

A Rationale for Mathematical Problem Solving on a Small Group — Focusing on Collaborative Interaction

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The purpose of this study is to examine a theoretical framework for the interactions of learning in a small group setting of mathematical problem solving. Many researchers already have described the theoretical background for the small group settings in problem solving. However, most of the literatures merely have reported findings of achievement and rising of test scores. They ignored the observation of process taken during the small group work and have not determined how various psychological, social and academic effects are created. As results of the study, two types, mutual collaboration and asymmetric collaboration, of interactions are observed as the interactions of learning, which are conceived as the cores of authentic mathematical activities.

1. INTRODUCTION

In this article, 'small group work' means the situation in which two or more learners attempt to engage in common task together¹.

Many mathematical educators are interested in the use of small group work. In Korea, educational practice has seen increasing interest in teaching-learning on a small group setting in the mathematics classroom. This trend concurs with the introduction of new differentiated curriculum² (Ministry of Education, 1998, p. 22).

Recently, many researchers of mathematical problem solving, who had traditionally focused on individual activity, have begun to consider problem solving as an aspect of the

¹ According to the use of the word 'group' in Webb & Palincsar (1996), a group is constituted by persons engaged in a common task who are interdependent in the performance of the task and interaction in its pursuit.

² To facilitate student's ability, needs, disposition and others, the curriculum is based on level-dependent learning and ability grouping instruction.

social thinking process³. Results (Noddings, 1985; Davidson & Kroll, 1991; Artzt & Armour-Thoman, 1992; Good, Mulryan, & McCaslin, 1992; Webb & Palincsar, 1996) from many researches have described the effects on academic achievement.

It, however, is not clear that under what conditions the use of small group problem solving is more effective. Lester (1994) suggested that research should focus on the role of the teacher, on the interaction between teacher-student and student-student, and on groups and whole classes rather than individuals. Therefore, to help identify these conditions, this paper has focused upon the dynamic process of collaborative interaction of mathematical problem solving in small group settings.

Within the above perspective, three questions are raised.

- 1) What is the theoretical basis for the use of a small group setting in mathematical problem solving?
- 2) What is the theoretical framework for the description and prescription of peer interactions during small group activity in mathematical problem solving?
- 3) What are the types of peer interactions that can be helpful for problem solving, particularly while doing authentic mathematical activities?

2. THEORETICAL FOUNDATION

In this section begins with examining the theoretical background for the use of a small group setting in mathematical problem solving.

The teaching-learning practices in mathematics possibly rest upon implicit epistemologies or philosophies of mathematics, and any philosophy of mathematics has powerful implications for theories on the teaching and learning of mathematics (Steiner, 1987, p. 8). That is, the association of the teaching-learning practices in mathematics and the philosophical perspectives of mathematics are bi-directional. Thus, based on the view on mathematics (Ernest, 1994) and the view on mathematical classroom practice (Lampert, 1990), we set out the theoretical foundation for the use of a small group setting in problem solving.

According to Ernest (1994), mathematics is basically conversational and dialogical. He sketches an account of mathematics based on the underlying explanatory metaphor of conversation and dialogue.

“A view of mathematics suggesting that it has a dialogical nature which encompasses its textual basis, some of its concepts, the origins and nature of proof, and the social processes

³ Nohda (1991) as well as Japanese mathematics education community regard mathematical problem solving as an aspect of the social thinking process and as an organizing principle of all levels of mathematics teaching-learning.

whereby mathematical knowledge is created, warranted and learnt. Taking conversation as epistemologically basic re-grounds mathematical knowledge in socially situated acts of human knowing and communication". (Ernest, 1994, p. 44)

This view of mathematics reflects the dialectic nature of mathematics and represents that mathematics is socially constructed and is on the process of development.

According to Lampert (1990), knowing mathematics through the practice of mathematical classroom is closer to knowing mathematics as a discipline. That is, learning mathematics in the way, which is congruent with disciplinary discourse, leads students to engage in authentic mathematical activity.

