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THE NIELSEN ROOT NUMBER FOR THE COMPLEMENT

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ABSTRACT. The purpose of this paper is to introduce the Nielsen root number for the complement $N(f; X - A, c)$ which shares such properties with the Nielsen root number $N(f; c)$ as lower bound and homotopy invariance.

1. INTRODUCTION

In Nielsen fixed point theory, the relative Nielsen number $N(f; X, A)$ is introduced in Schirmer [5], which is the better, and ideally sharp, lower bound for the minimum number of fixed points in the relative homotopy class of $f : (X, A) \to (X, A)$.

Zhao [7] considered the minimum number $MF[f; X - A]$ of fixed points on the complement $X - A$ and defined the Nielsen number on the complementary space, $N(f; X - A)$ which is a lower bound for $MF[f; X - A]$, and has the same basic properties as $N(f; X, A)$.

The Nielsen root number $N(f; c)$ of a map $f : X \to Y$ at a point $c \in Y$ is a homotopy invariant lower bound for the number of roots at c, that is, for the cardinality of $f^{-1}(c)$. Similarly, Yang [6] defined the relative root Nielsen number $N(f; X, A, c)$ which gives us a better lower bound for the minimum number $MR[f; X, A, c]$ of roots in the homotopy class of the map $f : (X, A) \to (Y, B)$, that is $N(f; X, A, c) \geq N(f; c)$.

It is the purpose of this paper, to determine the minimal number $MR[f; X - A, c]$ of roots on the complement $X - A$. The Nielsen root number on the complementary space, $N(f; X - A, c)$ is defined, which is a lower bound for $MR[f; X - A, c]$, and has the basic properties. The method used here follows that of Zhao [7].

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2. WEAKLY COMMON ROOT CLASSES

Let $f : (X, A) \to (Y, B)$ be a map of pairs of compact polyhedra with X, Y connected. We denote the set of roots of f at $c \in B$ by

$$\Gamma(f, c) = \{x \in X \mid f(x) = c\}.$$

We shall write $\bar{f} : A \to B$ be a restriction of $f : (X, A) \to (Y, B)$ to A and $\Gamma(\bar{f}, c) = \Gamma(f, c) \cap A$ if $c \in B$.

Throughout this paper c will be a point of $B \subset Y$. For this map f, let $\widehat{A} = \cup_{k=1}^{n} A_k$ be the disjoint union of all components of A such that for each k, A_k is mapped by f into some component B_c (containing c) of B.

Then we shall write $f_k : A_k \to B_c$ for the restriction of f to A_k. And we have a morphism of maps

$$\begin{array}{ccc} A_k & \xrightarrow{f_k} & B_c \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

where i_k, j are inclusions. The spaces X, Y are connected to have universal covering spaces. The universal covering will be denoted by $p : \tilde{X} \to X$, $q : \tilde{Y} \to Y$ and $p_k : \tilde{A}_k \to A_k, q_c : \tilde{B}_c \to B_c, k = 1, \cdots, n$. And γ denotes a covering translation.

For each k, we pick a lifting $(\tilde{i}_k, \tilde{j})$ of (i_k, j) such that the diagrams

$$\begin{array}{ccc} \tilde{A}_k & \xrightarrow{\tilde{i}_k} & \tilde{X} \\ {\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle p} \\ A_k & \xrightarrow{i_k} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{B}_c & \xrightarrow{\tilde{j}} & \tilde{Y} \\ {\scriptstyle q_c}\downarrow & & \downarrow{\scriptstyle q} \\ B_c & \xrightarrow{j} & Y \end{array}$$

commute.

This $\tilde{i}_k(\tilde{j})$ determines a correspondence $i_{k,\text{lift}}(j_{\text{lift}})$ from liftings of f_k to liftings of f.

Lemma 2.1. *Given $\tilde{f}_k$ of f_k and $\tilde{i}_k$ of i_k and $\tilde{j}$ of j there exist a unique lifting $\tilde{f}$ of f such that*

$$\begin{array}{ccccc} \tilde{A}_k & \xrightarrow{\gamma_k} & \tilde{A}_k & \xrightarrow{\tilde{f}_k} & \tilde{B}_c \\ {\scriptstyle \tilde{i}_k}\downarrow & & {\scriptstyle \tilde{i}_k}\downarrow & & \downarrow{\scriptstyle \tilde{j}} \\ \tilde{X} & \xrightarrow{\gamma} & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

commutes.

