Mode Decomposition of Three-Dimensional Mixed-Mode Cracks using the Solution for Penny-Shaped Crack

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ABSTRACT

A simple and convenient method of analysis for obtaining the individual stress intensity factors in a three-dimensional mixed mode crack is proposed. The procedures presented here are based on the path independence of J integral and mutual or two-state conservation integral, which involves two elastic fields. The problem is reduced to the determination of mixed mode stress intensity factor solutions in terms of conservation integrals involving known auxiliary solutions. Some numerical examples are presented to investigate the effectiveness and applicability of the method for a three-dimensional penny-shaped crack problem under mixed mode. This procedure is applicable to a three-dimensional mixed mode curved crack.

Key Words : Stress Intensity Factor, Conservation Integral, Mixed Mode Crack, Three-Dimensional Penny-Shaped Crack

1. Introduction

Conservation laws in elasticity have received much attention due to their wide range of application to the fracture mechanics. The property of path independency of J is the consequence of a conservation law as well as of the stress free condition on the crack surface. It is necessary to evaluate the individual stress intensity factors separately for mixed mode crack problems in order to investigate the crack growth and the crack propagation. However, the evaluation of the J integral alone does not determine the individual stress intensity factors, K_I , K_{II} and K_{III} separately.

Many works on mixed mode crack problems using the path-independent integrals have been reported. Bui[2] developed a technique using the J integrals associated with Mode I and Mode II, in which the symmetric and antisymmetric parts of the planar displacement, strain and stress fields around the crack plane are separated. Stern et al.[17] employed another conservation integral based on Betti's reciprocal work theorem with known auxiliary fields. Later the method was extended for the straight interfacial cracks by Hong and Stern[5]. For planar cracks, Yau et al.[20], Wang and Yau[18], Matos et al.[12] and Shih and Asaro[16] utilized a new class of conservation integral known as mutual integral or two-state conservation integral, proposed by Chen and Shield[3]: Nakamura and Park[24] and Nakamura[25] employed the same approach to determine the mixed-mode stress intensity factors for three-dimensional interface crack problems. Nikishkov and Atluri[26] developed a domain integral approach to calculate mixed-mode stress intensity factors for planar three-dimensional cracks. Nahta and Moran[14] and Gosz et al.[23] used asymptotic auxiliary fields for the plane problems to decompose the stress intensity factors in mixed-mode cracks, and proposed a method to evaluate the divergence term in the two-state integral. Kim[28] used numerical auxiliary fields to decompose the stress intensity factors in mixed-mode cracks. Choi and Earmme[4] employed the two-state L-integral to evaluate stress intensity factors in circular arc-shaped interfacial crack. Recently Im and Kim[8] showed that the two-state M-integral is applicable for computing the intensity of the singular near-tip field for a generic isotropic composite wedge including planar cracks. The main interest of fracture mechanics is shifting from two to three-dimensional crack problems and particularly it becomes increasingly important to study three-dimensional mixed mode crack problems for the crack growth and the crack propagation prediction.

This study presents a method to obtain the individual stress intensity factors in case of three-dimensional cracks in mixed mode using J and the associated two-state integral. The method is based on the path-independence of J and two-state J integral which involves two independent elastic fields. The path-independence of these conservation integrals enables one to obtain each stress intensity factor from the displacements and stresses remote from the crack tip. In this paper, we present a simple method to evaluate the two-state integral with exact auxiliary fields for penny-shaped crack problems. The purpose of this study is to show the validity and the effectiveness of the present method for mode decomposition utilizing the exact auxiliary fields for penny-shaped crack problems. In the next section the basic formulation for the method are described and the solution procedure is established. The implementation of the method is explained in Section 3. In Section 4 numerical examples are carried out in order to demonstrate the usefulness of the method.

