International Journal of Management Science
Vol 7, No 2, November 2001

A Paradox in an Indefinite Quadratic Transportation Problem

S. R. Arora
Department of Mathematics, Hans Raj College, University of Delhi,
Delhi-110007, India.

Archana Khurana
Department of Mathematics, University of Delhi,
Delhi-110007, India.

(Received Dec. 2000 ; Revised Sep. 2001 ; Accepted Nov. 2001)

ABSTRACT

This paper discusses a paradox in an Indefinite Quadratic Transportation Problem. Here, the objective function is the product of two linear functions. A paradox arises when the transportation problem admits of a total cost which is lower than the optimum cost, by transporting larger quantities of goods over the same route. A sufficient condition for the existence of a paradox is established. Paradoxical Range of Flow is obtained for any given flow in which the corresponding objective function value is less than the optimum value of the given transportation problem. It is illustrated with the help of a numerical example.

1. INTRODUCTION

Cost Minimizing Transportation Problem (CMTP) have extensively been studied by many researchers. F.Glover, D. Klingman and T.Ross [5] in 1974 considered a constrained transportation problem. D.Klingman and R.Russel [9] in 1975 introduced a specialized method for solving a transportation problem with several additional linear constraints. Brigden [2] in 1974 considered the transportation problem with mixed constraints. Brigden solved this problem by considering a related standard transportation problem having two additional supply points and two additional destinations.

W. Szwarc [10] in 1971 discussed a paradox that arises when there is a solution of a transportation problem involving lesser cost than the optimal cost and is available by shipping larger quantities of goods over the originally optimal routes. So a paradox arises in the sense that more is being shipped at a lesser cost. He developed the transportation scheme method for finding a paradoxical pair. His approach was confirmed in the problem in which the objective function is linear.

In this paper such a paradox is discussed for a transportation problem when the objective function is the product of two linear factors.

2. THEORETICAL DEVELOPMENT

We know linear functions are the type most useful and widely used in modelling of mathematical optimization problems. Also quadratic functions and quadratic problems are the least difficult to handle out of all non-linear programming problems. A fair number of functional relationships occurring in the real world are truly quadratic. For example, kinetic energy carried by a rocket or an atomic particle is proportional to the square of its velocity. In statistics, the variance of a given sample of observations is a quadratic function of the values that constitute the sample. So there are countless other non-linear relationships occurring in nature, capable of being approximated by quadratic functions.

Consider the following transportation problem when the objective function is a product of linear functions.

(P_0) Minimize $Z = N(X)\, D(X)$

$$= \left(\sum_i \sum_j c_{ij} x_{ij}\right)\left(\sum_i \sum_j d_{ij} x_{ij}\right)$$

subject to

$$\sum_j x_{ij} = a_i \qquad \forall\, i \in I$$

$$\sum_i x_{ij} = b_j \qquad \forall\, j \in J$$

$$x_{ij} \geq 0 \qquad \forall\, (i, j) \in I \times J$$

where I : Index set of supply points

J : Index set of destinations

$X = \{x_{ij}\}$

x_{ij} : quantity transported from the ith supply point to the jth destination

c_{ij} : per unit cost in transporting goods from the ith supply point to the jth destination

d_{ij} : per unit depreciation (wear and tear) in transporting goods from the ith supply point to the jth destination

In the above problem the total transportation cost of transporting one unit from ith origin to jth destination is $\sum_i \sum_j c_{ij} x_{ij}$, but while transporting goods from one origin to the other destination, some fraction of goods get damaged so the total cost of damaged goods is $\sum_i \sum_j d_{ij} x_{ij}$. Our aim is to minimize the two costs simultaneously; therefore we consider the product of two costs i.e., $(\sum_i \sum_j c_{ij} x_{ij})$ $(\sum_i \sum_j d_{ij} x_{ij})$.

It can easily be proved that the objective function $N(X) D(X)$ is quasiconcave. The minimum of a quasiconcave function is obtained at an extreme point of the feasible region. Hence to find an optimal solution of the given problem, our method searches for an optimal extreme point.