Proof. Pick a point $\tilde{x}_0 \in \tilde{A}_k$ and let $\tilde{c} = \tilde{f}_k(\tilde{x}_0) \in \tilde{B}_c$. Then there exists a unique lifting $\tilde{f}$ of f such that $\tilde{j}(\tilde{c}) = \tilde{f}(\tilde{i}_k(\tilde{x}_0))$. $\tilde{j} \circ \tilde{f}_k$ and $\tilde{f} \circ \tilde{i}_k$ are liftings of the same map $j \circ f_k = f \circ i_k : A_k \to Y$ and they agree at $\tilde{x}_0$.

By the unique lifting property of covering spaces, we have

$$\tilde{j} \circ \tilde{f}_k = \tilde{f} \circ \tilde{i}_k. \qquad \square$$

Definition 2.2. Two liftings $\tilde{f}_1$ and $\tilde{f}_2$ of $f : X \to Y$ are said to be *conjugate* if there exists a covering transformation $\gamma : \tilde{X} \to \tilde{X}$ such that $\tilde{f}_1 = \tilde{f}_2 \circ \gamma$. Lifting classes are the equivalence classes by conjugacy.

Lemma 2.3.

(a) $\Gamma(f;c) = \cup_{\tilde{f}}\, p\,\Gamma(\tilde{f};\tilde{c})$.
(b) $p\,\Gamma(\tilde{f}_1;\tilde{c}) = p\,\Gamma(\tilde{f}_2;\tilde{c})$ *if* $\tilde{f}_1 \sim \tilde{f}_2$ (*i. e.,* $[\tilde{f}_1] = [\tilde{f}_2]$).
(c) $p\,\Gamma(\tilde{f}_1;\tilde{c}) \cap p\,\Gamma(\tilde{f}_2;\tilde{c}) = \varnothing$ *if* $[\tilde{f}_1] \neq [\tilde{f}_2]$.

Proof.

(a) If $x_0 \in \Gamma(f;c)$, pick $\tilde{x}_0 \in p^{-1}(x_0)$. Then there exists $\tilde{f}$ such that $\tilde{f}(\tilde{x}_0) = \tilde{c}$. Hence $x_0 \in p\,\Gamma(\tilde{f};\tilde{c})$.
(b) Let $\tilde{f}_1 = \tilde{f}_2 \circ \gamma$. If $\tilde{x} \in \Gamma(\tilde{f}_1;\tilde{c})$, then $\tilde{f}_1(\tilde{x}) = \tilde{f}_2 \circ \gamma(\tilde{x}) = \tilde{c}$, so that $\gamma(\tilde{x}) \in \Gamma(\tilde{f}_2;\tilde{c})$. Thus $\gamma\,\Gamma(\tilde{f}_1;\tilde{c}) = \Gamma(\tilde{f}_2;\tilde{c})$. Hence $p\,\Gamma(\tilde{f}_1;\tilde{c}) = p\Gamma(\tilde{f}_2;\tilde{c})$.
(c) If $x \in p\,\Gamma(\tilde{f}_1;\tilde{c}) \cap p\,\Gamma(\tilde{f}_2;\tilde{c})$, then there are $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ such that $\tilde{f}_1(\tilde{x}_1) = \tilde{c}$ and $\tilde{f}_2(\tilde{x}_2) = \tilde{c}$. Suppose $\tilde{x}_1 = \gamma\,\tilde{x}_2$. Then $\tilde{f}_1(\gamma\,\tilde{x}_2) = \tilde{f}_2(\tilde{x}_2)(= \tilde{c})$. Thus

$$\tilde{f}_1\gamma = \tilde{f}_2, \quad \text{i. e.,} \quad [\tilde{f}_1] = [\tilde{f}_2]. \qquad \square$$

The subset $p\,\Gamma(\tilde{f};\tilde{c})$ of $\Gamma(f;c)$ is called the root class of f determined by the lifting class $[\tilde{f}]$. Define $\tilde{i}_{k,\text{lift}}(\tilde{f}_k) = \tilde{f}$ if $\tilde{j} \circ \tilde{f}_k = \tilde{f} \circ \tilde{i}_k$. And $\tilde{i}_{k,\text{lift}}$ induce a correspondence from lifting classes of f_k to lifting classes of f, which is independence of the choice of lifting $\tilde{i}_k$ of i_k and is determined by i_k itself. It is denoted

$$i_k^* : C(f_k) \to C(f)$$

where $C(f)$ is the lifting class of f.

