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BEST PARAMETRIC APPROXIMATION IN $C_1(X)$

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ABSTRACT. In some problems of abstract approximation theory the approximating set depends on some parameter p. In this paper, we make a set $M(f)$ depends on the element f, φ and then best approximations are sought from a subset $M(f)$ of M.

1. INTRODUCTION

A theory of best approximation from a finite dimensional subspace U of a normed linear space X is developed. In particular, to each $x \in X$, best approximations are sought from a subset $U(x)$ of U which depends on the element x being approximated.

Let X be a compact set in $\mathbb{R}^d$ and $X = \overline{\operatorname{int} X}$ and μ a non-atomic positive finite measure defined on X, such that every real-valued continuous function is μ-measurable. Furthermore, μ is assumed to have the property that if

$$\|f\|_1 = \int_X |f| d\mu = 0 \quad \text{for } f \in C(X)$$

then $f = 0$.

We let $C_1(X)$ denote the linear space $C(X)$ with norm $\|\cdot\|_1$. The space $C_1(X)$ is not a Banach space. It is a dense linear subspace of $L_1(X, \mu)$ and we have

$$(C_1(X))^* = L^\infty(X, \mu).$$

Let M be a finite dimensional subspace of $C(X)$ and let $\varphi \in (C_1(X))^*$ with $\|\varphi\| = 1$. For each $f \in C_1(X)$, let

$$M(f) := \{y \in M | \ \varphi(y) \leq \varphi(f)\}. \tag{1}$$

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The elements of $M(f)$ are said to *interpolate* f relative to φ. We are interested in the problem

$$\inf_{y\in M(f)} \|f-y\|_1 = \inf_{y\in M(f)} \int_X |f-y|d\mu.$$

Set

$$P_{M(f)}(f) = \{y^* \in M(f) |\ \|f-y^*\|_1 \leq \|f-y\|_1 \text{ for all } y \in M(f)\}.$$

By definition (1), $M(f)$ is a closed convex subset of M. Thus if $M(f)$ is not empty, then there exists $y^* \in P_{M(f)}(f)$, which will be called a *best approximation* to f from $M(f)$.

Firstly, if φ is constant, $M(f) = M$. We know that every finite dimensional subspace of a normed linear space is proximinal, that is, for every $f \in C_1(X)$, there exists $m^* \in P_M(f)$ which is a best approximation to f from M in $C_1(X)$. So $P_{M(f)}(f) = P_M(f) \neq \varnothing$. This is one of the classical problems of approximation theory.

As a special case, φ is an identical operator, i.e.,

$$M(f) = \{m \in M |\ m \leq f\}$$

was developed by Pinkus [5]. He called *one-sided L^1-approximation.* Park & Rhee [4] has studied earlier the one-sided simultaneous L^1-approximation problem, the approximating set depends on ℓ-tuple $(f_1, f_2, \cdots, f_\ell)$ is

$$M(f_1, f_2, \cdots, f_\ell) = \{m \in M |\ m \leq f_i;\ i = 1, 2, \cdots, \ell\}.$$

The one-sided simultaneous L^1-approximation problem was generalized to compact sets [4].

Secondly, the approximating set

$$M(f) = \{m \in M |\ \varphi(m) = \varphi(f)\}$$

where $\varphi \in (C_1(X))^*$, was considered by Deutsch & Mabizela [2]. They said $m^* \in P_{M(f)}(f)$ is the *best interpolator approximation* of f relative to φ.

Our first question is "When is $M(f)$ not empty?"

The result establishes conditions each of which is sufficient to the statement that $M(f) \neq \varnothing$ for each $f \in C_1(X)$.

2. Best parametric approximation

Proposition 1. *Let $f \in C_1(X)$ and $\varphi \in (C_1(X))^*$.*

(1) *If φ is increasing (that is, if $0 \le u - v$ then $\varphi(v) \le \varphi(u)$) and M contains a strictly positive function, then $M(f) \ne \varnothing$.*

(2) *If φ is linearly independent over M (i.e., $\alpha\varphi(y) = 0$ for all $y \in M$ implies $\alpha = 0$), then $M(f) \ne \varnothing$.*

Proof. (1) Since M contains a strictly positive function, there exists $m_0 \in M$ such that $m_0 \le f$. So $\varphi(m_0) \le \varphi(f)$. Hence $m_0 \in M(f) \ne \varnothing$.

(2) Since φ is linearly independent over M, there exists $m \in M$ such that $\varphi(m) \ne 0$. Without loss of generality, we may assume that $\varphi(m) > 0$ and $\varphi(f) > 0$. There exists a natural number N such that $\varphi(m) \le N \cdot \varphi(f)$. So $\varphi(m/N) \le \varphi(f)$. Hence $m/N \in M(f)$. □

Let

$$X_0 = \{f \in C_1(X) |\ \varphi(f) \le 0\}$$

and

$$M_0 = \{m \in M |\ \varphi(m) \le 0\} = M(0) = X_0 \cap M.$$

Then X_0 is a convex cone in $C_1(X)$, and M_0 is a convex cone in M.

Define $L : C_1(X) \longrightarrow M$ by $L(f) = \varphi(f)m_0$ where $\varphi(m_0) = 1, m_0 \in M$. Then L is a linear projection onto $\mathrm{span}(m_0)$.