Let an optimal feasible solution of (P_0) yield value $Z^0 = N^0 D^0$ of the objective function and let F^0 be the corresponding flow.

Clearly, $$F^0 = \sum_{i \in I} a_i = \sum_{j \in J} b_j$$

It can be observed that a paradox exists if more than F^0 is flown at an objective function value less than Z^0.

Flow can be increased if

(a) supply points are allowed to increase their supplies or

(b) destinations are allowed to receive more or

(c) both (a) and (b) hold.

These give rise to the following three problems

(P_1) Minimize $Z = N(X)\, D(X)$

subject to
$$\sum_j x_{ij} \geq a_i \qquad \forall\, i \in I$$
$$\sum_i x_{ij} = b_j \qquad \forall\, j \in J$$
$$x_{ij} \geq 0 \qquad \forall\, (i, j) \in I \times J$$

(P_2) Minimize $Z = N(X)\, D(X)$

subject to $\sum_j x_{ij} = a_i \quad \forall\, i \in I$

$\sum_i x_{ij} \geq b_j \quad \forall\, j \in J$

$x_{ij} \geq 0 \quad \forall\, (i, j) \in I \times J$

(P_3) Minimize $Z = N(X)\, D(X)$

subject to $\sum_j x_{ij} \geq a_i \quad \forall\, i \in I$

$\sum_i x_{ij} \geq b_j \quad \forall\, j \in J$

$x_{ij} \geq 0 \quad \forall\, (i, j) \in I \times J$

The feasible region in each case is larger than that of (P_0), so it follows that optimal objective function value in each of the three problems is not greater than that of (P_0). So more may be flown than that in (P_0) at an objective function value less that that of (P_0). Hence a paradox may arise in these cases.

3. DEFINITIONS

(1) **OBJECTIVE FUNCTION – FLOW PAIR:** If Z^0 be the objective function value and F^0 be the flow corresponding to a feasible solution X of (P_0), then the pair (Z^0, F^0) is called the objective function flow pair corresponding to a feasible solution X.

(2) **PARADOXICAL PAIR:** An objective function flow pair (Z, F) is called paradoxical pair if $Z < Z^0$ and $F > F^0$.

(3) **BEST PARADOXICAL PAIR:** The paradoxical pair (Z^*, F^*) is called the best paradoxical pair if $\forall$ paradoxical pairs (Z, F), either $Z^* < Z$, or $Z^* = Z$ but $F^* > F$.

(4) **PARADOXICAL RANGE OF FLOWS:** If F^* be the flow corresponding to the best paradoxical pair then $[F^0, F^*]$ is the paradoxical range of flows.

Theorem1: Let $X = \{x_{ij}\}$ be a basic feasible solution of (P_0) with basis matrix B.

Then it will be an optimal basic feasible solution if

$R_{ij} \geq 0 \quad \forall$ cells $(i, j) \notin B$
$\quad = 0 \quad \forall$ cells $(i, j) \in B$

where $R_{ij} = q_{ij}(z'_{ij} - d_{ij})(z_{ij} - c_{ij}) - Z_1(z'_{ij} - d_{ij}) - Z_2(z_{ij} - c_{ij})$

$u_i + v_j = z_{ij} \quad \forall$ cells $(i, j) \notin B$

$u'_i + v'_j = z'_{ij} \quad \forall$ cells $(i, j) \notin B$

Also
$$\left.\begin{aligned} u_i + v_j &= c_{ij} \quad \forall \text{ cells } (i, j) \in B \\ u'_i + v'_j &= d_{ij} \quad \forall \text{ cells } (i, j) \in B \end{aligned}\right\} \tag{1}$$

Z_1 = the value of $\sum_i \sum_j c_{ij} x_{ij}$, at the current basic feasible solution corresponding to B.

Z_2 = the value of $\sum_i \sum_j d_{ij} x_{ij}$, at the current basic feasible solution corresponding to B.

and θ_{ij} is the level at which a non basic cell (i, j) enters the basis replacing some basic cell of B.

