Robust Stability and Disturbance Attenuation for a Class of Uncertain Singularly Perturbed Systems

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Abstract: This paper considers the problem of robust stabilization and disturbance attenuation for a class of uncertain singularly perturbed systems with norm-bounded nonlinear uncertainties. It is shown that the state feedback gain matrices can be determined to guarantee the stability of the closed-loop system for all $\varepsilon \in (0,\infty)$. Based on this key result and some standard Riccati inequality approaches for robust control of singularly perturbed systems, a constructive design procedure is developed.

Keywords: Robust stability, disturbance attenuation, singularly perturbed systems

I. Introduction

Singularly perturbed systems often occur naturally because of the presence of small parasitic parameters multiplying the time derivatives of some of the system states. Singularly perturbed control systems have been intensively studied for the past three decades; see, e.g., [1]. A popular approach adopted to handle these systems is based on the so-called reduced technique [2]. The composite design based on separate designs for slow and fast subsystems has been systematically reviewed in [3]. Recently, the robust stabilization of singularly perturbed systems based on a new modeling approach has been investigated in [4].

The stability problem (ε-bound problem) in singularly perturbed systems differs from conventional linear systems, which can be designed as: characterizing an upper bound ε_0 of the positive perturbing scalar ε such that the stability of a reduced-order system would guarantee the stability of the original full-order system for all $\varepsilon \in (0,\varepsilon_0)$ [5]. It is known, by the lemma of Klimushchev and Krasovskii [1], [6], that if the reduced-order system is an asymptotically stable, then this upper bound ε_0 always exists. Researchers have tried various ways to find either the stability bound ε_0 or a less conservative lower bound for ε_0, as in [1], [5] and [7]. Recently, the robust stability analysis and stability bound improvement of perturbed parameter in the singularly perturbed systems by using linear fractional transformations and structured singular values approach (μ) has been investigated in [8].

Continuing the works of [9], [10] and [11], this paper presents new results on control synthesis for robust stabilization and robust disturbance attenuation for linear singularly perturbed systems with norm-bounded nonlinear uncertainties. The class of plants considered in this paper consists of systems in state-space form with linear nominal parts and norm-bounded nonlinear uncertainties on both types of states (slow and fast dynamics) and control inputs. Robust stabilization and disturbance attenuation of such systems is investigated using the Hamiltonian approach. The state feedback gain matrices can be constructed from the positive definite solutions to a certain Riccati inequalities. Another advantage to this approach is that we can preserve the characteristic of the composite controller [11], [12] and [13], i.e., the whole-dimensional process can be separated into two subsystems. Moreover, the presented stabilization design insures the stability for all $\varepsilon \in (0,\infty)$.

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II. Problem Statement

Consider a linear time-invariant singularly perturbed system with norm-bounded nonlinear uncertainties in the form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} w(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) + \begin{bmatrix} \Delta_1(x_1,x_2,u) \\ \Delta_2(x_1,x_2,u) \end{bmatrix} \tag{1}$$

$$z(t) = c_1\, x_1(t) + c_2\, x_2(t) + F\, u(t)$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $n\ (= n_1 + n_2)$ is the order of the whole system, $u \in R^m$, $w \in R^k$, $z \in R^l$ are control vector, disturbance and controlled output, respectively, $\Delta_i(x_1,x_2,u)$; $i = 1,2$ are nonlinear terms of the uncertainty space. The matrices $a_{11} \in R^{n_1\times n_1}$, $a_{12} \in R^{n_1\times n_2}$, $a_{21} \in R^{n_2\times n_1}$, $a_{22} \in R^{n_2\times n_2}$, $b_1 \in R^{n_1\times m}$, $b_2 \in R^{n_2\times m}$, $d_1 \in R^{n_1\times k}$ and $d_2 \in R^{n_2\times k}$ are constant and $\varepsilon \geq 0$ is scalar and real. For a vector v, v^T is its transpose, and $\|v\|$ its Euclidean norm. For a matrix $M \in R^{p\times q}$, $\bar{\sigma}(M)$ will denote its largest singular value and L^2 is the Lebesque space of square integrable functions. The uncertainty term $\Delta_i(x_1,x_2,u)$ is assumed to be norm-bounded for some $\varepsilon_{1i} \geq 0$, $\varepsilon_{2i} \geq 0$ and $\varepsilon_{3i} \geq 0$ ($i = 1,2$), i.e.,