Recall that a root class is always labelled by a lifting class, and we have

Proposition 2.4. *Every root class of* $f_k : A_k \to B_c$ *belongs to unique root class of* $f : X \to Y$. *When* $p_k\,\Gamma(\tilde{f}_k;c)$ *is non-empty,* $p_k\,\Gamma(\tilde{f}_k;c)$ *belongs to* $p\,\Gamma(\tilde{f};c)$ *if and only if* $i_k^*[\tilde{f}_k] = [\tilde{f}]$.

Let $H : f_0 \simeq f_1 : (X, A) \to (Y, B)$ be a relative homotopy between f_0 and f_1, $x_i \in \Gamma(f_i; c)$ and R_i a root class in $\Gamma'(f_i; c)$ containing x_i. If there exists in X a path α from x_0 to x_1 such that

$$[\Delta(H, \alpha)] = [c],$$

then x_0 and x_1 are said to be in correspondence under H and denoted by $x_0 H x_1$, where $\Delta(H, \alpha)$ is a diagonal path defined by $\Delta(H, \alpha)(t) = H(\alpha(t), t), 0 \leq t \leq 1$. This relation $x_0 H x_1$ induces a correspondence from R_o to R_1 under H, which is denoted by $R_0 H R_1$.

Let $f : (X, A) \to (Y, B)$ be a mapping under $H : f \simeq H(\cdot, 1) : (X, A) \to (Y, B)$ a relative homotopy. Let the root class in $\Gamma'(f; c)$ be denoted by R. If $R \in \Gamma'(f; c)$ corresponds to a root class $\in \Gamma'(H(\cdot\,, 1); c)$ under any such H, then R is called an *essential root class.*

As Zhao [7] has defined, we have a similar definition in the case of roots.

Definition 2.5. A root class $p\Gamma(\tilde{f}; c)$ of $f : X \to Y$ is a *weakly common root class* of f and $\bar{f}$ if it contains a root class of $f_k : A_k \to B_c$ for some k. It is an *essential weakly common root class* of f and $\bar{f}$ if it is an essential root class of f as well as a weakly common root class of f and $\bar{f}$. We write $E(f, \bar{f}; c)$ for the number of essential weakly common root classes of f and $\bar{f}$.

Theorm 2.6. *A root x_0 of f belongs to a weakly common root class of f and $\bar{f}$ (at c) if and only if there is a path α from x_0 to A such that $c \simeq f \circ \alpha : I, 0, 1 \to Y, c, B$.*

Proof. "$\Rightarrow$" Let x_0 belong to a weakly common root class $p\Gamma(\tilde{f}; c)$ of f and $\bar{f}$ at $c \in B_c$. Suppose $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{f}(\tilde{x}_0) = \tilde{c}$. By assumption, there is a lifting $\tilde{f}_k$ of $f_k : A_k \to B_c$ so that $(\tilde{i}_k)_{\text{lift}}(\tilde{f}_k) = \tilde{f}$. Pick a point $\tilde{a} \in \tilde{i}_k(\tilde{A}_k)$, then $\tilde{f}(\tilde{a}), \tilde{c} \in \tilde{j}(\tilde{B}_c)$. Take a path $\tilde{\alpha}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{a}$. Since $\tilde{Y}$ is 1-connected, there is a homotopy of the form

$$\tilde{c} \simeq \tilde{f} \circ \tilde{\alpha} : I, 0, 1 \to \tilde{Y}, \tilde{f}(\tilde{x}_0), \tilde{j}(\tilde{B}_c).$$

Projecting down to Y, we have

$$c \simeq f \circ \alpha : I, 0, 1 \to Y, c, B$$

where $\alpha = q \circ \tilde{\alpha}$.