Note: u_i, v_j, u'_i, v'_j are determined by using equations (1) and taking one of the u_i's or v_j's and u'_i's or v'_j's as zero.

Proof: Let Z^0 be the objective function value of the problem (P_0). Let $Z^0 = Z_1 Z_2$ Let $\hat{Z}$ be the value of the objective function at the current basic feasible solution $\hat{X} = \{x_{ij}\}$ corresponding to the basis B obtained on entering the cell (i, j) into the basis. Then $\hat{Z} = [Z_1 + q_{ij}(c_{ij} - z_{ij})]\,[Z_2 + q_{ij}(d_{ij} - z'_{ij})]$

$$\begin{aligned} \text{Now, } \hat{Z} - Z^0 &= [Z_1 + q_{ij}(c_{ij} - z_{ij})]\,[Z_2 + q_{ij}(d_{ij} - z'_{ij})] - Z_1 Z_2 \\ &= Z_1 Z_2 + Z_1 q_{ij}(d_{ij} - z'_{ij}) + Z_2 q_{ij}(c_{ij} - z_{ij}) + q_{ij}^2 (c_{ij} - z_{ij})(d_{ij} - z'_{ij}) - Z_1 Z_2 \\ &= Z_1 q_{ij}(d_{ij} - z'_{ij}) + Z_2 q_{ij}(c_{ij} - z_{ij}) + q_{ij}^2 (c_{ij} - z_{ij})(d_{ij} - z'_{ij}) \\ &= q_{ij}[Z_1(d_{ij} - z'_{ij}) + Z_2(c_{ij} - z_{ij}) + q_{ij}(c_{ij} - z_{ij})(d_{ij} - z'_{ij})] \end{aligned}$$

This basic feasible solution will give an improved value of Z if $\hat{Z} < Z^0$.
i.e., if $\hat{Z} - Z^0 < 0$ i.e., if $q_{ij}[Z_1(d_{ij} - z'_{ij}) + Z_2(c_{ij} - z_{ij}) + q_{ij}(c_{ij} - z_{ij})(d_{ij} - z'_{ij})] < 0$
since $q_{ij} \geq 0$

$$\therefore \ [Z_1(d_{ij}-z'_{ij})+Z_2(c_{ij}-z_{ij})+q_{ij}\,(c_{ij}-z_{ij})\,(d_{ij}-z'_{ij})] < 0 \tag{2}$$

$\Rightarrow$ One can move from one basic feasible solution to another basic feasible solution on entering the cell (i, j) into the basis for which condition (2) is satisfied.
It will be an optimal basic feasible solution if

$$q_{ij}(z'_{ij}-d_{ij})(z_{ij}-c_{ij})-Z_1(z'_{ij}-d_{ij})-Z_2(z_{ij}-c_{ij}) \geq 0$$

$$\text{or} \quad R_{ij} \geq 0 \quad \forall \text{ cells } (i, j) \notin B$$

where $R_{ij} = q_{ij}\,(z'_{ij}-d_{ij})(z_{ij}-c_{ij})-Z_1\,(z'_{ij}-d_{ij})-Z_2(z_{ij}-c_{ij})$

Also, it can easily be seen that

$$R_{ij} = 0 \qquad \forall \text{ cells } (i, j) \in B$$

Thus the solution will be an optimal basic feasible solution if

$$R_{ij} \geq 0 \qquad \forall \text{ cells } (i, j) \notin \text{B}$$
$$R_{ij} = 0 \qquad \forall \text{ cells } (i, j) \in \text{B}$$

Theorem 2 : In an optimal basic feasible solution $X = \{x_{ij}\}$ of the problem (P_0), if (*i*) $\exists$ a cell (p, q) where $p \in I$, $q \in J$ for which

$$Z_2(u_p + v_q) + Z_1(u'_p + v'_q) < 0$$

and $(u_p + v_q)\,(u'_p + v'_q) < 0$

OR (ii) out of all those cells for which

$$(u_i + v_j)\,(u'_i + v'_j) < 0$$

we choose a cell (p, q) for which

$$Z_2(u_p + v_q) + Z_1(u'_p + v'_q) < 0$$

and (iii) there exists a positive number λ such that the same basis B remain optimal when a_p and b_q are replaced by $a_p + \lambda$ and $b_q + \lambda$ respectively .
Then there exists a paradox.

