

STABILITY THEOREM FOR THE FEYNMAN INTEGRAL APPLIED TO MULTIPLE INTEGRALS

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ABSTRACT. In 1984, Johnson [A bounded convergence theorem for the Feynman integral, *J. Math. Phys.* **25** (1984), 1323–1326] proved a bounded convergence theorem for the Feynman integral. This is the first stability theorem of the Feynman integral as an $\mathcal{L}(L_2(\mathbb{R}^N), L_2(\mathbb{R}^N))$ theory. Johnson and Lapidus [Generalized Dyson series, generalized Feynman digrams, the Feynman integral and Feynman's operational calculus. *Mem. Amer. Math. Soc.* **62** (1986), no. 351] studied stability theorems for the Feynman integral as an $\mathcal{L}(L_2(\mathbb{R}^N), L_2(\mathbb{R}^N))$ theory for the functionals with arbitrary Borel measure. These papers treat functionals which involve only a single integral. In this paper, we obtain the stability theorems for the Feynman integral as an $\mathcal{L}(L_1(\mathbb{R}), L_\infty(\mathbb{R}))$ theory for the functionals which involve double integral with some Borel measures.

1. INTRODUCTION AND PRELIMINARIES

The theory of quantum mechanics is based on the Schrödinger wave equation. In 1948, to solve the wave equation, Feynman [3] introduced an integral, so called the Feynman integral (cf. Johnson and Lapidus [7]). In 1968, Cameron and Storvick [1] defined an integral, the operator valued function space integral, which the nearest concept to the original Feynman's suggestion (cf. [7]). Kim and Ryu [8] established the existence theorem for the operator valued function space integral as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$. The functionals introduced in the Feynman integral are defined in terms of measure and potentials. It is natural to ask if the corresponding operators are stable under perturbations of either of these objects.

In this paper, we obtain the stability theorems for the Feynman integral as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ for the functionals which involve double integrals with some Borel measures.

Received by the editors March 18, 2001.

2000 *Mathematics Subject Classification.* 28C20.

Key words and phrases. Feynman integral, operator-valued function space integral, stability.

This work was supported by the Daejin University Research Grants in 2000.

Now we present some necessary notations, lemma and definitions which are needed in our subsequent section.

- A. Let $C[a, b]$ will denote the space of all real-valued continuous functions on $[a, b]$ and the Wiener Space, $C_0[a, b]$, will consists of those x in $C[a, b]$ such that $x(a) = 0$, and m_w will denote Wiener measure on $C_0[a, b]$.
- B. Let $M[a, b]$ denote the space of all complex Borel measures on $[a, b]$ such that if μ is the continuous measure in $M[a, b]$ then the Radon-Nikodym derivative $d|\mu|/dm_l$ exists and is essentially bounded, where m_l is the Lebesgue measure (cf. Reed and Simon [9]).
- C. For $2 < r \leq \infty$, let $L_{1r} := L_{1r}([a, b]^2 \times \mathbb{R}^2)$ be the space of Borel measurable \mathbb{C} -valued functions θ on $[a, b]^2 \times \mathbb{R}^2$ such that

$$(1.1) \quad \|\theta\|_{1r} := \left\{ \int_a^b \int_a^b \|\theta(s, t, \cdot, \cdot)\|_1^r dm_l(s) dm_l(t) \right\}^{\frac{1}{r}} < \infty,$$

i.e., θ is in L_{1r} if and only if $\theta(s, t, \cdot, \cdot)$ is in $L_1(\mathbb{R}^2)$ for almost every (s, t) in $[a, b]^2$ and $\|\theta(s, t, \cdot, \cdot)\|_1$ is in $L_r([a, b]^2)$. Note that the mixed norm space L_{1r} , equipped with the norm $\|\cdot\|_{1r}$, becomes a Banach space and $L_{1r} \subseteq L_{1s}$ if $1 \leq s \leq r \leq \infty$.