$$\|\Delta_i(x_1,x_2,u)\| \leq \varepsilon_{1i}\|x_1\| + \varepsilon_{2i}\|x_2\| + \varepsilon_{3i}\|u\| \quad \forall x_1 \in R^{n_1}, x_2 \in R^{n_2}, u \in R^m \tag{2}$$

Denote the corresponding uncertainty set by

$$\Xi_i(x_1,x_2,u) = \{\Delta_i(x_1,x_2,u) : \|\Delta_i(x_1,x_2,u)\| \leq \varepsilon_{1i}\|x_1\| + \varepsilon_{2i}\|x_2\| + \varepsilon_{3i}\|u\|\} \tag{3}$$

Definition 1

1) A state feedback $u = -k_1\, x_1 - k_2\, x_2$, $k_1 \in R^{m\times n_1}$, $k_2 \in R^{m\times n_2}$ is said to achieve robust global asymptotic stability if for $w = 0$ and any $\Delta_i(x_1,x_2,u) \in \Xi_i(x_1,x_2,u)$; $i = 1, 2$ the closed-loop system

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon \ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} - b_1 k_1 & a_{12} - b_1 k_2 \\ a_{21} - b_2 k_1 & a_{22} - b_2 k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \Delta_1(x_1, x_2, -k_1 x_1 - k_2 x_2) \\ \Delta_2(x_1, x_2, -k_1 x_1 - k_2 x_2) \end{bmatrix} \quad (4)$$

is globally asymptotically stable in the Lyapunov sense for all $\varepsilon \in (0, \infty)$.

2) A state feedback $u = -k_1 x_1 - k_2 x_2$ is said to achieve robust disturbance attenuation if under zero initial condition there exists $0 \le \gamma < \infty$ for which the performance bound is such that:

$$\| z(t) \| < \gamma \ \| w(t) \| \ \forall \ w \in L^2, \Delta_i(x_1, x_2, u) \in \Xi_i(x_1, x_2, u) \ for \ i = 1, 2 \quad (5)$$

The main objective of the paper is to design $k_1 \in R^{m \times n_1}$, $k_2 \in R^{m \times n_2}$ such that the state feedback $u = -k_1 x_1 - k_2 x_2$ achieves simultaneously robust global asymptotic stability and robust disturbance attenuation for $\varepsilon \in (0, \infty)$. The main approach employed here is the standard HJI method. Hence, we define a quadratic energy function in the form:

$$E(x_1, x_2) = x_1^T P_1 x_1 + \varepsilon \ x_2^T P_2 x_2 \quad (6)$$

where $P_1 > 0$, $P_2 > 0$ are to be determined. Define the Hamiltonian function

$$H[u, w, \Delta_1(x_1, x_2, u), \Delta_2(x_1, x_2, u)] = z^T z - \gamma^2 w^T w + \frac{dE}{dt} \quad (7)$$

where derivative of $E(t)$ is evaluated along the trajectory of the closed-loop system. It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality

$$H[u, w, \Delta_1(x_1, x_2, u), \Delta_2(x_1, x_2, u)] < 0, \quad \forall \ w \in L^2, \Delta_i(x_1, x_2, u) \in \Xi_i(x_1, x_2, u), \ i = 1, 2 \quad (8)$$

results in an $E(x)$ which is strictly radially unbounded [9], $E(x)$ may be regulated as a Lyapunov function for the closed-loop systems, and hence, robust stability is guaranteed for all $\varepsilon \in (0, \infty)$.