Proof: For a basic feasible solution $X = \{x_{ij}\}$ of the problem (P_0), the dual variables u_i, v_j, u'_i, v'_j are given by

$$u_i + v_j = c_{ij}$$
$$u'_i + v'_j = d_{ij}$$

where (i, j) is a basic cell. The value of the objective function is given by

$$Z^0 = Z_1 Z_2 = \left(\sum_i \sum_j c_{ij}\, x_{ij}\right)\left(\sum_i \sum_j d_{ij}\, x_{ij}\right)$$

$$= \left(\sum_i \sum_j (u_i + v_j)\, x_{ij}\right)\left(\sum_i \sum_j (u'_i + v'_j)\, x_{ij}\right)$$

$$= \left[\sum_i \left(\sum_j x_{ij}\right) u_i + \sum_j \left(\sum_i x_{ij}\right) v_j\right]\left[\sum_i \left(\sum_j x_{ij}\right) u'_i + \sum_j \left(\sum_i x_{ij}\right) v'_j\right]$$

$$= \left[\sum_i a_i u_i + \sum_j b_j v_j\right]\left[\sum_i a_i u'_i + \sum_j b_j v'_j\right]$$

Suppose $\exists$ some cell (p, q) where $p \in I$, $q \in J$ satisfying hypothesis (i) or (ii) and (iii) . Then in this case problem (P_3) will emerge and quantity transported will be $(F^0 + \lambda)$. Let the corresponding value of the objective function be $\hat{Z}$,

where
$$\hat{Z} = \left[\sum_{i \neq p} u_i a_i + \sum_{j \neq q} v_j b_j + u_p (a_p + l) + v_q (b_q + l)\right]$$
$$\left[\sum_{i \neq p} u'_i a_i + \sum_{j \neq q} v'_j b_j + u'_p (a_p + l) + v'_q (b_q + l)\right]$$
$$= [Z_1 + l(u_p + v_q)]\,[Z_2 + l(u'_p + v'_q)]$$

Now,
$$\hat{Z} - Z^0 = [Z_1 + l(u_p + v_q)]\,[Z_2 + l(u'_p + v'_q)] - Z_1 Z_2$$
$$= lZ_1(u_p + v_q) + lZ_2(u'_p + v'_q) + l^2(u_p + v_q)\,(u'_p + v'_q)$$
$$= l[Z_2(u_p + v_q) + Z_1(u'_p + v'_q) + l(u_p + v_q)(u'_p + v'_q)]$$
$$< 0 \qquad \text{(using (i) and (ii))}$$
$$\Rightarrow \quad \hat{Z} - Z^0 < 0 \quad \text{or} \quad \hat{Z} < Z^0$$

Hence a paradox exists.

Theorem 3: Optimal basic solution of the problem (P_3) yields the best paradoxical pair.

Proof: let $X^* = \{x_{ij}\}$ be an optimal basic feasible solution of the problem (P_3), then

$$\sum_j x^*_{ij} = a^*_i \geq a_i \qquad \forall\, i \in I$$

$$\sum_i x_{ij}^* = b_j^* \geq b_j \qquad \forall\, j \in J$$

and let F^* and Z^* be the corresponding optimal flow and optimal value of the objective function .