2. Formulation of the problems

In three-dimensional infinitesimal deformation of homogeneous isotropic elastic bodies the J integral is given as [9]

$$J_k = \int_A \left(Wn_k - T_i u_{i,k}\right) dA \tag{1}$$

where A indicates a closed surface. Note that the pointwise value of J along the crack front is required for mode decomposition. According to Moran and Shih [13], $J(s)$ is given as

$$J(s) = \frac{-\lim_{\Gamma \to 0} \int_{S_t} l_k H_{kj} m_j d\Gamma}{\int_{L_c} l_k v_k ds} \tag{2}$$

where s denotes the coordinate along the crack

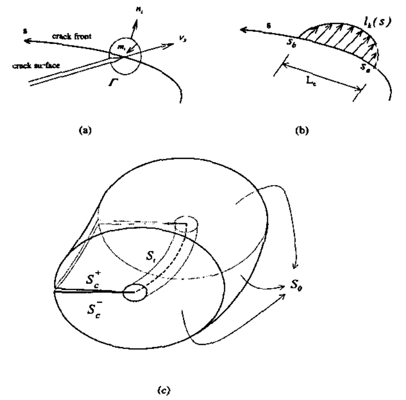

Fig. 1 (a) Conventions at curvilinear crack front.
(b) Virtual crack advance between s_a and s_b
(c) Inner tubular surface S_t and outer arbitrary surface S_0.

front of a three-dimensional crack; $H_{kj} = W\delta_{kj} - \sigma_{ij} u_{i,k}$, l_k, which is short for $l_k(s)$, indicates the component of crack advance vector; v_k, which is short for $v_k(s)$, represents an outward unit vector normal to crack front on the crack plane and m_j is the normal to Γ pointing towards the crack front. i.e. $m_j = -n_j$ on Γ as shown in Fig. 1a. Furthermore, S_t is an inner tubular surface and L_c is a small interval on the line of the crack front for virtual crack extension.

As the crack tip is approached in a three-dimensional crack, asymptotically the plain strain state prevails [15] and therefore we can use the following relationships for the three-dimensional mixed mode crack

$$J_I = \alpha K_I^2, \quad J_{II} = \alpha K_{II}^2, \quad J_{III} = \frac{\alpha}{1-\nu} K_{III}^2,$$
$$J = J_I + J_{II} + J_{III} \tag{3}$$

where $\alpha = \frac{1-\nu^2}{E}$ (E: Young's modulus, ν: Poisson's ratio)

The J integral in equations (2) and (3) alone does not provide adequate information for determining the individual stress intensity factors K_I, K_{II} and K_{III} in a mixed mode crack problem. Another information may be obtained from the two-state J-integral [3]. Consider two independent elastic states of a penny-shaped crack in an elastic medium, each denoted by superscript (1) and (2). Let the equilibrium state from the superposition of the two states be denoted by superscript (0). Then the J integral for the superimposed state is obtained in the following form:

$$J^{(0)} = J^{(1)} + J^{(2)} + J^{(1,2)} \quad (4)$$

in which $J^{(k)}(k=1,2)$ is defined by equation (2), $J^{(1,2)}$ is given as

$$J^{(1,2)} = -\lim_{\Gamma\to 0}\int_{\Gamma(s)} l_k H_{kj}^{(1,2)} m_j d\Gamma / \int_{L_c} l_k v_k ds \quad (5)$$

with $H_{kj}^{(1,2)} = W^{(1,2)}\delta_{kj} - \left(\sigma_{ij}^{(1)} u_{i,k}^{(2)} + \sigma_{ij}^{(2)} u_{i,k}^{(1)}\right)$ (6a)

and $W^{(1,2)} = C_{ijkl} u_{i,j}^{(1)} u_{k,l}^{(2)} = C_{ijkl} u_{i,j}^{(2)} u_{k,l}^{(1)}$ (6b)