In this paper we will establish conditions under which

$$Inf_u \ Sup_{\Delta_i \in \Xi_i} \ Sup_{w \in L^2} H[u, w, \Delta_1, \Delta_2] < 0 \ \ for \ \ i = 1, 2 \quad (9)$$

$$such\,that \ \ \Delta_i := \Delta_i(x_1, x_2, u), \ \Xi_i := \Xi_i(x_1, x_2, u).$$

III. Main Results

Before deriving the main results, some preliminary lemmas are reviewed.

Lemma 1 [10]

For any matrices X and Y with appropriate dimensions and for any constant $\eta > 0$, we have:

$$X^T \ Y + Y^T \ X \le \eta \ X^T X + \frac{1}{\eta} \ Y^T Y. \quad (10)$$

Lemma 2 [9]

Let Ψ_i be the following linear uncertainty set

$$\Psi_i(x_1, x_2, u) := \{ \varepsilon_{1i} \ M_{1i} \ x_1 + \varepsilon_{2i} \ M_{2i} \ x_2 + \varepsilon_{3i} \ M_{3i} \ u : M_{1i} \in R^{n_i \times n_1}, \ M_{2i} \in R^{n_i \times n_2}, \ M_{3i} \in R^{n_i \times m}, \ \bar{\sigma} \ (M_{ji}) \le 1 \} \ for \ j = 1, 2, 3 \ and \ i = 1, 2 \quad (11)$$

then, we have:

$$\Xi_i(x_1, x_2, u) = \Psi_i(x_1, x_2, u) \ for \ i = 1, 2. \quad (12)$$

Moreover, the existence of M_{ji} is obvious by following the Lemma.

Lemma 3 [9]

For $n \ge m$, suppose $v \in R^n$ with $\| v \| = 1$ and $u \in R^m$ with $\| u \| = 1$. Then, there exists $M \in R^{m \times n}$ with $\bar{\sigma}(M) \le 1$ such that $v = M u$ and $M^T M = I_{n \times n}$.

One of the key technical contributions of this paper is utilization of Lemma 2 which establishes a representation of the nonlinear uncertainty set by a linear uncertainty set. This observation leads to Theorem 1, which is the main result of this paper. The approach employed here is the standard method of Riccati inequalities, which have been used, extensively in linear control for state-space systems [9].

Theorem 1

If there exists the set of positive numbers

$$\Theta = \{ \varepsilon_a, \bar{\varepsilon}_a, \varepsilon_b, \bar{\varepsilon}_b, \varepsilon_c, \ \bar{\varepsilon}_c, \varepsilon_d \} \quad (13)$$

and positive definite solutions $P_1 > 0, P_2 > 0$ to the Riccati inequalities

$$a_{11}^T P_1 + P_1 a_{11} + P_1 [(\varepsilon_{11} \varepsilon_a + \varepsilon_{21} \varepsilon_c + \varepsilon_{31} \bar{\varepsilon}_a + \varepsilon_d) I_{n_1 \times n_1} + \gamma^{-2} (1 + \varepsilon_d) d_1 d_1^T] P_1 + (1 + \varepsilon_d) c_1^T c_1 + \varepsilon_d \ a_{21}^T a_{21} + (\frac{\varepsilon_{11}}{\varepsilon_a} + \frac{\varepsilon_{12}}{\bar{\varepsilon}_c}) I_{n_1 \times n_1} < 0 \quad (14)$$

and

$$a_{22}^T P_2 + P_2 a_{22} + P_2 [(\varepsilon_{22} \varepsilon_b + \varepsilon_{12} \bar{\varepsilon}_c + \varepsilon_{32} \bar{\varepsilon}_b + \frac{1}{\varepsilon_d}) I_{n_2 \times n_2} + \gamma^{-2} (1 + \frac{1}{\varepsilon_d}) d_2 d_2^T] P_2 + (1 + \frac{1}{\varepsilon_d}) c_2^T c_2 + \frac{1}{\varepsilon_d} a_{12}^T a_{12} + (\frac{\varepsilon_{21}}{\varepsilon_c} + \frac{\varepsilon_{22}}{\varepsilon_b}) I_{n_2 \times n_2} < 0 \quad (15)$$

then, the control law

$$u(t) = -(D^T D + (\frac{\varepsilon_{32}}{\bar{\varepsilon}_b} + \frac{\varepsilon_{31}}{\bar{\varepsilon}_a}) I_{m \times m})^{-1} ((b_1^T P_1 + D^T c_1) x_1 + (b_2^T P_2 + D^T c_2) x_2) \quad (16)$$

achieves robust global asymptotic stability and robust disturbance attenuation in the sense of (4) and (5), respectively.