- D. Let F be a real or complex functional defined on $C[a, b]$. Given $\lambda > 0$, $\psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(1.2) \quad (I_\lambda(F)\psi)(\xi) = \int_{C_0[a, b]} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(b) + \xi) dm_w(x).$$

If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \rightarrow I_\lambda(F)\psi$ gives an element of $\mathcal{L}(L_1(\mathbb{R}), L_\infty(\mathbb{R}))$, the space of bounded linear operators from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$, we say that the operator valued function space integral $I_\lambda(F)$ exists for λ .

For $\theta \in L_{1r}$, and β, η be continuous measures in $M[a, b]$, let

$$(1.3) \quad F(y) = \int_{[a, b]} \int_{[a, b]} \theta(s, t, y(s), y(t)) d\beta(s) d\eta(t)$$

for any $y \in C[a, b]$, for which the integral exists.

Then for every $\lambda > 0$ and every $\xi \in \mathbb{R}$, $F(\lambda^{-\frac{1}{2}}x + \xi)$ is defined for $m_w \times m_l$ -a.e. $(x, \xi) \in C_0[a, b] \times \mathbb{R}$ (cf. Kim and Ryu [8]).

Lemma 1.1. For any nonnegative integer q_0 , $r > 2$ and r' satisfying $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$(1.4) \quad A(2q_0 : \dots : r') := \left\{ \int_{\Delta_{2q_0}} [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2q_0})]^{-\frac{r'}{2}} d \times_{i=1}^{2q_0} r_i \right\}^{\frac{1}{r'}}$$

$$= \left\{ (b - a)^{2q_0 - (2q_0 + 1)\frac{r'}{2}} \frac{[\Gamma(1 - \frac{r'}{2})]^{2q_0 + 1}}{\Gamma((2q_0 + 1)(1 - \frac{r'}{2}))} \right\}^{\frac{1}{r'}}$$

where

$$(1.5) \quad \Delta_{2q_0} = \{(r_1, \dots, r_{2q_0}) \in [a, b]^{2q_0} : a < r_1 < \dots < r_{2q_0} < b\}.$$

Proof. See Kim and Ryu [8]. □

Definition 1.2. Let $A(\cdot, \lambda)$ be of class $L_\infty(\mathbb{R})$ for each λ in a domain Ω of the complex λ -plane. We shall say that $A(\cdot, \lambda)$ is a *weakly analytic vector valued function* of λ throughout Ω if for each $\phi \in L_1(\mathbb{R})$, $\int_{-\infty}^{\infty} A(\xi, \lambda)\phi(\xi)d\xi$ is an analytic function of λ in Ω .

Definition 1.3. Let Ω be a simply connected domain of the complex λ -plane whose intersection with the positive real axis is a single nonempty open interval (α, β) . Let F be a functional such that $I_\lambda(F)$ exists for $\lambda \in (\alpha, \beta)$. For each $\psi \in L_1(\mathbb{R})$ let a function $A(\lambda : \psi)$ exist as a weakly analytic vector-valued function of λ for $\lambda \in \Omega$, $A(\lambda : \psi) \in L_\infty(\mathbb{R})$ and let $A(\lambda : \psi) = I_\lambda(F)\psi$ for $\lambda \in (\alpha, \beta)$ and $\psi \in L_1(\mathbb{R})$; then we define

$$I_\lambda^{\text{an}}(F)\psi = A(\lambda : \psi)$$

for $\lambda \in \Omega$ and $\psi \in L_1(\mathbb{R})$. $I_\lambda^{\text{an}}(F)$ is called the *analytic operator valued function space integral*.

We note that, if $I_\lambda^{\text{an}}(F)$ exists, it is uniquely defined and is a linear operator that takes $L_1(\mathbb{R})$ into $L_\infty(\mathbb{R})$.

Definition 1.4. Let q be a real number, $\text{Re } \lambda > 0$, and F be a functional such that $I_\lambda^{\text{an}}(F)\psi$ exists for every $\psi \in L_1(\mathbb{R})$. If $Q(\cdot)$ is of class $L_\infty(\mathbb{R})$ and ψ is a given element of $L_1(\mathbb{R})$ such that

$$\lim_{\substack{\lambda \rightarrow -iq \\ \text{Re } \lambda > 0}} \int_{-\infty}^{\infty} [(I_\lambda^{\text{an}}(F)\psi)(\xi) - Q(\xi)]\phi(\xi)d\xi = 0$$

for every $\phi \in L_1(\mathbb{R})$; then we define

$$J_q(F)\psi = Q.$$

If $J_q(F)\psi$ exists for every $\psi \in L_1(\mathbb{R})$, $J_q(F)$ is called the operator-valued Feynman integral of F and we note that $J_q(F)$ is a linear operator, uniquely defined by the above equation which takes $L_1(\mathbb{R})$ into $L_\infty(\mathbb{R})$.