"$\Leftarrow$" Suppose $x_0 \in p\Gamma(\tilde{f}; c), \tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{f}(\tilde{x}_0) = \tilde{c}$. Lift a path α from $\tilde{x}_0$ to get a path $\tilde{\alpha}$ in $\tilde{X}$. Let $a = \alpha(1) \in A_k, b = f_k(a) \in B_c$, and pick $\tilde{a} \in (p_k)^{-1}(a), \tilde{b} \in$

$(q_c)^{-1}(b)$, then there are liftings $\tilde{i}_k, \tilde{j}$ (resp.) of i_k, j (resp.) such that

$$\begin{array}{ccc} (\tilde{A}_k, \tilde{a}) & \xrightarrow{\tilde{i}_k} & (\tilde{X}, \tilde{\alpha}(1)) \\ {\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle p} \\ (A_k, a) & \xrightarrow{i_k} & (X, a) \end{array} , \quad \begin{array}{ccc} (\tilde{B}_c, \tilde{b}) & \xrightarrow{\tilde{j}} & (\tilde{Y}, \widetilde{f\circ\alpha}(1)) \\ {\scriptstyle q_c}\downarrow & & \downarrow{\scriptstyle q} \\ (B_c, b) & \xrightarrow{j} & (Y, b) \end{array} \quad \text{(resp.)}$$

commute (see Jiang [2, p. 42, Proposition 1.2 (i)]). Let $H : I \times I \to Y$ be the homotopy from $f \circ \alpha$ to c, i.e., $H(t,0) = f\circ\alpha, H(t,1) = c$. Then $\widetilde{f\circ\alpha}$ determines a lifting $\tilde{H} : I \times I \to \tilde{Y}$ of H. Denote β the path $\{H(1,s)\}_{0\le s\le 1}$ in B_c. Lift the path $\beta : I \to B_c$ from $\tilde{b}$ to get a path $\tilde{\beta}$ in $\tilde{B}_c$, then $\tilde{j}\circ\tilde{\beta} : I \to \tilde{Y}$ is a lifting from $\widetilde{f\alpha}(1)$ in $\tilde{Y}$ of the path $j \circ \beta$.

By the unique lifting property of covering spaces, we have $\tilde{H}(1,s) = \tilde{j}\circ\tilde{\beta}(s)$. Then $\tilde{j}\circ\tilde{\beta}(0) = \tilde{H}(1,0) = \widetilde{f\circ\alpha}(1)$ and $\tilde{j}\circ\tilde{\beta}(1) = \tilde{H}(1,1) = \tilde{c}$, there exists a unique lifting $\tilde{f}_k$ of $f_k : A_k \to B_c$ such that $\tilde{f}_k(\tilde{a}) = \tilde{\beta}(0) = \tilde{b}$. Thus $\tilde{j}\circ\tilde{f}_k(\tilde{a}) = \tilde{f}\circ\tilde{i}_k(\tilde{a})$. By the unique lifting property of covering spaces, we have $\tilde{j}\circ\tilde{f}_k = \tilde{f}\circ\tilde{i}_k$, i.e., $\tilde{f} = (\tilde{i}_k)_{\text{lift}}(\tilde{f}_k)$. This implies $[\tilde{f}] = i_k^*[\tilde{f}_k]$, i.e., $p\Gamma(\tilde{f}; c)$ is a weakly common root class of f and $\bar{f}$. □

Corollary 2.7. *A root class of $f : X \to Y$ containing a root on A is a weakly common root class of f and $\bar{f}$.*

In Yang [6], the number $N(f,\bar{f};c)$ of essential common root classes of f and $\bar{f}$ is introduced, and we have

Proposition 2.8. $N(f,\bar{f};c) \le E(f;\bar{f},c) \le N(f;c)$.

Proof. By the Corollary 2.7 and Yang [6, Definition 2.1], we know that a common root class is always a weakly common root class. This implies the left inequality. The right one is obvious. □

Theorm 2.9 (Homotopy Invariance). *If two maps $f_0 \simeq f_1 : (X,A) \to (Y,B)$ are homotopic, then $E(f_0,\bar{f}_0;c) = E(f_1,\bar{f}_1;c)$.*

Proof. Let $(h_t,\bar{h}_t) : (X,A) \to (Y,B)$ be a (relative) homotopy between f_0 and f_1. There exist a bijection $\{h_t\} : \Gamma'(f_0) \to \Gamma'(f_1)$. It suffices to show $\{h_t\}$ sends weakly common root classes to weakly common root classes. Let $p\Gamma(\tilde{f}_0; c)$ be a weakly common root class of f_0 and $\bar{f}_0$, then there exist a component A_k of $\widehat{A}$ and a lifting class $[\widetilde{f_{0k}}]$ of $f_{0k} : A_k \to B_c$ such that $i_k^*[\widetilde{f_{0k}}] = [\tilde{f}_0]$.