Proof:

We will prove the Theorem by showing that the control law (16) will guarantee the inequality of (8).

Noting to the expression (6) and according to (7), we have:

$$H(u,w,\Delta_1,\Delta_2)=x_1^T(c_1^Tc_1+a_{11}^TP_1+P_1a_{11})x_1+x_2^T(c_2^Tc_2+a_{22}^TP_2+P_2a_{22})x_2+x_1^T(c_1^Tc_2+P_1a_{12}+a_{21}^TP_2)x_2+x_2^T(c_2^Tc_1+a_{12}^TP_1+P_2a_{21})x_1+u^T(b_1^TP_1+D^Tc_1)x_1+u^T(b_2^TP_2+D^Tc_2)x_2+x_1^T(c_1^TD+P_1b_1)u+x_2^T(c_2^TD+P_2b_2)u+u^TD^TDu-\gamma^2w^Tw+w^T(d_1^TP_1x_1+d_2^TP_2x_2)+(x_1^TP_1d_1+x_2^TP_2d_2)w+\Delta_1^TP_1x_1+x_1^TP_1\Delta_1+\Delta_2^TP_2x_2+x_2^TP_2\Delta_2 \quad (17)$$

It is easy to show that the worst case disturbance occurs when

$$w^*=\gamma^{-2}(d_1^TP_1x_1+d_2^TP_2x_2). \quad (18)$$

It follows that

$$H_1(u,\Delta_1,\Delta_2)=Sup_{w\in L^2}H(u,w,\Delta_1,\Delta_2)=F_1(x_1)+F_2(x_2)+F_3(x_1,x_2)+F_3^T(x_1,x_2)+u^T(b_1^TP_1+D^Tc_1)x_1+x_1^T(P_1b_1+c_1^TD)u+u^T(D^Tc_2+b_2^TP_2)x_2+x_2^T(P_2b_2+c_2^TD)u+u^TD^TDu+x_1^TP_1\Delta_1+\Delta_1^TP_1x_1+x_2^TP_2\Delta_2+\Delta_2^TP_2x_2 \quad (19)$$

where

$$F_1(x_1)=x_1^T(a_{11}^TP_1+P_1a_{11}+c_1^Tc_1+\gamma^{-2}P_1d_1d_1^TP_1)x_1$$
$$F_2(x_2)=x_2^T(a_{22}^TP_2+P_2a_{22}+c_2^Tc_2+\gamma^{-2}P_2d_2d_2^TP_2)x_2$$
$$F_3(x_1,x_2)=x_1^T(a_{21}^TP_2+P_1a_{12}+c_1^Tc_2+\gamma^{-2}P_1d_1d_2^TP_2)x_2$$

According to Lemma 2, we have

$$Sup_{\Delta\in\Xi}\ H_1(u,\Delta_1,\Delta_2)=Sup_{\Delta\in\Psi}H_1(u,\Delta_1,\Delta_2)$$
$$such\ that\ \Delta=[\Delta_1\ \ \Delta_2]^T, \Xi=[\Xi_1\ \ \Xi_2]^T, \Psi=[\Psi_1\ \ \Psi_2]^T \quad (20)$$