2. STABILITY IN THE POTENTIALS

Theorem 2.1. *Let β and η be continuous measures in $M[a, b]$ and let $H \in L_{1r}$, $r \in (2, \infty]$. Let $\theta^{(N)}$, $N = 1, 2, \dots$ be complex valued Borel measurable functions on $[a, b]^2 \times \mathbb{R}^2$ such that for $\beta \times \eta \times m_l \times m_l$ -a.e.,*

$$(2.1) \quad \theta^{(N)} \longrightarrow \theta \quad \text{as } N \rightarrow \infty \quad \text{and} \quad |\theta^{(N)}| \leq H.$$

Then θ and $\theta^{(N)}$ belong to L_{1r} . Moreover, let

$$(2.2) \quad F_n(y) = \left[\int_{[a,b]} \int_{[a,b]} \theta(s, t, y(s), y(t)) d\beta(s) d\eta(t) \right]^n$$

for y in $C[a, b]$. Let $F_n^{(N)}$ be defined in (2.2) with θ replaced by $\theta^{(N)}$. Then for all real $q > 0$, $J_q(F_n)$ and $J_q(F_n^{(N)})$ exist and for each $N \in \mathbb{N}$ and as $N \rightarrow \infty$, $J_q(F_n^{(N)}) \longrightarrow J_q(F_n)$ in the operator norm.

Proof. By (2.1), $\|\theta^{(N)}\|_{1r} \leq \|H\|_{1r}$ for $N = 1, 2, \dots$ and so $\theta^{(N)}$ and θ are in L_{1r} . From [8] and the above definitions, we can easily obtain, for each $N \in \mathbb{N}$, $J_q(F_n)$ and $J_q(F_n^{(N)})$ exist for all real $q > 0$. For each $\psi \in L_1(\mathbb{R})$ and $q > 0$,

$$(2.3) \quad \left| J_q(F_n^{(N)})\psi(\xi) - J_q(F_n)\psi(\xi) \right| \\ = \left| \sum_{(t_1, \dots, t_m, k_1, \dots, k_m) \in P} \int_{\Delta_{2m}} \left(\frac{-iq}{2\pi} \right)^{\frac{2m+1}{2}} [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2m})]^{-\frac{1}{2}} \right. \\ \cdot \left. \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^m \theta^{(N)}(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \psi(v_{2m+1}) \right. \\ \cdot \left. \exp \left\{ \sum_{j=1}^{2m+1} \frac{iq(v_j - v_{j-1})^2}{2(r_j - r_{j-1})} \right\} d \times_{n=1}^{2m+1} m_l(v_n) d \times_{n=1}^{2m} \tilde{\mu}_{p,n}(r_n) \right|$$

$$\begin{aligned}
 & - \sum_{(t_1, \dots, t_m, k_1, \dots, k_m) \in P} \int_{\Delta_{2m}} \left(\frac{-iq}{2\pi} \right)^{\frac{2m+1}{2}} [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2m})]^{-\frac{1}{2}} \\
 & \quad \cdot \overbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}^{2m+1 \text{ times}} \prod_{j=1}^m \theta(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \psi(v_{2m+1}) \\
 & \quad \cdot \exp \left\{ \sum_{j=1}^{2m+1} \frac{iq(v_j - v_{j-1})^2}{2(r_j - r_{j-1})} \right\} d \times_{n=1}^{2m+1} m_i(v_n) d \times_{n=1}^{2m} \tilde{\mu}_{p,n}(r_n) \Bigg| \\
 & \leq \|\psi\|_1 \sum_{(t_1, \dots, t_m, k_1, \dots, k_m) \in P} \left(\frac{q}{2\pi} \right)^{\frac{2m+1}{2}} \left(\int_{\Delta_{2m}} [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2m})]^{-\frac{1}{2}} \right. \\
 & \quad \cdot \left. \int_{\mathbb{R}^{2m}} \left| \prod_{j=1}^m \theta^{(N)}(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) - \prod_{j=1}^m \theta(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \right| \right. \\
 & \quad \quad \quad \cdot d \times_{n=1}^{2m} m_i(v_n) d \times_{n=1}^{2m} |\tilde{\mu}_{p,n}(r_n)| \Bigg)
 \end{aligned}$$

