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THE NEARLY H_1-STIELTJES REPRESENTABLE OPERATORS

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ABSTRACT. In this paper, we define the H_1-Stieltjes representable, nearly H_1-Stieltjes representable for vector-valued function, which is the generalization of Bochner representable and then study some properties of these operators.

1. INTRODUCTION AND PRELIMINARIES

It is well-known though not easily found in the literature that the Riemann integral can be defined by the Moore-Smith limit using divisions. Then many properties of the Riemann integral will have straightforward proofs. Garces, Lee and Zhao [2] defined the H_1-integral by means of the Moore-Smith limit involving δ-fine divisions for the Henstock integral and studied the properties of this integral.

The Bochner representable operator and the Pettis representable operator on the Banach space studied by many authors (cf. Bourgain [1]; Garces, Lee and Zhao [2]).

In this paper, we introduce the H_1-Stieltjes representable operator, nearly H_1-Stieltjes representable operator and investigate some properties of these operators. Throughout this paper, X and Y are a Banach spaces and α is an increasing function on $[a, b]$ unless otherwise stated.

A *division* D of $[a, b]$ is a finite set of point-interval pairs $(x, [c, d])$ such that the intervals $[c, d]$ are non-overlapping and their union is $[a, b]$, and $x \in [c, d]$. Let $D_1 = \{(x, [c, d])\}$ and $D_2 = \{(y, [s, t])\}$ be two divisions of $[a, b]$. Then D_2 is said to be *finer than* D_1 *in the Riemann sense*, or in symbols, $D_2 \sqsupseteq D_1$ if for each $(y, [s, t]) \in D_2$ we have $[s, t] \subset [c, d]$ for some $(x, [c, d]) \in D_1$ and when $[s, t] = [c, d]$ we have $x = y$.

Now, let $\mathcal{D}$ be the family of divisions of $[a, b]$. Then $(\mathcal{D}, \sqsupseteq)$ is a directed set of divisions D of $[a, b]$. More precisely, the following conditions are satisfied:

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(1) $D \sqsupseteq D$ for all $D \in \mathcal{D}$.
(2) If $D_1, D_2, D_3 \in \mathcal{D}$ with $D_1 \sqsupseteq D_2$ and $D_2 \sqsupseteq D_3$, then $D_1 \sqsupseteq D_3$.
(3) For every $D_1, D_2 \in \mathcal{D}$, there exists $D_3 \in \mathcal{D}$ such that $D_3 \sqsupseteq D_1$ and $D_3 \sqsupseteq D_2$.

Hence the Riemann integral of f on $[a,b]$ is the Moore-Smith limit of Riemann sums using $(\mathcal{D}, \sqsupseteq)$. In symbols

$$\int_a^b f = \lim_{D \in \mathcal{D}} (D) \sum f(x)(d-c).$$

A division D of $[a,b]$ is *δ-fine* if $x \in [c,d] \subset (x - \delta(x), x + \delta(x))$ for each $(x,[c,d]) \in D$.

We recall that a function f is *Henstock integrable* to A on $[a,b]$ if for every $\varepsilon > 0$ there exists $\delta(x) > 0$ such that for any δ-fine division D of $[a,b]$ we have

$$\left|(D) \sum f(x)(d-c) - A\right| < \varepsilon.$$

Now let $\mathcal{D}$ be the family of δ-fine divisions of $[a,b]$ for some given $\delta(x) > 0$. For $D_1, D_2 \in \mathcal{D}$, we write $D_2 \geq D_1$ and say that D_2 is *finer than D_1 in the Henstock sense* using δ if for every $(y,[s,t]) \in D_2$ we have

$$[s,t] \subset [c,d] \quad \text{for some } (x,[c,d]) \in D_1,$$

and

$$\{x : (x,[c,d]) \in D_1\} \subset \{y : (y,[s,t]) \in D_2\}.$$

Then $(\mathcal{D}, \geq)$ is a directed set.

A function $f : [a,b] \to X$ is said to be *H_1-Stieltjes integrable* to $z \in X$ with respect to α on $[a,b]$ if z is the Moore-Smith limit of the Riemann sums using the directed set $(\mathcal{D}, \geq)$. More precisely, there exists $\delta(x) > 0$ such that for every $\varepsilon > 0$ there exists a δ-fine division D_0 such that for every δ-fine division $D \geq D_0$ we have

$$\left\|(D) \sum f(x)(\alpha(d) - \alpha(c)) - z\right\| < \varepsilon.$$

We say that z is the *H_1-Stieltjes integral* of f on $[a,b]$ and that f is *H_1-Stieltjes integrable* on $[a,b]$ with respect to α using δ. We say that f is *H_1-Stieltjes integrable* on a measurable set $X \subset [a,b]$ with respect to α if $f\chi_X$ is H_1-Stieltjes integrable with respect to α on $[a,b]$.

It is easy to see that every H_1-Stieltjes integrable function on $[a,b]$ is also Henstock-Stieltjes integrable there.