"It might be possible to bring the practice of knowing mathematics in school closer to what it means to know mathematics within the discipline by deliberately altering the roles and responsibilities of teacher and students in classroom discourse". (Lampert, 1990, p. 29)

The foundations for this view on the practice of mathematical classroom are the teacher initiated and supported social interactions, appropriate to making mathematical arguments in response to the students' activities, which are asserting and examining conjectures about the mathematical structures that underlie their solutions to the problems.

Both Ernest (1994) and Lampert (1990) explained that mathematics learning is a process of social construction, and that the patterns of interaction are influential in solving mathematics problems and participating in a mathematics discourse during a mathematics lesson. Problem solving stimulates mathematical activity and provides the context for working out the relationships between mathematical ideas. The descriptions of doing mathematics have centered on solving genuine mathematical problems.

Working with problem solving is central at all levels of mathematics teaching and learning (Nohda, 1990), and doing mathematics is a collaborative activity. It depends upon communication and social interaction. Peer interaction is especially helpful because the differences in thinking are likely to be within a range that will generate genuine and fruitful conflict. Students often challenge each other in ways that they can make sense of and deal with productively for the purpose of sharing and developing mathematical thinking (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne, 1997, pp. 44–46). According to these perspectives, the theoretical basis for problem solving in a small group setting has been set. Particularly, there are two reasons to use a small group setting in mathematical problem solving; one is for the theoretical reason and the other is for the practical reason.

Theoretically, using the small group setting for problem solving enhances learning opportunities (Yackel, Cobb & Wood, 1991). In Korea, the most common way of teaching-learning of mathematics in school is in the form of whole-classroom teaching, a teaching style in which a teacher teaches the same contents to all students at the same time. Problem solving is one of the objectives of mathematics education in Korea

(Ministry of Education, 1998). Korean educators try to attain this objective in whole-classroom teaching, but this task is very difficult because there are too many students in a class and too few opportunities for the students to present and share their ideas. Also teachers are not patient with students to finish the problem solving.

According to the statistics of the Korean Educational Development Institute (2000), the average number of students in a classroom in Korea is over 38. For this reason, practicing the discourse community described by Lampert (1990) involves many problems in Korea. To resolve these practical problems, a small group setting may provide an environment that is more conducive than a whole-class arrangement.

3. A FRAMEWORK OF PROCESS-OBSERVATIONAL STUDY

A theoretical framework of the process-observational study has been developed. In order to find out a theoretical basis for analyzing the process of mathematical problem solving in a small group, the interaction model developed by Granott (1993) and the idea of "problem transformation" identified by Shimizu (1989) are suitable for the description of the integration.

In traditional way, if we give a mathematical problem to the class in the individual basis, most students just give up and only one or two students are working on. As the problem solving in a small group setting progresses, all the students make at least some progress toward forms of solution requiring more coordination and toward increased involvement in their partner's action. That is, during the process of problem solving in a small group setting, qualitatively, there are two different types of activities, which the students are engaged in. One is solving problem and the other is interacting with partners. These two activities are closely related to each other. Therefore, an integrated framework is necessary for mathematical problem solving in a small group.

3.1 Interaction Model

"Collaboration is a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem". (Roschelle & Teasley, 1995, p. 70)

The interaction model developed by Granott (1993) suggests two major dimensions from which interactions can be analyzed.

One dimension is the degree of collaboration. Collaborative patterns can involve a high level of mutuality and evolve through shared understanding and a common focus of attention. Yet, other interactions may only involve some exchange of ideas and turn-taking activity.

The other dimension refers to the participants' relative knowledge and expertise in the context of the interaction.

For analysis, these two dimensions are conceptually continuous. So we can modify the interaction model to analyze mathematical problem solving in a small group.