Consider the following problem

(P_3') Minimize $Z = N(X)\,D(X)$

subject to
$$\sum_j x_{ij}^* = a_i^* + p_i \qquad \forall\, i \in I$$
$$\sum_i x_{ij}^* = b_j^* + q_j \qquad \forall\, j \in J$$
where
$$\sum_j q_j = 0 = \sum_i p_i$$
$$a_i^* + p_i \geq a_i \qquad \forall\, i \in I$$
$$b_j^* + q_j \geq b_j \qquad \forall\, j \in J$$

Let $X' = \{x'_{ij}\}$ be an optimal basic feasible solution of problem (P_3'). Then X' will be a feasible solution of problem (P_3).

i.e., $N(X')\,D(X') \geq N(X^*)\,D(X^*) = Z^*$.

$\Rightarrow$ No optimal basic feasible solution of the problem (P_3') can yield the objective function value less than Z^*.i.e., $Z^*$is the optimal objective function value.

$\Rightarrow \nexists$ any other distribution of supplies a_i's and demands b_j's better than a_i^* and b_j^* $\forall$ i and j, for which the corresponding problem (P_3') has objective function value less than Z^* and flow greater than F^*. Hence, optimal basic feasible solution of the problem (P_3) yields the best paradoxical pair.

'Paradoxical Solution' for a Specified Flow $\overline{F}$ in $[F^0, F^*]$

For a specified flow $\overline{F}$ in, $[F^0, F^*]$ problem (P_4) given below is studied whose optimal feasible solution provides the best value of the objective function.

(P_4) $\underset{x}{\text{Minimize}}$ $Z = N(X)\,D(X)$

subject to
$$\sum_j x_{ij} \geq a_i \qquad \forall\, i \in I$$
$$\sum x_{ij} \geq b_j \qquad \forall\, j \in J$$

$$\sum_i \sum_j x_{ij} = \overline{F} \qquad \forall\ (i, j) \in I \times J$$

$$x_{ij} \geq 0 \qquad \forall\ (i, j) \in I \times J$$

The associated equivalent transportation problem (P_5) is

(P_5) Minimize $\left(\sum_i \sum_j c'_{ij}\, y_{ij}\right)\left(\sum_i \sum_j d'_{ij}\, y_{ij}\right)$

subject to $\sum_j y_{ij} = a'_i \qquad \forall\ i \in I'$

$\sum_i y_{ij} = b'_j \qquad \forall\ j \in J'$

$y_{ij} \geq 0 \qquad \forall\ (i, j) \in I' \times J'$

where $I' = I \cup \{ m + 1, m + 2 \}$

$J' = J \cup \{ n + 1, n + 2 \}$

$c'_{ij} = c_{ij}$ and $d'_{ij} = d_{ij} \qquad \forall\ (i, j) \in I \times J$

Let $c'_{m+1,j} = c_{lj}$ and $d'_{m+1,j} = d_{lj}$

such that $c_{lj}\, d_{lj} = \min_{i \in I}(c_{ij}\, d_{ij})$

$c'_{i,n+1} = c_{ik}$ and $d'_{i,n+1} = d_{ik}$

such that $c_{ik}\, d_{ik} = \min_{j \in J}(c_{ij}\, d_{ij})$

$c'_{m+2,j} = M \qquad \forall\ j \in J$

$c'_{i,n+2} = M \qquad \forall\ i \in I$

where M (> 0) is number.

Similarly, $d'_{m+2,j} = M \qquad \forall\ j \in J$

$d'_{i,n+2} = M \qquad \forall\ i \in I$

Also, $c'_{m+1,n+1} = c'_{m+2,n+2} = M$

$c'_{m+1,n+2} = c'_{m+2,n+1} = 0$

likewise , $d'_{m+1,n+1} = d'_{m+2,n+2} = M$

$d'_{m+1,n+2} = d'_{m+2,n+1} = 0$

$a'_i = a_i, \quad i \in I\ ; = \quad a'_{m+1} = \sum_j b_j$

$b'_j = b_j, \quad j \in J\ ; \quad b'_{n+1} = \sum a_i$

$$a'_{m+2} = b'_{n+2} = \sum_i a_i + \sum_j b_j - \overline{F}$$

We have introduced two additional sources and two additional destinations, the method is such that it moves from an $(m + 2) \times (n + 2)$ transportation problem to an $(m + 1) \times (n + 1)$ transportation problem and finally to an $m \times n$ transportation problem. The $(m + 1)$th row and $(n + 1)$th column is introduced to balance the problem and $(m + 2)$th row and $(n + 2)$th column helps us in making the total flow equal to $\overline{F}$.