From equation (3), one finds that the J integral for the elastic state (0) may be written as

$$J^{(0)} = J^{(1)} + J^{(2)} + 2\alpha\left(K_I^{(1)}K_I^{(2)} + K_{II}^{(1)}K_{II}^{(2)} + \frac{1}{1-\nu}K_{III}^{(1)}K_{III}^{(2)}\right) \quad (7)$$

and comparison to equation (4) yields

$$J^{(1,2)} = 2\alpha\left(K_I^{(1)}K_I^{(2)} + K_{II}^{(1)}K_{II}^{(2)} + \frac{1}{1-\nu}K_{III}^{(1)}K_{III}^{(2)}\right) \quad (8)$$

Equation (5) together with equation (8) provides, in fact, sufficient information for determining the individual stress intensity factors for a mixed mode fracture problem when two known auxiliary solutions of three modes are introduced.

Let the first auxiliary solutions be denoted by a superscript (2a). The auxiliary state (2a) is chosen to be a pure mode I state, i.e.,

$$K_I^{(2a)} \neq 0,\ K_{II}^{(2a)} = 0, \text{ and } K_{III}^{(2a)} = 0 \quad (9)$$

Equation (7) can be simplified as

$$K_I^{(1)} = \frac{J^{(1,2a)}}{2\sqrt{\alpha J^{(2a)}}} \quad \text{and} \quad J_I^{(1)} = \frac{\left[J_I^{(1,2a)}\right]^2}{4J_I^{(2a)}} \quad (10)$$

where $J^{(2a)}$ and $J^{(1,2a)}$ can be obtained by equations (2) and (5).

In equation (5), $\sigma_{ij}^{(1)}$ and $u_i^{(1)}$ can be determined by any convenient method along a properly selected integration domain, which will be discussed later. In a penny-shaped crack problem subjected to the loading condition of equation (9), $\sigma_{ij}^{(2a)}$ and $u_i^{(2a)}$ can be calculated from the following exact expressions in case of pure mode I.

$$2\mu u_r^{(2a)} = \frac{2p_0}{\pi}a\left[(1-2\nu)(C_2^1 - S_1^1) - z(C_3^1 - S_2^1)\right] - \frac{p_0\nu r a}{1+\nu} \quad (11a)$$

$$2\mu u_z^{(2a)} = \frac{2p_0}{\pi}a\left[2(1-\nu)(C_2^0 - S_1^0) + z(C_3^0 - S_2^0)\right] + \frac{p_0 z a}{1+\nu} \quad (11b)$$

$$\sigma_r^{(2a)} = \frac{2p_0}{\pi}\left[\begin{array}{l} C_2^0 - S_1^0 - z(C_3^0 - S_2^0) - \\ \frac{(1-2\nu)}{r}(C_1^1 - S_0^1) + \frac{z}{r}(C_2^1 - S_1^1) \end{array}\right] \quad (11c)$$

$$\sigma_z^{(2a)} = \frac{2p_0}{\pi}\left[C_2^0 - S_1^0 + z(C_3^0 - S_2^0)\right] + p_0 \quad (11d)$$

$$\sigma_{rz}^{(2a)} = \frac{2p_0}{\pi} z\left[C_3^1 - S_2^1\right] \quad (11e)$$

The details are explained in Appendix A.

The second auxiliary solution denoted by the superscript, 2b, for pure mode-III deformation of the cracked body is also introduced so that

$$K_I^{(2b)} = 0,\ K_{II}^{(2b)} = 0, \text{ and } K_{III}^{(2b)} \neq 0 \quad (12)$$

This gives

$$K_{III}^{(1)} = \sqrt{\frac{E}{1+\nu}}\frac{J^{(1,2b)}}{2\sqrt{J^{(2b)}}} \quad \text{and} \quad J_{III}^{(1)} = \frac{\left[J^{(1,2b)}\right]^2}{4J^{(2b)}} \quad (13)$$

The quantities $\sigma_{ij}^{(2b)}$ and $u_i^{(2b)}$ can be calculated from the following exact expressions for a penny-shaped crack in case of pure mode III.

$$u_\theta^{(2b)} = \frac{\tau_0}{\mu}az + \frac{\tau_0}{\mu}a(I_0^1 - S_1^1/2) \quad (14a)$$

$$\sigma_{\theta z}^{(2b)} = \tau_0\left\{1 - (Z_1^1 - C_1^1 - S_2^1/2)\right\} \quad (14b)$$

$$\sigma_{r\theta}^{(2b)} = \tau_0\left\{-\frac{2}{r}(IC_0^1 - S_1^1) + IC_1^0 - S_2^0/2\right\} \quad (14c)$$

The details are explained in Appendix A.