Hence, we only need to consider

$$H_1(u,\Delta_1,\Delta_2)=F_1(x_1)+F_2(x_2)+F_3(x_1,x_2)+F_3^T(x_1,x_2)+F_4(x_1,x_2,u)+F_4^T(x_1,x_2,u)+F_5(x_1,x_2,u)+F_5^T(x_1,x_2,u)+F_6(x_1,x_2,u)+F_6^T(x_1,x_2,u)+u^TD^TDu \quad (21)$$

where

$$F_4(x_1,x_2,u)=(x_1^TP_1b_1+x_2^TP_2b_2+x_1^Tc_1^TD+x_2^Tc_2^TD)u$$
$$F_5(x_1,x_2,u)=(\varepsilon_{11}M_{11}x_1+\varepsilon_{21}M_{21}x_2+\varepsilon_{31}M_{31}u)^TP_1x_1$$
$$F_6(x_1,x_2,u)=(\varepsilon_{12}M_{12}x_1+\varepsilon_{22}M_{22}x_2+\varepsilon_{32}M_{32}u)^TP_2x_2$$

Now, by Lemma 1, it is trivial to show that for any member of (13) the following inequalities hold:

$$x_1^TP_1(\varepsilon_{11}M_{11}x_1)+(\varepsilon_{11}M_{11}x_1)^TP_1x_1\le\varepsilon_{11}\varepsilon_ax_1^TP_1^2x_1+\frac{\varepsilon_{11}}{\varepsilon_a}x_1^Tx_1$$
$$x_2^TP_2(\varepsilon_{22}M_{22}x_2)+(\varepsilon_{22}M_{22}x_2)^TP_2x_2\le\varepsilon_{22}\varepsilon_bx_2^TP_2^2x_2+\frac{\varepsilon_{22}}{\varepsilon_b}x_2^Tx_2$$
$$x_1^TP_1(\varepsilon_{21}M_{21}x_2)+(\varepsilon_{21}M_{21}x_2)^TP_1x_1\le\varepsilon_{21}\varepsilon_cx_1^TP_1^2x_1+\frac{\varepsilon_{21}}{\varepsilon_c}x_2^Tx_2$$
$$x_2^TP_2(\varepsilon_{12}M_{12}x_1)+(\varepsilon_{12}M_{12}x_1)^TP_2x_2\le\varepsilon_{12}\bar{\varepsilon}_cx_2^TP_2^2x_2+\frac{\varepsilon_{12}}{\bar{\varepsilon}_c}x_1^Tx_1$$
$$x_1^TP_1(\varepsilon_{31}M_{31}u)+(\varepsilon_{31}M_{31}u)^TP_1x_1\le\varepsilon_{31}\bar{\varepsilon}_ax_1^TP_1^2x_1+\frac{\varepsilon_{31}}{\bar{\varepsilon}_a}u^Tu$$
$$x_2^TP_2(\varepsilon_{32}M_{32}u)+(\varepsilon_{32}M_{32}u)^TP_2x_2\le\varepsilon_{32}\bar{\varepsilon}_bx_2^TP_2^2x_2+\frac{\varepsilon_{32}}{\bar{\varepsilon}_b}u^Tu$$

Consequently,

$$Sup_{\Delta\in\Xi}H_1(u,\Delta_1,\Delta_2)\le F_1(x_1)+F_2(x_2)+F_3(x_1,x_2)+F_3^T(x_1,x_2)+F_4(x_1,x_2,u)+F_4^T(x_1,x_2,u)+x_1^T[(\varepsilon_{11}\varepsilon_a+\varepsilon_{21}\varepsilon_c+\varepsilon_{31}\bar{\varepsilon}_a)P_1^2+(\frac{\varepsilon_{11}}{\varepsilon_a}+\frac{\varepsilon_{12}}{\bar{\varepsilon}_c})I_{n1\times n1}]x_1+x_2^T[(\varepsilon_{22}\varepsilon_b+\varepsilon_{12}\bar{\varepsilon}_c+\varepsilon_{32}\bar{\varepsilon}_b)P_2^2+(\frac{\varepsilon_{21}}{\varepsilon_c}+\frac{\varepsilon_{22}}{\varepsilon_b})I_{n2\times n2}]x_2+u^T[D^TD+(\frac{\varepsilon_{32}}{\bar{\varepsilon}_b}+\frac{\varepsilon_3}{\bar{\varepsilon}_a})I_{m\times m}]u \quad (22)$$