Let $\{h_{k,t}\}$, which is the restriction of h_t to A_k, send $[\widetilde{f_{0k}}]$ to $[\widetilde{f_{1k}}]$, then we have a commutative diagram

$$\begin{array}{ccc} [\widetilde{f_{0k}}] & \xrightarrow{\{h_{k,t}\}} & [\widetilde{f_{1k}}] \\ {\scriptstyle i_k{}^*}\downarrow & & \downarrow{\scriptstyle i_k{}^*} \\ [\tilde{f_0}] & \xrightarrow{\{h_t\}} & [\tilde{f_1}] \end{array}$$

Thus, $\{h_t\}$ send $[\tilde{f_0}]$ to $[\tilde{f_1}] = i_k{}^*[\widetilde{f_{1k}}]$, we get the conclusion. □

In general, $E(f, \bar{f}; c)$ is different from $N(f, \bar{f}; c)$. A simple example is the identity map $f : (S^1, J) \to (S^1, J)$ of the pair of unit circle and its subset $J = e^{i\theta}\,(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$. Let $c = (1, 0) \in J$, it is easy to see that $N(f; c) = 1, N(\bar{f}; c) = 0$. And by definition $N(f, \bar{f}; c) = 0$. But $E(f, \bar{f}; c) = 1$.

Definition 2.10. The number of essential root classes of $f : X \to Y$ which are not weakly common root classes is called the Nielsen root number of f on the complementary space $X - A$, denoted $N(f; X - A, c)$.

By definition $N(f; X - A, c)$ is a non-negative integer, and

$$N(f; X - A, c) + E(f; \bar{f}, c) = N(f; c).$$

And we also have the basic properties of $N(f; X - A, c)$.

Theorm 2.11 (Lower Bound). *Any map $f : (X, A) \to (Y, B)$ has at least $N(f; X - A, c)$ roots on $X - A$. Thus $N(f; X - A, c) \leq MR[f; X - A, c]$.*

Proof. Recall that each essential root class contains at least one root. By Corollary 2.7, we shall get the conclusion. □

Theorm 2.12 (Homotopy Invariance). *Let $(X, A), (Y, B)$ be pairs of compact polyhedra with X, Y connected. If the maps $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then*

$$N(f_0; X - A, c) = N(f_1; X - A, c).$$

Proof. It is known (cf. Kiang [3, p. 129, Theorem 4.14]) that $N(f_0; c) = N(f_1; c)$, and $E(f_0, \bar{f_0}; c) = E(f_1, \bar{f_1}; c)$ from Theorem 2.9.

By definition of $N(f; X - A, c)$, we have the conclusion. □

3. Computation of $N(f; X - A, c)$

Theorm 3.1. *Let $f : (X, A) \to (Y, B)$ be a map of pairs of compact polyhedra. If there is a component A_k of $\widehat{A}$ such that $j_\pi : \pi_1(B_c) \to \pi_1(Y)$ is onto, then $N(f; X - A, c) = 0$.*

Proof. By Jiang [2, p. 46, Theorem 1.14 (i)], j_π is surjective. Then every root class of $f : X \to Y$ is a weakly common root class of f and $\bar{f}$. □

Pick a base point $a_k \in A_k$ for each $A_k \subset \widehat{A}$ such that $f(a_k) = c, x_0 \in X$ and $c \in B_c \subset Y$. Recall that points of universal covering spaces $\tilde{A}_k$ and $\tilde{X}$ of A_k and X are respectively in one-to-one correspondence with the path classes in A_k and X starting from a_k to x_0. Under this identification, let $\tilde{a_k} \in {p_k}^{-1}(a_k), \tilde{x_0} \in p^{-1}(x_0), \tilde{b} \in {q_c}^{-1}(c)$ and $\tilde{y}_0 \in q^{-1}(c)$ be the constant paths. Then there are unique liftings $\tilde{f}_k$ of f_k and $\tilde{f}$ of f such that $\tilde{f}_k(\tilde{a}_k) = \tilde{b}$ and $\tilde{f}(\tilde{x}_0) = \tilde{y}_0$.