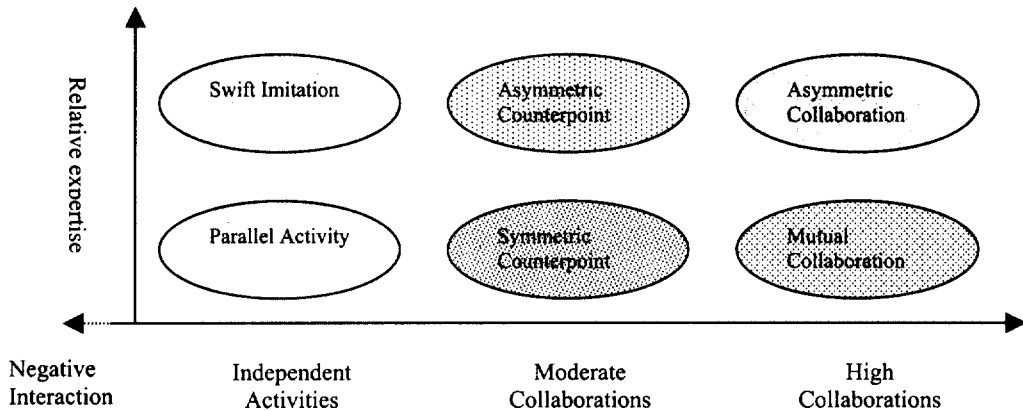


Figure 1. Interaction Model

Figure 1 shows the framework that is set up to categorize different types of peer interaction. The horizontal dimension represents the level of collaboration. The vertical dimension characterizes relative knowledge and expertise between the participants.

- 1) Mutual collaboration is characterized by a high level of interaction between peers of equal expertise.
- 2) Symmetric counterpoint occurs between peers of equal expertise who interact while alternating dominance on an activity.
- 3) Parallel activity is an interaction among peers of symmetric expertise, engaged in an activity that is mostly independent.
- 4) Asymmetric collaboration represents a collaborative interaction between peers of some asymmetric expertise.
- 5) Asymmetric counterpoint is a moderately collaborative interaction among peers of some asymmetric expertise.
- 6) Swift imitation is an interaction among peers of moderately asymmetric expertise, engaged in an activity that is mostly independent (see Granott, 1993, for details).

These diverse types of peer interactions can promote the participants' cognitive growth in different ways. The interactions of a high level of collaboration, mutual collaboration and asymmetric collaboration are characterized by a united effort and continuous sharing. These high level interactions are defined as 'collaborative

interaction' when the peers' action must merge into a single activity with the students working toward a common shared goal. This modified framework for analyzing interactions according to the level of collaboration and relative expertise can be used as protocol for small group activity in mathematical problem solving.

3.2 Problem Transformation

"Actual thinking is a process; it occurs, goes on; in short, it is in continual change as long as a person thinks". (Dewey, 1933, p. 72)

Working on a problem, the student may transform the problem at hand to an easier one, for example, an auxiliary problem. Thus, problem solving is regarded as repeated transformations of the original problem (Shimizu, 1989). Problem solving in a small group acknowledges that problems arise for students as they attempt to solve their original problem. The situations that students find problematic generate a variety of forms in the problem transformation, some content related and some non-content related. These include "resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts and procedures, accounting for a surprising outcome (particularly when two alternative procedures lead to the same result), verbalizing their mathematical thinking, explaining or justifying a solution, resolving conflicting points of view, developing a framework that accommodates alternative solution methods, and formulating an explanation to clarify another child's solution attempts" (Yackel et al., 1991, p. 395).

3.3 Theoretical Observational Framework

To analyze the process of mathematical problem solving by using a theoretical framework, we use the protocols of small group activity. Protocols of small group activity in mathematical problem solving are partitioned into macroscopic chunks of consistent problem contents. Each macroscopic chunk is the recurring change of problem contents and is termed 'episode' in this paper. That is, an episode is a period of time during which a problem-solving group is engaged in the original problem.

Once a protocol has been parsed into episodes, each episode is characterized by the following types of interactions: mutual collaboration, symmetric counterpoint, parallel activity, asymmetric collaboration, asymmetric counterpoint, and swift imitation. In the process of problem-solving activity of a small group, every episode requires different levels of mutual coordination of action, with parallel activity and swift imitation needing the least coordination, and mutual collaboration and asymmetric collaboration needing the most.

To illustrate the utility of this theoretical observation framework, a case study for pilot

testing was implemented (see Lee & Nohda, 2000, for further details). Figure 2 shows an analysis of a case study which was undertaken by a group of two college students (L and J) working on the “painted cube” problem.

“Painted Cube” Problem: Suppose a large cube is built from 1,000 small cubes and then painted on all six faces. When the large cube is disassembled, how many of the small cubes will be unpainted? Separate the remaining small cubes into groups according to the number of faces painted and compute the number of cubes in each group.