The optimal control law, which minimizes the right-hand side of (22), is given by

$$u(t)=-(D^TD+(\frac{\varepsilon_{32}}{\bar{\varepsilon}_b}+\frac{\varepsilon_{31}}{\bar{\varepsilon}_a})I_{m\times m})^{-1}((b_1^TP_1+D^Tc_1)x_1+(b_2^TP_2+D^Tc_2)x_2).$$

As a result, we have:

$$Inf_u\ Sup_{\Delta\in\Xi}\ H_1(u,\Delta_1,\Delta_2)\le x_1^TR_1x_1+x_2^TR_2x_2+F_3(x_1,x_2)+F_3^T(x_1,x_2)-S(x_1,x_2) \quad (23)$$

where

$$R_1=a_{11}^TP_1+P_1a_{11}+P_1[(\varepsilon_{11}\varepsilon_a+\varepsilon_{21}\varepsilon_c+\varepsilon_{31}\bar{\varepsilon}_a)I_{n1\times n1}+\gamma^{-2}d_1d_1^T]P_1+c_1^Tc_1+(\frac{\varepsilon_{11}}{\varepsilon_a}+\frac{\varepsilon_{12}}{\bar{\varepsilon}_c})I_{n1\times n1}$$
$$R_2=a_{22}^TP_2+P_2a_{22}+P_2[(\varepsilon_{22}\varepsilon_b+\varepsilon_{12}\bar{\varepsilon}_c+\varepsilon_{32}\bar{\varepsilon}_b)I_{n2\times n2}+\gamma^{-2}d_2d_2^T]P_1+c_2^Tc_2+(\frac{\varepsilon_{21}}{\varepsilon}+\frac{\varepsilon_{22}}{\varepsilon_b})I_{n2\times n2}$$

$$S(x_1,x_2)=L^T(x_1,x_2)\,(D^TD+(\frac{\varepsilon_{32}}{\bar{\varepsilon}_b}+\frac{\varepsilon_{31}}{\bar{\varepsilon}_a})I_{m\times m})^{-1}\times L(x_1,x_2)$$

$$L(x_1,x_2)=(b_1^TP_1+F^Tc_1)x_1+(b_2^TP_2+F^Tc_2)x_2.$$

By Lemma 1, it is clear that

$$F_3(x_1,x_2)+F_3^T(x_1,x_2)\le\varepsilon_d\,x_1^T(a_{21}^T\,a_{21}+c_1^Tc_1+P_1^2+\gamma^{-2}P_1\,d_1\,d_1^TP_1)x_1+\frac{1}{\varepsilon_d}x_2^T(a_{12}^T\,a_{12}+c_2^T\,c_2+P_2^2+\gamma^{-2}P_2\,d_2\,d_2^T\,P_2)x_2 \quad (24)$$

From (23), (24) and by noting that the expression of $S(x_1,x_2)$ is positive definite, we have

$$Inf_{\,u}\ Sup_{\Delta\in\Xi}\ H(u,\Delta_1,\Delta_2)\le x_1^T\,\widetilde{R}_1\,x_1+x_2^T\,\widetilde{R}_2\,x_2 \quad (25)$$

where

$$\widetilde{R}_1=a_{11}^T\,P_1+P_1\,a_{11}+P_1[(\varepsilon_{11}\varepsilon_a+\varepsilon_{21}\varepsilon_c+\varepsilon_{31}\bar{\varepsilon}_a+\varepsilon_d)I_{n_1\times n_1}+\gamma^{-2}(1+\varepsilon_d)d_1d_1^T]P_1+(1+\varepsilon_d)c_1^Tc_1+\varepsilon_d\,a_{21}^T\,a_{21}+(\frac{\varepsilon_{11}}{\varepsilon_a}+\frac{\varepsilon_{12}}{\bar{\varepsilon}_c})I_{n_1\times n_1}$$