2. H_1-Stieltjes representable operators

Definition 2.1 (Petrakis [7]). A bounded linear operator $T : L_1[a,b] \to X$ is the *Bochner* (resp. *Pettis, Henstock*) *representable operator* if there exists an *essentially bounded Bochner integrable* (resp. *Pettis integrable, scalarly essentially bounded Henstock integrable*) function $g : [a,b] \to X$ such that

$$T(f) = \int_a^b fgd\mu \quad (\text{resp.} \quad T(f) = (P)\int_a^b fgd\mu, \quad T(f) = (H)\int_a^b fgd\mu)$$

for all $f \in L_1[a,b]$.

We define the H_1-Stieltjes representable operator, which is the generalization of Pettis representable operator.

Definition 2.2. A bounded linear operator $T : L_1[a,b] \to X$ is *H_1-Stieltjes representable* with respect to α if there exists a scalarly essentially bounded H_1-Stieltjes integrable function $g : [a,b] \to X$ with respect to α such that

$$T(f) = \int_a^b fgd\alpha$$

for all $f \in L_1[a,b]$.

Theorm 2.3. *If $f : [a,b] \to X$ is a H_1-Stieltjes integrable function with respect to α and if $T : X \to Y$ is a bounded linear operator, then the composition $T \circ f : [a,b] \to Y$ is a H_1-Stieltjes integrable function with respect to α and*

$$T(\int_a^b fd\alpha) = \int_a^b T \circ fd\alpha.$$

Proof. If $T = 0$, then it is clear. Suppose that $T \neq 0$. Let $\int_a^b fd\alpha = z$. Then there exists a positive function δ on $[a,b]$ such that for$\varepsilon > 0$ there exists a δ-fine division D_0 such that $\|z - f_\alpha(\mathcal{P})\| < \varepsilon/\|T\|$ whenever every δ-fine division $D \geq D_0$. Hence

$$\|T(z) - (T \circ f)_\alpha(\mathcal{P})\| \leq \|T\| \, \|z - f_\alpha(\mathcal{P})\| < \varepsilon$$

for every δ-fine division $D \geq D_0$. Therefore $T \circ f : [a,b] \to Y$ is a H_1-Stieltjes integrable function with respect to α and

$$\int_a^b T \circ fd\alpha = Tz = T(\int_a^b fd\alpha). \qquad \square$$

Theorm 2.4. *If $T : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α and $S : X \to Y$ is any bounded linear operator, then the composition $S \circ T : L_1[a,b] \to Y$ is H_1-Stieltjes representable with respect to α.*

Proof. Suppose that $T : L_1[a,b] \to Y$ is a bounded linear operator and there exists a scalarly essentially bounded H_1-Stieltjes integrable function $g : [a,b] \to X$ with respect to α such that

$$T(f) = \int_a^b fg d\alpha \quad \text{for all } f \in L_1[a,b].$$

By Theorem 2.3, $S \circ g : [a,b] \to Y$ is also scalarly essentially bounded H_1-Stieltjes integrable with respect to α. For each $f \in L_1[a,b]$,

$$(S \circ T)(f) = S\left(\int_a^b fg d\alpha\right) = \int_a^b f(S \circ g) d\alpha.$$

$S \circ g$ is H_1-Stieltjes representable with respect to α. Therefore $S \circ T : L_1[a,b] \to Y$ is Henstock-Stieltjes representable with respect to α. □

Theorm 2.5. *If a bounded linear operator $T : L_1[a,b] \to X$ and $G : L_1[a,b] \to X$ are H_1-Stieltjes representable with respect to α, then $k_1T + k_2G : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α for all k_1, k_2 in $\mathbb{R}$.*

Proof. We will show that kT and $T + G$ are H_1-Stieltjes representable with respect to α. Suppose that a bounded linear operator $T : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α, then there exists a scalarly essentially bounded H_1-Stieltjes integrable function $g : [a,b] \to X$ with respect to α such that

$$T(f) = \int_a^b fg d\alpha \quad \text{for all } f \in L_1[a,b].$$

Since $T : L_1[a,b] \to X$ is bounded linear operator, $kT : L_1[a,b] \to X$ is a bounded linear operator for all k in R and $kG : [a,b] \to X$ is H_1-Stieltjes integrable with respect to α. Hence

$$(kT)(f) = kT(f) = k\int_a^b fg d\alpha = \int_a^b k(fg) d\alpha = \int_a^b f(kg) d\alpha.$$

Thus $kT : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α.

