $a_1, a_2, \ldots$ in I there is a positive integer n such that $a_1 a_2 \cdots a_n = 0$ ($a_n \cdots a_2 a_1 = 0$). Nilpotent subsets of a ring are both right and left T-nilpotent obviously but the converse does not hold in general by [7, Example 2.7.38]. Left (right) T-nilpotent subsets are nil, but nil ideals need not be right (or left) T-nilpotent by [7, Example 2.7.38]; and T-nilpotence is not left-right symmetric also by [7, Example 2.7.38]. The following example shows that Theorem 2.9(2) does not hold in general for the T-nilpotence.

EXAMPLE 2.11. We use the ring R in Example 1.3. Note that $P(R)$ is right T-nilpotent. $P(R)[[x]] \nsubseteq P(R[[x]])$ by the argument in Example 1.3.

COROLLARY 2.12. *Suppose that a ring R satisfies ascending chain condition on right annihilators. Then we have the following assertions:*

(1) *$P(R)$ is right T-nilpotent if and only if R satisfies (*) if and only if $R[X]$ satisfies (*).*

(2) *$P(R)$ is right T-nilpotent if and only if R satisfies (*) if and only if $R[[X]]$ satisfies (*) with $P(R[[X]]) = P(R)[[X]]$.*

Proof. (1) By Theorem 2.9 and [1, Lemma 1.33]. □

A ring is called *reduced* if it has no nonzero nilpotent elements. The *index* of nilpotency of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$. The *index* of nilpotency of a subset I of R is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index* of nilpotency. Given a ring R, the set of all nilpotent elements is denoted by $N(R)$. The ring R in Example 2.11 is not of bounded index of nilpotency, but every nilpotent element generates a nilpotent right ideal in R; so we check the following case.

PROPOSITION 2.13. *Suppose that a ring R is of bounded index of nilpotency.*

(1) *If every nilpotent element generates a nil right (or left) ideal in R, then $P(R) = N(R)$.*

(2) *If R satisfies (*) and every nilpotent element generates a nil right ideal in R, then $R[X]/P(R[X])$ and $R[[X]]/P(R[[X]])$ are reduced rings.*

Proof. (1) First note that since R is of bounded index of nilpotency by hypothesis and $P(R)$ is nil, $R/P(R)$ is also of bounded index of nilpotency. So we may suppose that R is semiprime. Assume to the contrary that there is $0 \neq a \in N(R)$. Then aR is a nil right ideal by the condition, set $I = aR$. Next since R is of bounded index of nilpotency by hypothesis, there is the bounded index of nilpotency of I, say n. By [2, Lemma 11], $b^{n-1}Rb^{n-1} = 0$ for all $b \in I$, and then $(Rb^{n-1}R)^2 = 0$; hence $Rb^{n-1}R$ is a nonzero nilpotent ideal for some $b \in I$, a contradiction. Thus $P(R) = N(R)$. The left case can be obtained by the symmetry.

(2) Since $R[X] \,/\, P(R)\,[X] \cong R/P(R)\,[X]$ and $R[[X]] \,/P(R)\,[[X]] \cong R/P(R)\,[[X]]$, we have the results by Theorem 2.9 and (1). □

Given a ring that satisfies (*), it is also natural to conjecture that every homomorphic image of it also satisfies (*). But it is not true in general by the following.

EXAMPLE 2.14. Let K be a field and $X = \{x_n \mid n = 1, 2, \dots\}$ be a countably infinite set of commuting indeterminates over F. Set $R_n = K[x_n]$ for all n, and I_n be the ideal of R_n generated by x_n^{n+1}. Next let $R = \prod_{n=1}^{\infty} R_n$. Then R is reduced and satisfies (*) obviously. Define $I = \prod_{n=1}^{\infty} I_n$, then I is an ideal of R with $R/I \cong \prod_{n=1}^{\infty} R_n/I_n$. Notice that $R_n x_n/I_n$ is a nilpotent ideal in R_n for every n, and so we have

$$\oplus_{n=1}^{\infty} R_n x_n/I_n \subseteq P(\prod_{n=1}^{\infty} R_n/I_n).$$

Note that $\oplus_{n=1}^{\infty} R_n x_n/I_n$ is not nilpotent, so $P(\prod_{n=1}^{\infty} R_n/I_n)$ is not nilpotent. Thus R/I does not satisfy (*).

In the following argument we may obtain a condition for which the preceding conjecture holds.

PROPOSITION 2.15. *Given a ring R, the following conditions are equivalent:*
(1) *R satisfies (*).*
(2) *R/I satisfies (*) for every nilpotent ideal I.*
(3) *R/I satisfies (*) for some nilpotent ideal I.*
(4) *Every nonzero homomorphic image of R satisfies (*) with nilpotent kernel.*
(5) *Some nonzero homomorphic image of R satisfies (*) with nilpotent kernel.*

Proof. (2)⇒(3) and (4)⇒(5): Obvious.

(1)⇒(2): Note that $I \subseteq P(R)$ since I is nilpotent, and so $P(R/I) = P(R)/I$; hence R/I satisfies (*) if R satisfies (*).

(2)⇒(4): Note that every nonzero homomorphic image of R is isomorphic to R/K with K the kernel, so (2) applies if the kernel is nilpotent.

(3)⇒(1): Note that

$$P(R/I) = \cap_{J \in \mathrm{Spec}(R), J \supseteq I} J/I = (\cap_{J \in \mathrm{Spec}(R), J \supseteq I} J)/I.$$

Since R/I satisfies (*), $P(R/I)$ is nilpotent and $(\cap_{J \in \mathrm{Spec}(R), J \supseteq I} J)^n \subseteq I$ for some positive integer n. But I is nilpotent and so $\cap_{J \in \mathrm{Spec}(R), J \supseteq I} J$ is also nilpotent. Consequently $\bigcap_{J\in \mathrm{Spec}(R), J\supseteq I} J$ is a nilpotent semiprime ideal in R, and thus R satisfies (*) by Lemma 2.1.

(5)$\Rightarrow$(1): Let K be the nilpotent kernel, then R/K satisfies (*) by the condition. The remainder of the proof is similar to one of (3)$\Rightarrow$(1).□

In Proposition 2.15, notice that the condition "nilpotent" is not superfluous by the rings R in Examples 1.3, [6, Example 1.1], and 2.14.

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