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DEMI-CLOSED PRINCIPLE AND WEAK CONVERGENCE PROBLEMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. A demi-closed theorem and some new weak convergence theorems of iterative sequences for asymptotically nonexpansive and nonexpansive mappings in Banach spaces are obtained. The results presented in this paper improve and extend the corresponding results of [1], [8]-[10], [12], [13], [15], [16], and [18].

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space and E^* is the dual space of E. Let D be a nonempty subset of E and $F(T)$ denote the set of fixed points of T.

DEFINITION 1.1. Let $T : D \to D$ be a mapping.

(1) T is said to be *asymptotically nonexpansive* ([7]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \tag{1.1}$$

for all $x, y \in D$ and $n \geq 1$.

(2) T is said to be *uniformly L-Lipschitzian* if

$$||T^n x - T^n y|| \leq L||x - y|| \tag{1.2}$$

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for all $x, y \in D$ and $n \geq 1$, where L is a positive constant.

REMARK 1.1. (1) It is easy to see that, if $T : D \to D$ is a nonexpansive mapping, then T is an asymptotically nonexpansive mapping with a constant sequence $\{1\}$.

(2) If $T : D \to D$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$, then it must be uniformly L-Lipschitzian with $L = \sup_{n\geq 1}\{k_n\}$.

DEFINITION 1.2. Let E be a real Banach space and D be a closed subset of E. A mapping $T : D \to D$ is said to be *demi-closed* at the origin if, for any sequence $\{x_n\}$ in D, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

Recall that E is said to satisfy *Opial's condition* if, for each sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{1.3}$$

for all $y \in E$ with $y \neq x$, and also that E is said to have the *Fréchet differentiable norm* if, for each $x \in S(E)$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in S(E)$, where $S(E)$ denotes the unit sphere of E.

The weak and strong convergence problems to the fixed points for nonexpansive and asymptotically nonexpansive mappings have been studied by many authors (for example, see [1], [3]-[18] and the references therein).

In 1972, Goebel and Kirk [7] proved that, if D is a bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive mapping $T : D \to D$ has a fixed point in D. Subsequently, Bose [1] first proved that, if D is a bounded closed convex subset of a uniformly convex Banach space E satisfying Opial's condition and $T : D \to D$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in D$, i.e.,

$$\lim_{n\to\infty} ||T^n x - T^{n+1} x|| = 0.$$

Furthermore, Passty [9] and Xu [18] proved that, if Opial's condition of E is replaced by the condition that E has the Fréchet differentiable norm, then this conclusion still also holds. In addition, Tan and Xu [15], [16] proved that, in both cases, the asymptotic regularity of T at x can be weakened to the weakly asymptotic regularity of T at x, i.e.,

$$w - \lim_{n\to\infty} (T^n x - T^{n+1}x) = 0.$$

Recently, Tan and Xu [13] proved the following theorem:

THEOREM A. *Let E be a real uniformly convex Banach space which satisfies Opial's condition or has the Fréchet differentiable norm, D be a nonempty bounded closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=0}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $0 < a_1 \le \alpha_n \le a_2 < 1, \quad n \ge 0,$

(ii) $0 \le \beta_n \le b, \quad n \ge 0,$

where $a_1, a_2, b \in (0, 1)$ are some constants. For any given $x_0 \in D$, let $\{x_n\} \subset D$ be a sequence defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, & n \ge 0, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, & n \ge 0. \end{cases} \tag{1.4}$$

Then the sequence $\{x_n\}$ converges weakly to a fixed point of T in D.

The purpose of this paper is to prove a new demi-closed principle and then, by using this principle, to prove some convergence theorems for asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces without assuming any one of the following conditions:

(1) E satisfies Opial's condition or E has the Fréchet differentiable norm,

(2) T is asymptotically regular or weakly asymptotically regular,

(3) D is a bounded subset of E.

Our main results presented in this paper improve and generalize the corresponding results of [1], [8]-[10], [12], [13], [15], [16] and [18].