The problem (P_5) is solved in the same way as (P_0). The optimal solution of (P_4) is then obtained from the optimal solution of (P_5).

4. CONCLUSIONS

1. It is concluded that if the objective function is the product of two linear functions and in an optimal basic feasible solution $X = \{x_{ij}\}$ if there exists a cell (p, q) with $Z_2(u_p+v_q) + Z_1(u'_p+v'_q) < 0$ and $(u_p + v_q)\ (u'_p + v'_q) < 0$ then we can increase the flow and reduce the cost i.e., a paradox exists.
2. Problem (P_3) determines the best paradoxical pair.
3. To find an optimal solution for a specified flow, problem (P_4) is solved by forming a problem (P_5) by introducing two additional rows and columns.

NUMERICAL ILLUSTRATION

Consider a balanced transportation problem

				$a_i \downarrow$
$c_{11} \leftarrow$ $d_{11} \leftarrow$	2 -4	4 -3	1 -2	4
	3 -6	2 -4	1 -2	7
	1 -2	1 -2	3 -6	3
$b_j \rightarrow$	5	5	4	

The objective function is $\underset{x}{\text{Min}}\, Z = N(X)\ D(X)$. The initial basic feasible solution is:

				$a_i \downarrow$
	2 (4) -4	4 -3	1 -2	4
	3 (1) -6	2 (5) -4	1 (1) -2	7
	1 -2	1 -2	3 (3) -6	3
$b_j \rightarrow$	5	5	4	

Optimal table is

				$u_i \downarrow$	$u'_i \downarrow$
	2 (3) (-1) -4	4 (4) -3	1 2 (1) -2	0	5
	3 (5) -6	2 (1) -4	1 (1) -2	-2	4
	1 (4) (-8) -2	1 (3) (-6) -2	3 (3) -6	0	0
$v_j \rightarrow$	5	4	3		
$v'_j \rightarrow$	-10	-8	-6		

$Z_1 = 16 + 15 + 2 + 1 + 9 = 43$,

$Z_2 = -12 - 30 - 4 - 2 - 18 = -66$

$\theta_{11} = 4,\ \theta_{13} = 1,\ \theta_{31} = 3,\ \theta_{32} = 1$

$R_{11} = 4\ (-1)\ (3) - 43\ (-1) - (-66)\ (3) > 0$,

$R_{13} = 1\ (1)\ (2) - 43\ (1) - (-66)\ (2) > 0$

Also $R_{31}, R_{32} > 0 \quad \therefore R_{ij} \geq 0\ \forall\ (i, j) \notin B$

$\therefore$ Optimality condition is satisfied.

Flow $F^0 = 4 + 5 + 1 + 1 + 3 = 14$ and Optimal solution is $x_{12} = 4$, $x_{21} = 5$, $x_{22} = 1$ $x_{23} = 1$, $x_{33} = 3$ with value of the objective function = (43) (-66) = -2838

Now, we check the sign of $(u_i + v_j)(u'_i + v'_j)$

For $i = 1, j = 1$, $(u_1 + v_1)(u'_1 + v'_1) = (3 + 2)\ (-6 + 1) = (5)\ (-5) < 0$

Also, $Z_2(u_1 + v_1) + Z_1(u'_1 + v'_1) = -66\ (5) + 43\ (-5) < 0$

$\therefore a_1 \rightarrow a_1 + \lambda,\ b_1 \rightarrow b_1 + \lambda$

The new optimal solution is

				$a_i \downarrow$
	2; -4	4; (4) + l; -3	1; -2	4 + l
	3; (5) + l; -6	2; (1) - l; -4	1; (1); -2	7
	1; -2	1; -2	3; (3); -6	3
$b_j \rightarrow$	5 + l	5	14	

$l = 1$ at most, otherwise feasibility of cell (2, 2) is violated.