Equations (10) and (13) provide sufficient information to determine $K_I^{(1)}$ and $K_{III}^{(1)}$ directly. The mode II stress intensity factor can be calculated by equation (3) and above $K_I^{(1)}$ and $K_{III}^{(1)}$ as follows

$$K_{II}^{(1)} = \frac{1}{\sqrt{\alpha}}\sqrt{J^{(1)} - \alpha K_I^{(1)2} - \alpha\frac{1}{1-\nu}K_{III}^{(1)2}} \qquad (15)$$

It should be noted that, in solving for $K_I^{(1)}$, $K_{II}^{(1)}$ and $K_{III}^{(1)}$, the integrals $J^{(1)}$, $J^{(1,2a)}$ and $J^{(1,2b)}$ have to be evaluated accurately and explicitly. Note that the path-independent integrals may be calculated accurately, transforming equations (2) and (5) into the domain integral expression [11, 14], as will be shown in the next section.

3. Finite Element Implementation

The values of $J(s)$ and the two-state integral along a three-dimensional crack front are given by the limiting contour integral as seen in equations (2) and (5). Following Moran and Shih[13], we can show that the domain integral representation is obtained as

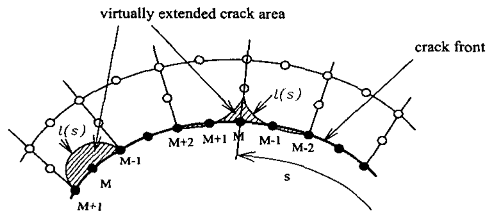

Fig. 2 Local advance of the crack front for 20-node 3D element.

$$J(s) = \frac{-\int_V H_{kj} q_{k,j} dV}{\int_{L_c} l_k(s)\nu_k(s)ds} \qquad (16)$$

where "L_c" is the crack front line from S_a to S_b (see Fig. 1b); the weight function "q_k" is smooth enough for the indicated operations to be carried out and is defined as follows

$$q_k = \begin{cases} l_k & \text{on } S_t \\ 0 & \text{on } S_0 \\ \text{othogonal to } m_i & \text{on } S_c^+ \text{ and } S_c^- \\ \text{otherwise arbitrary.} & \end{cases} \qquad (17)$$

where S_t, S_0, S_c^+, S_c^-, and m_i are indicated in Fig. 1.

In a three-dimensional analysis the virtual crack extension has to be applied to a single node point on the crack front for evaluating the local value of the energy release rate. For the 20-node three-dimensional isoparametric element, a new crack front is defined by the quadratic interpolation function as shown in Fig. 2. The volume V is identified with the collection of elements which contain the line L_c. Thus, in the finite element framework, for considering the increase in cracked area due to the shift of a given particular node M we take L_c to be the line connecting the nodes M-1, M and M+1 for mid nodes and the nodes M-2, M-1, M M+1 and M+2 for corner nodes as shown in Fig. 2. Then, the local crack front advance is therefore taken as follows [11]

$$l = \sum_{k=1}^{20} N^k Q^k \qquad (18)$$

where N^k is the triquadratic shape function and Q^k is the nodal values for the k-th node. Note that $Q^k = 0$ if the k-th node is on S_0. For nodes inside V, Q^k is given by interpolation between the nodal values on L_c and S_0.