The following four theorems are the main results of this paper:

THEOREM 1 (Demi-closed Principle). *Let E be a uniformly convex Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in D, if the sequence $\{x_n\}$ converges weakly to $q \in D$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

THEOREM 2. *Let E be a real uniformly convex Banach space, D be a nonempty closed convex subset of E, $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Suppose that the set $F(T)$ of fixed points of T in D is nonempty and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:*

(i) *There exist positive integers n_0, n_1 and $\epsilon > 0$, $0 < b < \min\{1, \frac{1}{L}\}$, where $L = \sup_{n \geq 0} k_n$, such that*

$$\begin{aligned} &0 < \epsilon \leq \alpha_n \leq 1 - \epsilon, \quad n \geq n_0, \\ &0 \leq \beta_n \leq b, \quad n \geq n_1, \end{aligned} \tag{1.5}$$

(ii) $\sum_{n=0}^{\infty}(k_n^2 - 1) < \infty$.

Then the Ishikawa iterative sequence $\{x_n\}$ defined by (1.4) converges weakly to some fixed point x^* of T in D.

THEOREM 3. *Let E be a real uniformly convex Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Suppose that the set $F(T)$ of fixed points of T in D is nonempty and $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying the following conditions:*

(i) *There exist positive integers n_0 and $\epsilon > 0$ such that*

$$0 < \epsilon \leq \alpha_n \leq 1 - \epsilon, \quad n \geq n_0,$$

(ii) $\sum_{n=0}^{\infty}(k_n^2 - 1) < \infty$.

Then the Mann iterative sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in D, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

converges weakly to some fixed point x^* of T in D.

THEOREM 4. *Let E be a real uniformly convex Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be an nonexpansive mapping. Suppose that the set $F(T)$ of fixed points of T in D is nonempty and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:*

There exist positive integers n_0, n_1 and $\epsilon > 0$, $0 < b < 1$ such that

$$\begin{aligned} &0 < \epsilon \le \alpha_n \le 1 - \epsilon, \quad n \ge n_0, \\ &0 \le \beta_n \le b, \quad n \ge n_1. \end{aligned} \tag{1.7}$$

Then the Ishikawa iterative sequence $\{x_n\}$ defined by (1.4) converges weakly to some fixed point x^* of T in D.

The following lemmas play an important role in proving our main results:

LEMMA 1 ([17]). *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^p \le \lambda||x||^p + (1-\lambda)||y||^p - \omega_p(\lambda)g(||x-y||)$$

for all $x, y \in B(0,r)$ and $0 \le \lambda \le 1$, where $B(0,r)$ is the closed ball of E with center zero and radius r and

$$\omega_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p. \tag{1.8}$$

LEMMA 2 ([11]). *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$. Let $\{x_n\}$ be the Ishikawa iterative sequence defined by (1.4). Then the condition $||x_n - T^n x_n|| \to 0$ as $n \to \infty$ implies that $||x_n - Tx_n|| \to 0$ as $n \to \infty$.*

LEMMA 3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following conditions is satisfied:*

(i) $a_{n+1} \le a_n + b_n, \quad n \ge 1,$
(ii) $a_{n+1} \le (1 + b_n)a_n, \quad n \ge 1,$

then the limit $\lim_{n\to\infty} a_n$ *exists.*

Proof. If the condition (i) is satisfied, then the conclusion of Lemma 3 can be obtained from Tan-Xu [14, Lemma 1].

Next, if the condition (ii) is satisfied, then we have

$$\begin{aligned} a_{n+1} &\leq (1+b_n)a_n \leq (1+b_n)(1+b_{n-1})a_{n-1} \\ &\leq \cdots \leq \prod_{j=1}^{n}(1+b_j)a_1 \leq \prod_{j=1}^{\infty}(1+b_j)a_1 \\ &= M < \infty, \end{aligned}$$

which implies that $\{a_n\}$ is a bounded sequence and so we have

$$a_{n+1} \leq (1+b_n)a_n \leq a_n + Mb_n, \quad n \geq 1.$$

Therefore, the conclusion (ii) can be obtained from the conclusion (i). This completes the proof. □

LEMMA 4. *Let E be a normed space, D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ and $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$. If $F(T)$ is nonempty in D and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$. Then the limit $\lim_{n\to\infty}||x_n - q||$ exists for all $q \in F(T)$, where $\{x_n\}$ is the Ishikawa iterative sequence defined by (1.4).*