Let $l = 1$, then

				$u_i \downarrow$	$u_i' \downarrow$
	2; (3); (1); -4	4; (5); -3	1; (2); (1); -2	0	5
	3; (6); -6	2; (0); -4	1; (1); -2	-2	4
	1; (4); (-8); -2	1; (3); (-6); -2	3; (3); -6	0	0
$v_j \rightarrow$	5	4	3		
$v_j' \rightarrow$	-10	-8	-6		

$Z_1 = 20 + 18 + 1 + 9 = 48$, $Z_2 = - 15 - 36 - 2 - 18 = - 71$

$\theta_{11} = 5$, $\theta_{13} = 1$, $\theta_{31} = 3$, $\theta_{32} = 0$

$R_{11} = 5\,(1)\,(3) - 41\,(1) - (-71)\,(3) > 0$

Also $R_{13}, R_{31}, R_{32} \geq 0 \quad \therefore R_{ij} \geq 0 \quad \forall (i, j) \notin B$

Hence the optimality condition is satisfied. Flow = 5 + 6 + 1 + 3 = 15

Optimal objective function value = (48) (-71) = - 3408

Best Paradoxical Pair

To find the best paradoxical pair, we form the problem (P_3). Applying Klingsman and Russell's approach and solving,

The optimal table is

					$u_i \downarrow$	$u'_i \downarrow$
	2 (1) (-2) -4	4 (0) -3	1 (2) (-4) -2	4 (4) -3	0	0
	3 (-1) (-3) -6	2 (1) (1) -4	1 (-1) (-7) (-2)	3 (7) -6	-1	-3
	1 (1) (-7) -2	1 (2) (-4) -2	3 (-1) (-3) -6	3 (3) -6	-1	-3
	3 (5) -6	4 (5) -3	3 (4) -6	0 (4) -3 0	0	0
$v_j \rightarrow$	3	4	3	4		
$v'_j \rightarrow$	-6	-3	-6	-3		

$Z_1 = 16 + 21 + 9 + 15 + 20 + 12 = 93$, $Z_2 = -12 - 42 - 18 - 24 - 15 - 30 = -141$

$\theta_{21} = 0$, $\theta_{22} = 0$, $\theta_{33} = 0$

$R_{21} = 0\ (-3)\ (-1) - (-93)\ (-3) - (-141)\ (-1) = 279 - 141 > 0$

$R_{22} = 0\ (1)\ (1) - 93\ (1) - (-141)\ (1) = -93 + 141 > 0$

$R_{33} = 0\ (-3)\ (-1) - 93\ (-3) - (-141)\ (-1) = 279 - 141 > 0$

Also, $R_{11}, R_{13}, R_{23}, R_{31}, R_{32}, R_{44} > 0 \quad \therefore R_{ij} \geq 0 \quad \forall (i, j) \notin B$

Flow, $F^* = 28 > F^0$

also, Optimal objective function value is $Z_1 Z_2 = (93)\ (-141) = -13113$

and the optimal table is

2 -4	4 (9) -3	1 -2
3 (12) -6	2 -4	1 -2
1 -2	1 -2	3 (7) -6

and optimal solution is $x_{12} = 9$, $x_{21} = 12$, $x_{33} = 7$
So the best paradoxical pair is (-13113,28) and paradoxical range of flow is [14, 28].