4. Numerical examples and discussion

The procedures just outlined have been programmed for studying a three-dimensional penny-shaped crack under mixed mode in isotropic. solids. The numerical calculations for mixed mode crack problems are carried out with the finite element code ABAQUS[22]. This code supplies the required displacements and stresses at the Gaussian points to a separate program developed for evaluating J-integral and two-state conservation integrals.

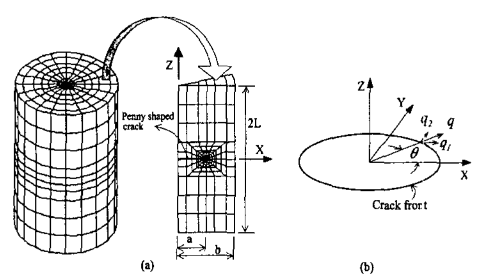

Fig. 3 (a) Three-dimensional finite element model for penny-shaped crack under nonaxsymmetric loading

(b) q definition at node M.

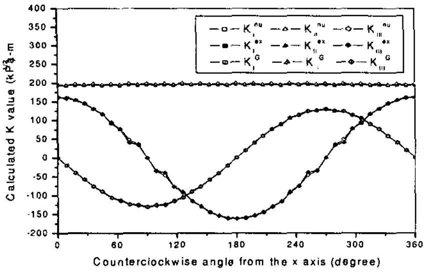

Fig. 4 Comparison of stress intensity factors with the results of other methods

To examine the accuracy and the convergence of the decomposition method using auxiliary fields of the exact solutions for a penny-shaped crack we first compare the results from the decomposition method for three-dimensional mixed mode cracks with those from the decomposition method using numerical auxiliary fields proposed by Kim[28]. The penny-shaped crack embedded in a cylinder under fields. As shown in Fig. 3 we take the same material and geometry as those in Kim[28] ($E = 2.069\times10^{11}\ N/m^2$, ν =0.3, $a = 0.0254\,m$, $b = 0.0508\,m$ and $2L = 0.152\,m$). The bottom surface is fixed not to move in any direction, and the displacement $u_x = 2.54\times10^{-6}\ m$ and $u_z = 7.62\times10^{-7}\ m$ are applied on the top surface. The stress intensity factors obtained using numerical auxiliary fields are taken as reference solution to compare with the results obtained using the exact auxiliary fields. The three stress intensity factors K_I , K_{II} , and K_{III} are obtained from the two-state integrals with two exact auxiliary fields for Mode I and Mode III, respectively. The stress intensity factors calculated in the areas containing annular rings for domain integral (16) along the crack front show path independence with respect to the integration paths or rings. In Fig. 4, the average values of the stress intensity factors are plotted in terms of the angle from x axis for three solutions: the stress intensity factors marked by "nu" is the reference solution calculated using the numerical solution being taken for the auxiliary field; the stress intensity factors marked by "ex" is the solution with using auxiliary fields of the exact solutions for a penny-shaped crack; the stress intensity factors marked by "G" is the solution with Gosz method[23]. The results for the stress intensity factors decomposed by the present method are found to be in an excellent agreement with the reference solutions obtained with the numerical auxiliary fields and Gosz method.

Considering the fact that the present method provides a more straightforward scheme for obtaining the individual stress intensity factors in a three-dimensional mixed mode crack, it may be more attractive in view of numerical computation. mixed mode is adopted to study the mode decomposition using exact auxiliary

5. Conclusions

A method of analysis, based on the conservation laws of elasticity and the fundamental relationships in fracture mechanics, has been proposed for studying three-dimensional mixed mode crack problems. The method is based on the path-independence of J integral and two-state integral. Path-independence of the J and two-state integrals enables us to compute the individual stress intensity factors accurately and effectively from the domain integral expression.

The exact auxiliary fields for the penny-shaped crack are successfully implemented to decompose three-dimensional mixed-mode crack. The solution procedure has been established and shown to be computationally efficient and operationally simple, involving only the choice of appropriate auxiliary solutions and the calculation of the J and two-state integrals with the aid of the domain integral expression. Furthermore, this procedure is applicable to a three-dimensional mixed mode arbitrary curved crack.