Time	(3' 12")	(7' 25")	(15' 24")	(21' 29")	(39' 52")	(54' 58")
EPISODE 1	EPISODE 2	EPISODE 3	EPISODE 4	EPISODE 5	EPISODE 6	
Moves* 1–41	Moves 42–70	Moves 71–129	Moves 130–156	Moves 157–234	Moves 235–274	
Symmetric Counterpoint	Mutual Collaboration	Asymmetric Counterpoint	Symmetric Counterpoint	Asymmetric Counterpoint	Asymmetric Collaboration	

* The turns of speaking are numbered sequentially and referred to as ‘Moves’.

Figure 2. Protocol Analysis of Case Study Using the Process-Observational Framework

Here, we present two episodes that were identified as “mutual collaboration” and “asymmetric collaboration”.

Episode 2 (Moves 42-70): Mutual Collaboration

During this episode L and J worked in a ‘collaborative completion’ type turn-taking⁴ on finding a large cube which is built from 1000 small cubes. J imagined a $2 \times 2 \times 2$ cube and found the fact that it is constructed by 8 small cubes: two layers which consisted of 4 small cubes.

Following J’s approach, L extended it to the case of $3 \times 3 \times 3$ and found the property of the cube, that is, the cube is built from a product of a number multiplied by its square. By the mutual collaboration, L and J could build a large cube from 1000 small cubes and abstract constructing cube.

Episode 6 (Moves 235-274): Asymmetric Collaboration

The structure of the original problem dictates that there are sub-problems to be solved.

⁴ ‘Collaborative completion’ type turn-taking is as follows: “One partner’s turn begins a sentence or an idea, and the other partner uses their next turn to complete it.” (Roschelle & Teasley, 1995, p. 77)

In this episode, L and J took a while to get started. Despite the fact that L directed the sequence of activity and that she was the major source of new ideas and procedures throughout this episode, J's vague utterance of 'position' was the key suggestion to L.

240. J: Three groups. Are there? Isn't there a relation to position?

241. L: Position? Of course, they are related to position. There is a three painted face cube at vertex, two painted face cube here (pointing to the edge), and one painted face cube here (pointing to the face).

With J's help, L began to systematically organize groups according to the number of painted faces based on their position (i.e., vertex, edge, or face). Then she computed the number of cubes in each group with J checking (see Table 1).

Table 1. The number of painted faces and the number of unit cubes with paint

The number of painted faces	The number of unit cubes with paint	Generalization $n \times n \times n$
No painted face	$(10 - 2)^3 = 512$	$(n - 2)^3$
One painted face	$(10 - 2)^2 \times 6 = 384$	$6(n - 2)^2$
Two painted faces	$(10 - 2) \times 12 = 96$	$12(n - 2)$
Three painted faces	8	8

4. CONCLUSION

In fact, there are so many terms related to learning based on a small group setting: small group learning, peer tutoring, cooperative learning, collaborative learning, and so on. Frequently, they are not properly defined and used interchangeably. Also, many people imagine a small group work as a cooperative learning.

In this article, we differentiate between cooperative and collaborative learning. In the literature (Webb & Palinscsar, 1996), the characteristics of the cooperative learning approaches are different roles assigned to group members by a teacher and the use of external rewards that attempt to foster cooperative skills and to promote a high achievement score. In contrast, the intention of the collaborative learning approaches is to work together willingly to attain a common goal, without external role assignments and extrinsic reward systems⁵. Therefore, in collaborative mathematical problem solving, the interlocutors are working together willingly to solve a shared problem with mutuality and equality. For this reason, collaborative interactions that involve the mutual engagement

⁵ 'Although certain forms of cooperative learning can occur without collaboration, collaborative learning is generally assumed to subsume cooperation' (Webb & Palinscsar, 1996, p. 848)

of participants in a coordinated effort are the essential to solve the mathematical problem together in a small group setting.

However, for prescribing into the mathematical classroom, it needs a more thorough examination which links the ability to work collaboratively with the students' development of problem solving. That is, the effect of interaction on cognitive development. Furthermore, to nurture collaborative interaction, the most suitable conditions need to be identified in the aspects of task, tool, and classroom culture.