By Brown [1, Lemma 1.13], $\tilde{f}_{k,\pi} = f_{k,\pi}$ and $\tilde{f}_\pi = f_\pi$. Throughout this section, the liftings $\tilde{f}_k$ of f_k and $\tilde{f}$ of f are chosen as references.

Lemma 3.2. *There exist one-to-one correspondence*

$$\phi_k : C(f_k) \to {\pi_1}'(B_c, c)$$

$$\phi : C(f) \to {\pi_1}'(Y, c)$$

defined by

$$\phi_k[\alpha_k \circ \tilde{f}_k] = [\alpha_k]$$

$$\phi[\alpha \circ \tilde{f}] \to [\alpha]$$

where $\alpha_k \in \pi_1(B_c, c), \alpha \in \pi_1(Y, c)$, $C(f)$ is the set of all lifting classes of f and ${\pi_1}'(Y, c)$ is the coset of $\mathrm{Im} f_\pi$.

If α_k is coset in $\pi_1(B_c, c)$, then $j_\pi(\alpha_k)$ is coset in $\pi_1(Y, c)$. Then the homomorphism $j_\pi : \pi_1(B_c, c) \to \pi_1(Y, c)$ induces a transformation $\nu : {\pi_1}'(B_c, c) \to {\pi_1}'(Y, c)$ and ν is independent of the choice of the α_k.

Theorm 3.3. *Let $f : (X, A) \to (Y, B)$ be a map of pairs of compact polyhedra. A root class of f is a weakly common root class of f and $\bar{f}$ if and only if it corresponds to an element in the image of ν.*

Proof. By Lemma 3.2, it suffices to check that the diagram

$$\begin{array}{ccc} C(f_k) & \xrightarrow{i_k{}^*} & C(f) \\ {\scriptstyle \phi_k}\downarrow & & \downarrow{\scriptstyle \phi} \\ \pi_1{}'(B_c, c) & \xrightarrow{\nu} & \pi_1{}'(Y, c) \end{array}$$

commutes. Let $[\alpha_k \circ \tilde{f}_k] \in C(f_k)$, then $\phi \circ i_k{}^*[\alpha_k \circ \tilde{f}_k] = \phi[\alpha \circ \tilde{f}] = [\alpha]$ and $\nu \circ \phi_k[\alpha_k \circ \tilde{f}_k] = \nu[\alpha_k] = [\alpha]$. □

Consider the commutative diagram

$$\begin{array}{ccccc} \pi_1(B_c, c) & \xrightarrow{\theta_c} & H_1(B_c) & \xrightarrow{\eta_A} & \mathrm{coker}(-f_{k,*} : H_1(A) \to H_1(B)) \\ \downarrow{\scriptstyle j_\pi} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \pi_1(Y, c) & \xrightarrow{\theta} & H_1(Y) & \xrightarrow{\eta} & \mathrm{coker}(-f_* : H_1(X) \to H_1(Y)) \end{array}$$

where θ_c, θ are abelianization and η_A, η are the natural projections.

By some modification of Lee [4, Lemma 3.4], we have the following lemma.

Lemma 3.4. *The compositions $\eta_A \circ \theta_c$ and $\eta \circ \theta$ induce correspondences*

$$\tau_k : \pi_1{}'(B_c, c) \to \mathrm{coker}(-f_{k,*})$$

$$\tau : \pi_1{}'(Y, c) \to \mathrm{coker}(-f_*)$$

and the diagram

$$\begin{array}{ccc} \pi_1{}'(B_c, c) & \xrightarrow{\tau_k} & \mathrm{coker}(-f_{k,*}) \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle j_*} \\ \pi_1{}'(Y, c) & \xrightarrow{\tau} & \mathrm{coker}(-f_*) \end{array}$$

commutes.

Theorm 3.5. *Let $f : (X, A) \to (Y, B)$ be as in Theorem 3.3. Suppose Y and B_c are Jiang spaces. Then if $N(f; X, c) \neq 0$, then*

$$N(f; X - A, c) = \#\{\mathrm{coker}(-f_*)\} - \#\{\bigcup_{k=1}^{n} j_* \,\mathrm{coker}(-f_{k,*})\}.$$

Proof. Since $\pi_1(B_c, c)$ and $\pi_1(Y, c)$ are abelian groups, the correspondence τ is bijective. Apply Theorem 3.3 and Lemma 3.4 to get the conclusion. □

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