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Appendix A

Solution of Penny-Shaped Crack in an infinite body

The basic solution used for the infinite body containing a penny-shaped crack is obtained as follows;

A characteristic length is taken to be a, and all length coordinates are expressed as the product of a and a dimensionless coordinate. The cylindrical coordinates (ar, θ, az) involve the dimensionless variables r and z.

(a) Mode I

The displacement u satisfy the Navier's equation of equilibrium:

$$\frac{1}{1-2\nu}\nabla e+\nabla^2 u=0,\ e=\nabla^2 u \tag{A1}$$

where ν is the Poisson ratio and e denotes the dilatation. General solution of (A1) for an axisymmetric problem is expressed in terms of two harmonic functions as

$$2\mu u_r(r,z)=z\frac{\partial\phi}{\partial r}+\frac{\partial\chi}{\partial r} \tag{A2a}$$

$$2\mu u_z(r,z)=z\frac{\partial\phi}{\partial z}-(3-4\nu)\phi+\frac{\partial\chi}{\partial z} \tag{A2b}$$

where is μ shear modulus and ϕ and χ are the harmonic functions.

The corresponding stress components are as follows;

$$\sigma_r(r,z)=z\frac{\partial^2\phi}{\partial r^2}-2\nu\frac{\partial\phi}{\partial z}+\frac{\partial^2\chi}{\partial r^2} \tag{A3a}$$

$$\sigma_\theta(r,z)=-z\frac{\partial^2\phi}{\partial r^2}-z\frac{\partial^2\phi}{\partial z^2}-2\nu\frac{\partial\phi}{\partial z}-\frac{\partial^2\chi}{\partial r^2}-\frac{\partial^2\chi}{\partial z^2} \tag{A3b}$$

$$\sigma_z(r,z)=z\frac{\partial^2\phi}{\partial z^2}-2(1-\nu)\frac{\partial\phi}{\partial z}+\frac{\partial^2\chi}{\partial z^2} \tag{A3c}$$

$$\sigma_{rz}(r,z)=z\frac{\partial^2\phi}{\partial r\partial z}-(1-2\nu)\frac{\partial\phi}{\partial r}+\frac{\partial^2\chi}{\partial r\partial z} \tag{A3d}$$

The boundary conditions on the penny-shaped crack surface is

$$\sigma_z(r,0)=p_0,\quad 0\le r<1 \tag{A4}$$

The continuity of the tractions and displacements requires in the plane z=0, r>1,

$$\begin{aligned} u_r(r,0^+)&=u_r(r,0^-)\\ u_z(r,0^+)&=u_z(r,0^-)\\ \sigma_{rz}(r,0^+)&=\sigma_{rz}(r,0^-)\\ \sigma_z(r,0^+)&=\sigma_z(r,0^-)\end{aligned} \tag{A5}$$

The appropriate form of the two harmonic functions ϕ and χ are obtained from Fourier-Hankel relations as

$$\phi(r,z)=\int_0^\infty\frac{A(s)}{s}e^{-zs}J_0(rs)ds \tag{A6a}$$

$$\chi(r,z)=\int_0^\infty\frac{B(s)}{s}e^{-zs}J_0(rs)ds \tag{A6b}$$

where J_0 denotes the Bessel function of order 0 and $A(s)$, and $B(s)$ are to be determined so as to satisfy the boundary conditions.