and

$$\widetilde{R}_2=a_{22}^T\,P_2+P_2\,a_{22}+P_2[(\varepsilon_{22}\varepsilon_b+\varepsilon_{12}\bar{\varepsilon}_c+\varepsilon_{32}\bar{\varepsilon}_b+\frac{1}{\varepsilon_d})I_{n_2\times n_2}+\gamma^{-2}(1+\frac{1}{\varepsilon_d})d_2\,d_2^T]P_2+(1+\frac{1}{\varepsilon_d})c_2^Tc_2+\frac{1}{\varepsilon_d}a_{12}^T\,a_{12}+(\frac{\varepsilon_{21}}{\varepsilon_c}+\frac{\varepsilon_{22}}{\varepsilon_b})I_{n_2\times n_2}$$

Consequently, if there exists positive definite solutions to the Riccati inequalities

$$\widetilde{R}_1<0\ and\ \widetilde{R}_2<0$$

then we have

$$H[u,w,\Delta_1(x_1,x_2,u),\Delta_2(x_1,x_2,u)]<0,\quad \forall\ w\in L^2,\ \Delta_i(x_1,x_2,u)\in\Xi_i(x_1,x_2,u),\ i=1,2 \quad (26)$$

Hence, (26) completes the proof.

IV. Example

Consider a fourth-order singularly perturbed system [11]

$$\begin{bmatrix}\dot{x}_{s1}\\ \dot{x}_{s2}\\ \varepsilon\dot{x}_{f1}\\ \varepsilon\dot{x}_{f2}\end{bmatrix}=\begin{bmatrix}-9&0&0&0.1\\0.1&-8&0.05&0.1\\0&0&-15&0\\0.01&0.003&0&-1\end{bmatrix}\begin{bmatrix}x_{s1}\\x_{s2}\\x_{f1}\\x_{f2}\end{bmatrix}+\begin{bmatrix}1&0\\0&1\\1&0.5\\0.5&0\end{bmatrix}u(t)+\begin{bmatrix}0.1\\0.2\\0.2\\0.5\end{bmatrix}w(t)+\begin{bmatrix}\Delta_1(x_1,x_2,u)\\ \Delta_2(x_1,x_2,u)\end{bmatrix}$$

$$z(t)=[0.4\quad 0.15]\begin{bmatrix}x_{s1}\\x_{s2}\end{bmatrix}+[0.2\quad -0.2]\begin{bmatrix}x_{f1}\\x_{f2}\end{bmatrix}+[0.25\quad 0.5]u(t) \quad (27)$$

where $x_1=[x_{s1}\quad x_{s2}]^T$, $x_2=[x_{f1}\quad x_{f2}]^T$ and the uncertainty terms $\Delta_i(x_1,x_2,u)$; i =1, 2, are assumed to be norm bounded with $\varepsilon_{ji}=0.01$ (for $j=1,2,3$ and $i=1,2$), consider also $\gamma=0.1$ as the performance bound and $\varepsilon=0.1$ as the perturbed parameter. From (14), (15), we can choose the positive definite solutions $P_1>0$, $P_2>0$ as follows:

$$P_1=\begin{bmatrix}0.1845&0.0411\\0.0411&0.0750\end{bmatrix},\quad P_2=\begin{bmatrix}0.0377&-0.0015\\-0.0015&0.0477\end{bmatrix}.$$

Also, the set of positive numbers of Θ are obtained as follows: $\varepsilon_a=\bar{\varepsilon}_a=\varepsilon_b=\bar{\varepsilon}_b=\varepsilon_c=\bar{\varepsilon}_c=\varepsilon_d=1.0047$.