To show that $T + G$ is H_1-Stieltjes representable with respect to α. Suppose that bounded linear operators $T : L_1[a,b] \to X$ and $G : L_1[a,b] \to X$ are H_1-Stieltjes representable with respect to α. Then there exist scalarly essentially bounded H_1-Stieltjes integrable functions $g : [a,b] \to X$ and $h : [a,b] \to X$ with respect to α such that

$$T(f) = \int_a^b fg d\alpha, \ G(f) = \int_a^b fh d\alpha$$

for all $f \in L_1[a,b]$. Since $T : L_1[a,b] \to X$ and $G : L_1[a,b] \to X$ are bounded linear operators, so $T + G : L_1[a,b] \to X$ is also a bounded linear operator and $g + h : [a,b] \to X$ is scalarly essentially bounded H_1-Stieltjes representable with respect to α. Hence

$$(T+G)(f) = T(f) + G(f) = \int_a^b fg d\alpha + \int_a^b fh d\alpha = \int_a^b f(g+h) d\alpha.$$

Thus $T+G : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α. Therefore $k_1T + k_2G : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α. □

3. Nearly H_1-Stieltjes representable operators

Definition 3.1 (Petrakis [7]). A bounded linear operator $T : X \to Y$ is *nearly Pettis representable* if the composition $T \circ D : L_1[a,b] \to Y$ is the Pettis representable for every Dunford-Pettis operator $D : L_1[a,b] \to X$.

We now give the definition of the nearly H_1-Stieltjes representable operator, which is the generalization of the H_1-Stieltjes representable operator.

Definition 3.2. A bounded linear operator $T : X \to Y$ is *nearly H_1-Stieltjes representable* if the composition $T \circ D : L_1[a,b] \to Y$ is H_1-Stieltjes representable with respect to α for every Dunford-Pettis operator $D : L_1[a,b] \to X$.

Theorm 3.3. *If $T : X \to Y$ is nearly H_1-Stieltjes representable with respect to α and $U : Y \to Z$ (or $V : Z \to X$) is any bounded linear operator, then $U \circ T$ (or $T \circ V$) is nearly H_1-Stieltjes representable with respect to α.*

Proof. Assume that $T : X \to Y$ is nearly H_1-Stieltjes representable with respect to α and $U : Y \to Z$ is any bounded linear operator. Let $D : L_1[a,b] \to X$. be Dunford-Pettis operator. Then $T \circ D : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α.

By Theorem 2.4, $U \circ T \circ D : L_1[a,b] \to Z$ is also H_1-Stieltjes representable with respect to α, and hence $U \circ T : X \to Z$ is nearly H_1-Stieltjes representable with respect to α.

Let $V : Z \to X$ be any bounded linear operator. Let $D : L_1[a,b] \to Z$ be a Dunford-Pettis operator. Then $V \circ D : L_1[a,b] \to X$ is the Dunford-Pettis operator.

Since $T : X \to Y$ is nearly H_1-Stieltjes representable with respect to α, $T \circ V \circ D : L_1[a,b] \to Y$ is H_1-Stieltjes representable with respect to α. Therefore $T \circ V : Z \to Y$ is nearly H_1-Stieltjes representable with respect to α. □

Note that a bounded linear operator $T : L_1[a,b] \to L_1[a,b]$ is said to be positive if $T(f) \geq 0$ whenever $f \in L_1[a,b]$ and $f \geq 0$. This gives a lattice ordering of the class $L(L_1[a,b], L_1[a,b])$ of all bounded linear operators from $L_1[a,b]$ to $L_1[a,b]$. Define

$$T^+(f) = \sup\{T(g) : 0 \leq g \leq f\}$$

for $f \in L_1[a,b]$ and $f \geq 0$.

Bourgain [1] showed that if $T : L_1[a,b] \to L_1[a,b]$ is Dunford-Pettis operator, then the positive part T^+ of T is also a Dunford-Pettis operator.

By Gordon [3], a bounded linear operator $T : L_1[a,b] \to X$ is nearly representable if and only if $T \circ D : L_1[a,b] \to X$ is representable for all positive Dunford-Pettis operators $D : L_1[a,b] \to L_1[a,b]$.

Theorm 3.4. *A bounded linear operator $T : L_1[a,b] \to X$ is nearly Henstock-Stieltjes representable with respect to α if and if only $T \circ D : L_1[a,b] \to X$ is nearly H_1-Stieltjes representable with respect to α for all positive Dunford-Pettis operator $D : L_1[a,b] \to L_1[a,b]$.*

Proof. Suppose that a bounded linear operator $T : L_1[a,b] \to X$ is nearly H_1-Stieltjes representable with respect to α. Then it is clear that the composition $T \circ D : L_1[a,b] \to Y$ is H_1-Stieltjes representable with respect to α for all positive Dunford-Pettis operators $D : L_1[a,b] \to L_1[a,b]$.

Conversely, assume that $T \circ D : L_1[a,b] \to X$ is H_1-Stieltjes representable with respect to α for all positive Dunford-Pettis operators $D : L_1[a,b] \to L_1[a,b]$. Let $S : L_1[a,b] \to L_1[a,b]$ be any Dunford-Pettis operator.

By Bourgain[1], the positive part S^+ of S and the negative part S^- of S are both Dunford-Pettis operators. Hence $T \circ S^+$ and $T \circ S^-$ are both H_1-Stieltjes operators with respect to α and $T \circ S = T \circ (S^+ - S^-) = T \circ S^+ - T \circ S^-$ is H_1-Stieltjes representable with respect to α. Therefore $T : L_1[a,b] \to X$ is nearly H_1-Stieltjes representable with respect to α. □

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