The solution for a penny-shaped crack with normal loading

$$\begin{aligned} A(s)&=-sD_1(s)\\ B(s)&=(1-2\nu)D_1(s)\end{aligned} \tag{A7}$$

where $D_1(s)=\sqrt{\frac{2s}{\pi}}\int_0^\infty J_{1/2}(st)\sqrt{t}\left\{\int_0^t\frac{rp_1(r)}{\sqrt{t^2-r^2}}dr\right\}dt$

The normal loading $p_1(r)=-p_0$ is applied on crack surface and using the relations

$$\begin{aligned} J_{1/2}(st)&=\sqrt{\frac{2}{\pi st}}\sin(st)\\ \int_0^t\frac{r(-p_0)}{\sqrt{t^2-r^2}}dr&=-p_0t\end{aligned} \tag{A8}$$

the following result is obtained[31].

$$D_1(s)=-\frac{2p_0}{\pi s^2}\{\sin(s)-s\cos(s)\} \tag{A9}$$

The required solution of the stress free condition on the crack surface can be obtained by the superposition of the above solution and the solution for simple tension.

Considering the superposition and the dimensionless variable a , the resultant displacement and stress components are as follows

$$2\mu u_r = -a\int_0^\infty D_1(s)(1-2\nu-zs)e^{-zs}J_1(rs)ds - \frac{p_0\nu ra}{1+\nu} \tag{A10a}$$
$$= \frac{2p_0}{\pi}a\left[(1-2\nu)(C_2^1 - S_1^1) - z(C_3^1 - S_2^1)\right] - \frac{p_0\nu ra}{1+\nu}$$

$$2\mu u_z = -a\int_0^\infty D_1(s)\{2(1-\nu)+zs\}e^{-zs}J_0(rs)ds + \frac{p_0 za}{1+\nu} \tag{A10b}$$
$$= \frac{2p_0}{\pi}a\left[2(1-\nu)(C_2^0 - S_1^0) + z(C_3^0 - S_2^0)\right] + \frac{p_0 za}{1+\nu}$$

$$\sigma_r = -\int_0^\infty sD_1(s)\left\{(1-zs)J_0(rs) - (1-2\nu-zs)\frac{J_1(rs)}{rs}\right\}e^{-zs}ds \tag{A10c}$$
$$= \frac{2p_0}{\pi}\left[C_2^0 - S_1^0 - z(C_3^0 - S_2^0) - \frac{(1-2\nu)}{r}(C_1^1 - S_0^1) + \frac{z}{r}(C_2^1 - S_1^1)\right]$$

$$\sigma_z = -\int_0^\infty sD_1(s)(1+zs)e^{-zs}J_0(rs)ds + p_0 \tag{A10c}$$
$$= \frac{2p_0}{\pi}\left[C_2^0 - S_1^0 + z(C_3^0 - S_2^0)\right] + p_0$$

$$\sigma_{rz} = -\int_0^\infty s^2 D_1(s)ze^{-zs}J_1(rs)ds \tag{A10d}$$
$$= \frac{2p_0}{\pi}\left[C_3^1 - S_2^1\right]z$$

where

$$C_n^m = \int_0^\infty s^{n-2}\cos(s)e^{-zs}J_m(rs)ds$$
$$S_n^m = \int_0^\infty s^{n-2}\sin(s)e^{-zs}J_m(rs)ds$$
$$\rho^2 = 1+z^2, \quad R^2 = (r^2+z^2-1)+4z^2$$
$$\theta = \arctan\left(\frac{1}{z}\right), \quad \phi = \arctan\left(\frac{2z}{r^2+z^2-1}\right)$$
$$C_2^0 = R^{-1/2}\cos(\phi/2), \quad S_2^0 = R^{-1/2}\sin(\phi/2)$$
$$C_3^0 = \rho R^{-3/2}\cos(3\phi/2-\theta), \quad S_3^0 = \rho R^{-3/2}\sin(3\phi/2-\theta)$$
$$C_1^1 = \frac{1}{r}\left(R^{1/2}\cos(\phi/2) - z\right), \quad S_1^1 = \frac{1}{r}\left(1 - R^{1/2}\sin(\phi/2)\right)$$
$$C_2^1 = \frac{1}{r} - \frac{\rho}{r}R^{-1/2}\cos(\theta-\phi/2), \quad S_2^1 = \frac{\rho}{r}R^{-1/2}\sin(\theta-\phi/2)$$
$$C_3^1 = rR^{-3/2}\cos(3\phi/2), \quad S_3^1 = rR^{-3/2}\sin(3\phi/2)$$