The required state feedback control law is given by

$$u=-k_1x_1-k_2x_2$$

$$k_1=\begin{bmatrix}7.0509&0.3724\\-2.3722&0.3832\end{bmatrix},\ k_2=\begin{bmatrix}1.3006&0.8517\\-0.1621&-0.5824\end{bmatrix}$$

Robust stability and disturbance attenuation of the slow and fast dynamics in the presence of disturbance (Gussian noise) have been depicted in Figures 1 and 2. Therefore, we conclude that system (27) can be stabilized by the control law (16) for all $\varepsilon\in(0,\infty)$ which has been depicted in Figure 3 and the correctness of the attenuation level has been depicted in Figure 4.

V. Conclusion

This paper has three major contributions for the robust stability and robust disturbance attenuation of the linear singularly perturbed systems with norm-bounded nonlinear uncertainties. One is that the type of norm-bounded nonlinear uncertainties considered in this class of systems coincides with a set of linear uncertainties by utilization of Lemma 2. The other is that the state feedback gain matrices can be determined independently from two certain Riccati inequalities, and the last is that the closed-loop system is stable for all $\varepsilon\in(0,\infty)$. In this paper, the results are presented on the two-time-scale case, and the extension of results to multiple-time-scale is a topic currently under study.

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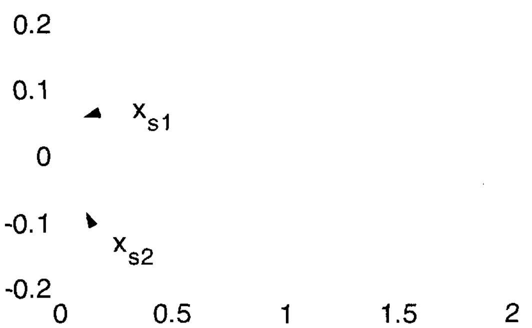

Fig. 1. Robust stability and disturbance attenuation of slow dynamics by means of state feedback.

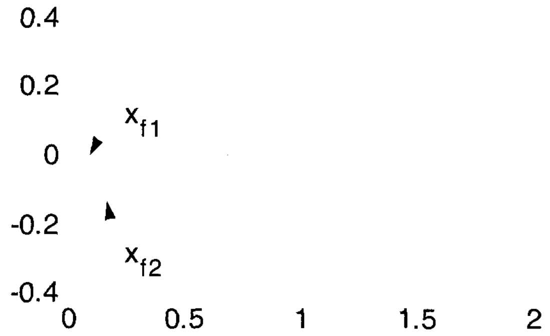

Fig. 2. Robust stability and disturbance attenuation of fast dynamics by means of state feedback.

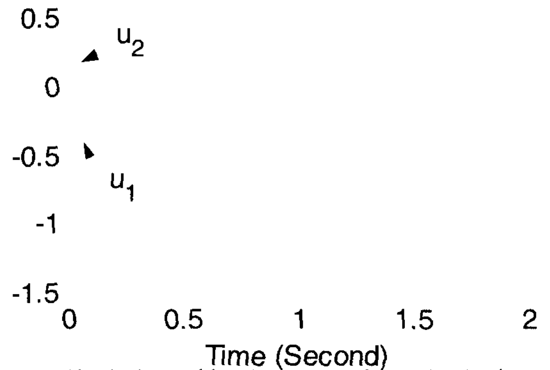

Fig. 3. Control law by means of state feedback.

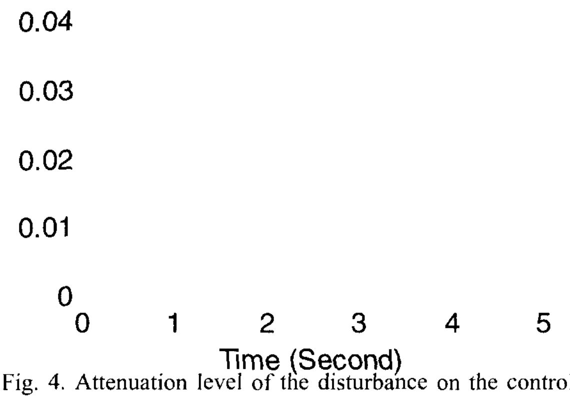

Fig. 4. Attenuation level of the disturbance on the controlled output.

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