where $\{r_1, \dots, r_{2q_0}\}$ is the set of numbers $s_1, \dots, s_{q_0}, t_1, \dots, t_{q_0}$ in some rearrangement, P is the set of all permutations of $\{1, 2, \dots, 2q_0\}$, $s_j := r_{m_j}$, $t_j := r_{k_j}$, and $\int f d\tilde{\mu}_{p,i}(r_i)$ means that $\int f d\beta(r_i)$ when $r_i = r_{m_j}$ for some r_{m_j} and $\int f d\tilde{\mu}_{p,i}(r_i)$ means that $\int f d\eta(r_i)$ when $r_i = r_{k_j}$ for some r_{k_j} . Since

$$\|J_q(F_n^{(N)})\psi - J_q(F_n)\psi\|_{\infty} \leq \|\psi\|_1 \left(\|J_q(F_n^{(N)})\| + \|J_q(F_n)\| \right) < \infty,$$

we obtain the inequality in (2.3) from Kim and Ryu [8], elementary calculus and Fubini theorem. Thus we have

$$\begin{aligned}
 (2.4) \quad & \|J_q(F_n^{(N)})\psi - J_q(F_n)\psi\|_{\infty} \\
 & \leq \|\psi\|_1 \sum_{(t_1, \dots, t_m, k_1, \dots, k_m) \in P} \left(\frac{q}{2\pi} \right)^{\frac{2m+1}{2}} \int_{\Delta_{2m}} L(2m; r_1, \dots, r_{2m}) d \times_{n=1}^{2m} |\tilde{\mu}_{p,n}(r_n)
 \end{aligned}$$

where

$$\begin{aligned}
 & L(2m; r_1, \dots, r_{2m}) \\
 & = [(r_1 - a)(r_2 - r_1) \cdots (r_{2m} - r_{2m-1})(b - r_{2m})]^{-\frac{1}{2}} \\
 & \quad \cdot \int_{\mathbb{R}^{2m}} \left| \prod_{j=1}^m \theta^{(N)}(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) - \prod_{j=1}^m \theta(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \right| d \times_{n=1}^{2m} m_i(v_n)
 \end{aligned}$$

We know that by (2.4), as $N \rightarrow \infty$,

$$(2.5) \quad \prod_{j=1}^m \theta^{(N)}(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \longrightarrow \prod_{j=1}^m \theta(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \quad \text{a.e.}$$

Since for every $N \in \mathbb{N}$,

$$|\theta^{(N)}(s, t, u, v)| \leq H(s, t, u, v) \quad \text{for } \beta \times \eta \times m_l \times m_l\text{-a.e. } (s, t, u, v),$$

we have

$$(2.6) \quad \left| \prod_{j=1}^m \theta^{(N)}(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) - \prod_{j=1}^m \theta(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) \right| \leq 2 \prod_{j=1}^m H(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}).$$

In view of (2.5) and (2.6), the dominated convergence theorem gives

$$L(2m : r_1, r_2, \dots, r_{2m}) \longrightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Now, we claim that

$$(2.7) \quad \int_{\Delta_{2m}} L(2m : r_1, r_2, \dots, r_{2m}) d \times_{n=1}^{2m} |\tilde{\mu}_{p,n}|(r_n) \longrightarrow 0 \quad \text{as } N \rightarrow \infty.$$