(b) Mode III

We now consider the problem of calculating the stress field in the vicinity of the penny-shaped crack $0 \le r \le 1, z = 0$ in a solid under torsion so that, in terms of cylindrical coordinates (r, θ, z),

$$\sigma_{\theta z} \to \tau_0 \text{ as } r \to \infty \tag{A11}$$

In such a stress distribution the components u_θ and u_z of the displacement field will be identically zero and the remaining component u_θ will satisfy the equation

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \tag{A12}$$

If we take the solution of this equation in the form

$$u_\theta(r,z) = \frac{\tau_0}{\mu}z + \frac{\tau_0}{\mu}\int_0^\infty A(s)e^{-zs}J_1(rs)ds \tag{A13}$$

where μ is the rigidity modulus, we find that

$$\sigma_{\theta z}(r,z) = \mu\frac{\partial u_\theta}{\partial z} = \tau_0 - \tau_0\int_0^\infty A(s)e^{-zs}J_1(rs)ds \tag{A14}$$
$$\sigma_{r\theta}(r,z) = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\tau_0\int_0^\infty sA(s)e^{-zs}J_2(rs)ds \tag{A15}$$

boundary conditions of the plane z=0 is

$$\sigma_{\theta z}(r,0) = 0, \quad 0 \le r \le 1$$
$$u_\theta = 0, \qquad r \ge 1 \tag{A16}$$

The solution of $A(s)$ satisfying the above boundary conditions is obtained as follows

$$A(s) = \frac{1-\cos(s)}{s^2} - \frac{\sin(s)}{2s} \tag{A17}$$

Considering the dimensionless variable a , the resultant displacement and stress components are as follows

$$u_\theta(r,z) = \frac{\tau_0}{\mu}az + \frac{\tau_0}{\mu}a\int_0^\infty\left(\frac{1-\cos(s)}{s^2} - \frac{\sin(s)}{2s}\right)e^{-zs}J_1(rs)ds \tag{A18a}$$
$$= \frac{\tau_0}{\mu}az + \frac{\tau_0}{\mu}a(I_0^1 - S_1^1/2)$$

$$\sigma_{\theta z}(r,z) = \tau_0 - \tau_0\int_0^\infty\left(\frac{1-\cos(s)}{s^2} - \frac{\sin(s)}{2s}\right)e^{-zs}J_1(rs)ds \tag{A18b}$$
$$= \tau_0\left\{1 - (Z_1^1 - C_1^1 - S_2^1/2)\right\}$$

$$\sigma_{r\theta}(r,z) = -\tau_0\int_0^\infty s\left(\frac{1-\cos(s)}{s^2} - \frac{\sin(s)}{2s}\right)e^{-zs}J_2(rs)ds \tag{A18c}$$
$$= \tau_0\left\{-\frac{2}{r}(IC_0^1 - S_1^1) + IC_1^0 - S_2^0/2\right\}$$

where

$$Z_1^1 = \frac{1}{r}\left[\left(r^2+z^2\right)^{1/2} - z\right]$$
$$IC_0^1 = \frac{1}{2r}\left[\begin{array}{l}\rho R^{1/2}\cos(\theta+\phi/2) - \rho^2\cos(2\theta) + \\ r^2\log\left[\left\{\begin{array}{l}\left(\rho\sin(\theta) + R^{1/2}\sin(\phi/2)\right)^2 + \\ \left(\rho\cos(\theta) + R^{1/2}\cos(\phi/2)\right)^2\end{array}\right\}/r\right]\end{array}\right]$$
$$IC_1^0 = \log\left[\left\{\begin{array}{l}\left(\rho\sin(\theta) + R^{1/2}\sin(\phi/2)\right)^2 + \\ \left(\rho\cos(\theta) + R^{1/2}\cos(\phi/2)\right)^2\end{array}\right\}/r\right]$$