Remark 2. (1) L is a bounded linear operator with $\|L\| = \|m_0\|$. In fact,

$$\begin{aligned}\|L\| &= \sup_{\|f\|_1=1} \|L(f)\| \\ &= \sup_{\|f\|_1=1} \|\varphi(f)m_0\| \\ &= \sup_{\|f\|_1=1} \|m_0\|\,|\varphi(f)| \\ &= \|m_0\| \sup_{\|f\|_1=1} |\varphi(f)| \\ &= \|m_0\| \cdot \|\varphi\| \\ &= \|m_0\|.\end{aligned}$$

(2) For any $m \in \mathrm{span}(m_0)$, $L(m) = m$.

(3) L is idempotent, that is,

$$L(L(f)) = L(\varphi(f)m_0) = \varphi(f)L(m_0) = \varphi(f)m_0 = L(f)$$

for any $f \in C_1(X)$.

(4) $L^{-1}(0) = \{f - L(f) |\ f \in C_1(X)\} \subset X_0$.

(5) $C_1(X) = X_0 \oplus \mathrm{span}(m_0)$.

Theorem 3. *For any $f \in C_1(X), M(f) = M_0 + L(f)$, and*

$$P_{M(f)}(f) = P_{M_0}(f - L(f)) + L(f).$$

Proof. For any $f \in C_1(X), y \in M(f)$ if and only if $y \in M$ and $\varphi(y) \leq \varphi(f)$ if and only if $y \in M$ and $\varphi(y - f) \leq \varphi(0) = 0$ if and only if $y \in M$ and $\varphi(y) - \varphi(L(f)) \leq 0$ if and only if $y \in M$ and $\varphi(y - L(f)) \leq 0$ if and only if $y \in M_0 + L(f)$. And

$$\begin{aligned}
&P_{M(f)}(f)\\
&= P_{M_0+L(f)}(f)\\
&= \{x^* + L(f) \mid x^* \in M_0, \|f - (x^* + L(f))\|_1 \leq \|f - (x + L(f))\| \text{ for all } x \in M_0\}\\
&= \{x^* \mid x^* \in M_0, \|(f - L(f)) - x^*\|_1 \leq \|(f - L(f)) - x\| \text{ for all } x \in M_0\} + L(f)\\
&= P_{M_0}(f - L(f)) + L(f)
\end{aligned}$$

where $y = x + L(f) \in M_0$ and $y^* = x^* + L(f) \in M_0$. □

Corollary 4. *Suppose that M is a finite dimensional subspace of $C_1(X)$ and φ is linearly independent over M. Then the closed subset $P_{M(f)}(f) \neq \varnothing$ for each $f \in C_1(X)$. And $P_{M(f)}(f)$ is singleton for each $f \in C_1(X)$ if and only if M_0 is Chebyshev.*

Let

$$M_0^* = \{m \in M \mid \varphi(m) = 0\}.$$

Then M_0^* is an $(n-1)$-dimensional subspace in M_0 where $n = \dim M$ (cf. Deutsch and Mabizela [2]).

Theorem 5. *For any $f \in C_1(X), d(f, M(f)) = d(f, M'(f))$ and*

$$P_{M(f)}(f) = P_{M'(f)}(f)$$

where $M'(f) = \{m \in M \mid \varphi(m) = \varphi(f)\}$.

Proof. For any $f \in C_1(X), M'(f) \subset M(f)$. So $d(f, M(f)) \leq d(f, M'(f))$. We must show that

$$d(f, M(f)) \geq d(f, M'(f)).$$

If $m_0 \in M(f)$ with $d(f, m_0) = d(f, M(f))$ then $m_0 \in M'(f)$. In fact, for any $\varepsilon > 0$, there exist u and v in ε-neighborhood of m_0 such that $\varphi(u) \leq \varphi(f)$ and $\varphi(v) > \varphi(f)$ that is, $\varphi(m_0) = \varphi(f)$. So, $d(f, M(f)) \geq d(f, M'(f))$ and

$$d(f, M'(f)) \leq d(f, m_0) = d(f, M(f)) \leq d(f, M'(f))$$

Hence $m_0 \in P_{M'(f)}(f)$.

Conversely, if $m' \in P_{M'(f)}(f)$ then $m' \in M'(f) \subset M(f)$ and

$$d(f,m') = d(f,M'(f)) = d(f,M(f)).$$

Thus $m' \in P_{M(f)}(f)$. □

Corollary 6. *Suppose that $f \in C_1(X)$ with $\varphi(f) = 0$. Then*

$$P_{M(f)}(f) = P_{M_0^*}(f).$$

Theorem 7. *Let $f \in C_1(X)$. Then the followings are equivalent:*

(1) $m \in P_{M(f)}(f)$.

(2) $m - L(f) \in P_{M_0^*}(f')$, *where* $f' = f - L(f)$.

(3) *There exist k extremal points $f_i \in S_{(C_1(X))^*}$ where $1 \leq k \leq n$, and k scalars $\lambda_i > 0$ such that*

$$\sum_{i=1}^{k} \lambda_i f_i \in {M_0^*}^{\perp}.$$

(4) $0 \in \mathrm{co}\{(f^*(m_1), \cdots f^*(m_{n-1}))|\ f^*$ *is an extremal point of* $S_{(C_1(X))^*}\}$ *where* $\{m_1, \cdots m_{n-1}\}$ *is any basis for* M_0^*.

Proof. For any $f \in C_1(X), (1) \Leftrightarrow (2)$ by Theorem 3. And $(2) \Leftrightarrow (3) \Leftrightarrow (4)$ by Singer [6, Theorem 1.1, Chap. 2]. □

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