For a.e. $(r_1, \dots, r_{2m}) \in \Delta_{2m}$, by (2.6) and Fubini theorem,

$$(2.8) \quad |L(2m : r_1, r_2, \dots, r_{2m})| \\ \leq 2 [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2m})]^{-\frac{1}{2}} \int_{\mathbb{R}^{2m}} \prod_{j=1}^m H(r_{t_j}, r_{k_j}, v_{t_j}, v_{k_j}) d \times_{n=1}^{2m} m_l(v_n) \\ \leq 2 [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2m})]^{-\frac{1}{2}} \prod_{j=1}^m \|H(r_{t_j}, r_{k_j}, \cdot, \cdot)\|_1.$$

Then, by Lemma 1.1 and the Hölder's inequality, the right hand side of the second inequality in (2.8) is $d \times_{n=1}^{2m} \tilde{\mu}_{p,n}$ -integrable. Hence, by the dominated convergence theorem, (2.7) is established.

Therefore, $J_q(F_n^{(N)}) \longrightarrow J_q(F_n)$ as $N \rightarrow \infty$ in the operator norm. \square

Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be an entire function of growth $(1, \tau)$ where $\tau < \infty$ and let

$$(2.9) \quad F(y) = f \left[\int_{[a,b]} \int_{[a,b]} \theta(s, t, y(s), y(t)) d\beta(s) d\eta(t) \right]$$

for y in $C[a, b]$ and

$$(2.10) \quad F^{(N)}(y) = f \left[\int_{[a,b]} \int_{[a,b]} \theta^{(N)}(s, t, y(s), y(t)) d\beta(s) d\eta(t) \right]$$

for y in $C[a, b]$

By Theorem 2.1 and Lemma 1.1 we have the following theorem.

Theorem 2.2. *Let the hypotheses of Theorem 2.1 be satisfied. Let F and $F^{(N)}$ are given by (2.9) and (2.10), respectively. Then*

Case I (growth (1, 0)): *In this case for all real $q > 0$, $J_q(F)$ and $J_q(F^{(N)})$ exist for each $N \in \mathbb{N}$.*

Case II (order one, type τ , $0 < \tau < \infty$): *In this case for all real q such that $0 < |q| < \lambda_0$, $J_q(F)$ and $J_q(F^{(N)})$ exist for each $N \in \mathbb{N}$ where*

$$(2.11) \quad \lambda_0 = \frac{\pi \left[\frac{2(b-a)}{2-r'} \right]^{\frac{r'-2}{r'}}}{\tau \left(\left\| \frac{d|\beta|}{dm_i} \right\|_{\infty} \left\| \frac{d|\eta|}{dm_i} \right\|_{\infty} \right) \left[\Gamma\left(1 - \frac{r'}{2}\right) \right]^{\frac{2}{r'}} \|\theta\|_{1r}}$$

and Γ is the gamma function.

Moreover $J_q(F^{(N)}) \rightarrow J_q(F)$ in the operator norm.

Proof. We omit the proof of Theorem 2.2, since it is essentially like the proof of our above result. \square

Corollary 2.3. *Under the hypotheses of Theorem 2.2, let*

$$(2.12) \quad F(y) = \exp \left[\int_{[a,b]} \int_{[a,b]} \theta(s, t, y(s), y(t)) d\beta(s) d\eta(t) \right]$$

for y in $C[a, b]$ and

$$(2.13) \quad F^{(N)}(y) = \exp \left[\int_{[a,b]} \int_{[a,b]} \theta^{(N)}(s, t, y(s), y(t)) d\beta(s) d\eta(t) \right].$$

for y in $C[a, b]$.

Then $J_q(F)$ and $J_q(F^{(N)})$ exist for each real $q \neq 0$, $|q| < \lambda_0$, where

$$\lambda_0 = \frac{\pi \left[\frac{2(b-a)}{2-r'} \right]^{\frac{r'-2}{r'}}}{\left(\left\| \frac{d|\beta|}{dm_i} \right\|_{\infty} \left\| \frac{d|\eta|}{dm_i} \right\|_{\infty} \right) \left[\Gamma\left(1 - \frac{r'}{2}\right) \right]^{\frac{2}{r'}} \|\theta\|_{1r}}$$

moreover $J_q(F^{(N)}) \rightarrow J_q(F)$ in the operator norm.

Proof. Since the order of $f(z) = e^z$ is one and type of e^z is one, it holds by Theorem 2.2. \square

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