Consequently, there is still a great deal of confusion and disagreement about why collaborative mathematical problem solving affects achievement. Thus, in order to implement this framework into the mathematical classroom, future research is needed to gain a better understanding of how interrelationships between the ability to work collaboratively and the students' development of problem solving affect, and to outline under what conditions collaborative mathematical problem solving is effective.

REFERENCES

- Artzt, A. F. & Armour-Thoman, E. (1992): Development of a cognitive-metacognitive framework for protocol analysis of mathematical problem solving in small groups. *Cogn. Instr.* **9(2)**, 137–175. MATHDI 1993e.00587
- Davidson, N. & Kroll, D. L. (1991): An overview of research on cooperative learning related to mathematics. *Journal for Research in Mathematics Education* **22(5)**, 362–365. MATHDI 1992h.00251
- Dewey, J. (1933): *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: Heath.
- Ernest, P. (1994): The dialogical nature of mathematics. In: P. Ernest (Ed.), *Mathematics, education and philosophy: An international perspective* (pp. 33–48), Studies in Mathematics Education Series, v. 3. London: Falmer Press. MATHDI 1994f.03133
- Good, T. M.; Mulryan, C. & McCaslin, M. (1992): Grouping for instruction in mathematics: A call for programmatic research on small-group processes. In: D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 165–196). A project of the National Council of Teachers of Mathematics. New York, NY: Macmillan. MATHDI 1993f.01809
- Granott, N. (1993): Patterns of interaction in the co-construction of knowledge: Separate minds, joint efforts, and weird creatures. In: R. H. Fischer (Ed.), *Development in context* (pp. 183–207). Hillsdale, NJ: LEA.
- Hiebert, J.; Carpenter, T. P.; Fennema, E.; Fuson, K. C.; Wearne, D.; Murray, H.; Olivier, A. & Human, P. (1997): *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann. MATHDI 1999b.01311

- Korean Educational Development Institute (2000): *Statistical yearbook of education 2000*. Seoul: Korean Educational Development Institute.
- Lampert, M. (1990): When the problem is not the question and solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal* **27**(1), 29–63.
- Lee, Y.-S. & Nohda, N. (2000): The process of collaborative mathematical problem solving: Focusing on emergent goals perspective. *Journal of Science Education in Japan* **24**(3), 159–169.
- Lester, F. K. (1994): Musing about mathematical problem solving research: 1970–1994. *Journal for Research in Mathematics Education* **25**(6), 660–675. MATHDI **1995d.02520**
- Ministry of Education (1998): *7th National Curriculum (Mathematics)*. Seoul: Ministry of Education.
- Noddings, N. (1985): Small groups as a setting for research on mathematical problem solving. In: E. A. Silver (Ed.), *Teaching and learning mathematics problem solving: Multiple research perspectives* (pp. 345-59). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Nohda, N. (1990): Actual communication in the mathematical classroom. In: L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: An international perspective* (pp. 266–271). Hillsdale, NJ: Erlbaum. MATHDI **1991h.03354**
- ____ (1991): Paradigm of the “open-approach” method in mathematics teaching: Focus on mathematical problem solving. *ZDM, Zentralbl. Didakt. Math.* **23**(2), 32–37. MATHDI **1991c.00860**
- Roschelle, J. & Teasley, S. D. (1995): The construction of shared knowledge in collaborative problem solving. In: C. O'Malley (Ed.), *Computer-supported collaborative learning* (pp. 69–97). New York, NY: Springer-Verlag.
- Shimizu, Y. (1989): Metacognition in solution processes of junior high school students on a construction problem. *Journal of Japan Society of Mathematical Education* **52**, 3–25.
- Steiner, H. G. (1987): Philosophical and epistemological aspects of mathematics and their interaction with theory and practice in mathematics education. *Learn. Math.* **7**(1), 7–13. MATHDI **1987k.00584**
- Webb, N. M. & Palincsar, A. S. (1996): Group processes in the classroom. In: D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 841–873). New York, NY: Macmillan.
- Yackle, E.; Cobb, P. & Wood, T. (1991): Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education* **22**(5), 390–408. MATHDI **1992h.00253**