To find a Paradoxical Solution for a specified flow $\overline{F} = 20 \in [F^0, F^*]$
In order to obtain the paradoxical solution when flow $\overline{F}$ = 20 is specified , we solve the following problem:

				$a_i \downarrow$
	2 -4	4 -3	1 -2	≥ 4
	3 -6	2 -4	1 -2	≥ 7
	3 -2	3 -2	3 -4	≥ 3
$b_j \rightarrow$	≥ 5	≥ 5	≥ 4	

We reduce it into a related transportation problem (RTP) such that

$$a_{m+1} = \sum_{j=1}^{n} b_j \,,\ b_{n+1} = \sum_{i=1}^{m} a_i$$

$$a'_{m+2} = b'_{n+2} = \sum_i a_i + \sum_j b_j - \overline{F}$$

Optimal solution is given in the next table:

						$u_i \downarrow$	$u'_i \downarrow$
	2 (1) (-2) -4	4 (4) -3	1 (2) (-4) -2	4 (-1) (-3) -3	M (-M) (-M) M	3	-6
	3 (1) -6	2 (2) (1) -4	1 (2) (-4) -2	3 (6) -6	M (-M) (-M) M	3	-6
	1 (2) (-4) -2	1 (3) (-2) -2	3 (3) -6	3 (0) (0) -6	M (-M) (-M) M	3	-6
	3 (4) -6	4 (1) -3	3 (1) -6	M (3-M) (-6-M) M	0 (8) 0	3	-6
	M (-M) (-M) M	M (1-M) (3-M) M	M (-M) (-M) M	0 (8) 0	M (-3-M) (6-M) M	0	0
$v_j \rightarrow$	0	1	0	0	-3		
$v'_j \rightarrow$	0	3	0	0	6		

$Z_1 = 65$, $Z_2 = -105$; $\theta_{11} = 4$, $\theta_{13} = 1$

$R_{11} = 4\ (-2)\ (1) - 65\ (-2) - (-105)\ (1) > 0$, $R_{13} = 1\ (-4)\ (2) - 65\ (-4) - (-105)\ (2) > 0$

also, $R_{ij} \geq 0 \quad \forall\ (i, j) \notin B$

$\therefore$ we have obtained the optimality condition and the optimal table is

2 -4	4 (5) -3	1 -2
3 (11) -6	2 -4	1 -2
1 -2	1 -2	3 (4) -6

Flow = 5 + 11 + 4 = 20
and optimal objective function value = (65) (-105) = - 6825

Acknowledgements: We are thankful to the referee's for their valuable comments and suggestions.

REFERENCES

[1] Appa, G. M., "The Transportation Problem and its Variants," Operational Research Quarterly 24 (1973), No. 1, 79-99.

[2] Brigden, M. E. B., "A Variant of Transportation Problem in which the constraints are of mixed type," Operational Research Quarterly 25 (1974), No. 3, 437-445.

[3] Charnes, A and D. Klingman, "The more for less paradox in the Distribution Model," Cahiers du Centre d' Etudes de Recherche Operationnelle 13 (1971), 11-22.

[4] Dinkelbach, W., "On Non-Linear Fractional Programming," Management Science 13 (1967), No. 7, 492-498.

[5] Glover, F., D. Kingman, and T. Ross, "Finding Equivalent Transportation Formulations for Constrained Transportation Problems," Naval Research Logistics Quarterly 21 (1974), 247-254.

[6] Gupta, Khanna, Puri, "A Paradox in Linear Fractional Transportation Problems with Mixed Constraints," Optimization 27 (1993), 375-387.

[7] Khanna, Saroj, and M. C. Puri, "Solving a Transportation Problem with mixed constraints and a Specified Transportation Flow," Opsearch 20 (1983), No.1, 16-24 (India).

[8] Klingman, D. and R. Russel, "The Transportation Problem with mixed constraints," Operational Research Quarterly 25 (1974), No. 3, 447-455.

[9] Klingman, D. and R. Russel, "Solving Constrained Transportation Problems," Operations Research 23 (1975), No. 1, 91-105.

[10] Szwarc, W., "The Transportation Paradox," Naval Research Logistics Quarterly 18 (1971), No. 2, 185-202.

[11] Verma, Vanita and M. C. Puri, "On a Paradox in Linear Fractional Transportation Problem. Recent Developments in Mathematical Programming," published on behalf of Australian Society for Operational Research (ed. Santosh Kumar), Gordan and Breach Science Publishers (1991), 413-424.