Proof. Taking $q \in F(T)$, by (1.4), we have

$$\begin{aligned} ||x_{n+1} - q|| &= ||(1-\alpha_n)(x_n - q) + \alpha_n(T^n y_n - q)|| \\ &\leq (1-\alpha_n)||x_n - q|| + \alpha_n k_n ||y_n - q|| \\ &= (1-\alpha_n)||x_n - q|| \\ &\quad + \alpha_n k_n ||(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - q)|| \\ &\leq (1-\alpha_n)||x_n - q|| \\ &\quad + \alpha_n k_n \{(1-\beta_n)||x_n - q|| + \beta_n k_n ||x_n - q||\} \\ &= \{1 + \alpha_n(k_n - 1)(k_n\beta_n + 1)\}||x_n - q|| \\ &\leq \{1 + (k_n^2 - 1)\}||x_n - q||, \quad n \geq 1. \end{aligned}$$

Taking $a_n = ||x_n - q||$ and $b_n = k_n^2 - 1$ in Lemma 3, the conclusion of Lemma 4 can be obtained from Lemma 3 immediately. This completes the proof. □

LEMMA 5 ([2]). *Let E be a uniformly convex Banach space, C be a nonempty bounded closed convex subset of E. Then there exists a strictly increasing continuous convex function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ such that, for any Lipschitzian mapping $T : C \to E$ with the Lipschitz constant $L \geq 1$, any finite many elements $\{x_i\}_{i=1}^n$ in C and any finite many nonnegative numbers $\{t_i\}_{i=1}^n$ with $\sum_{i=1}^n t_i = 1$, the following inequality holds:*

$$\begin{aligned}&\Big\|T\Big(\sum_{i=1}^n t_i x_i\Big) - \sum_{i=1}^n t_i T x_i\Big\| \\ &\leq L f^{-1}\{\max_{1\leq i,j\leq n}(\|x_i - x_j\| - L^{-1}\|Tx_i - Tx_j\|)\}.\end{aligned}$$

2. The proofs of Theorems 1∼4

Now we are ready to prove our main results in this paper.

Proof of Theorem 1. Since $\{x_n\}$ converges weakly to $q \in D$, $\{x_n\}$ is a bounded sequence in D. Therefore, there exists $r > 0$ such that $\{x_n\} \subset C =: D \cap B(0,r)$, where $B(0,r)$ is a closed ball of E with center 0 and radius r and so C is a nonempty bounded closed convex subset in D.

Next, we prove that, as $n \to \infty$,

$$T^n q \to q. \tag{2.1}$$

In fact, since $\{x_n\}$ converges weakly to q, by Mazur's theorem, for each positive integer $n \geq 1$, there exists a convex combination $y_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}$ with $t_i^{(n)} \geq 0$ and $\sum_{i=1}^{m(n)} t_i^{(n)} = 1$ such that

$$\|y_n - q\| < \frac{1}{n}. \tag{2.2}$$

Again, since $\{(I-T)x_n\} \to 0$ strongly, for any given $\epsilon > 0$ and positive integer $j \geq 1$, there exists a positive integer $N = N(\epsilon, j)$ such that $\frac{1}{N} < \epsilon$ and

$$\|(I-T)x_n\| \leq \frac{1}{(1 + k_1 + \cdots + k_{j-1})} < \epsilon, \quad n \geq N.$$

Hence, for any $n \geq N$, we have

$$\begin{aligned} ||(I-T^j)x_n|| &\leq ||(I-T)x_n|| + ||(T-T^2)x_n|| \\ &\quad + \cdots + ||(T^{j-1}-T^j)x_n|| \\ &\leq (1+k_1+\cdots+k_{j-1})||(I-T)x_n|| \\ &< \epsilon. \end{aligned} \tag{2.3}$$

Since $T : D \to D$ is asymptotically nonexpansive, $T : C \to D$ is also an asymptotically nonexpansive mapping. Therefore, $T^j : C \to D$ is a Lipschitzian mapping with the Lipschitz constant $k_j \geq 1$. Next, we consider the following inequality:

$$\begin{aligned} ||T^j y_n - y_n|| &= \Big\| T^j y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} \\ &\quad + \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} - \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n} \Big\| \\ &\leq ||T^j y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n}|| \\ &\quad + \sum_{i=1}^{m(n)} t_i^{(n)} ||T^j x_{i+n} - x_{i+n}||. \end{aligned} \tag{2.4}$$

By (2.3), we know that

$$\sum_{i=1}^{m(n)} t_i^{(n)} ||T^j x_{i+n} - x_{i+n}|| < \epsilon, \quad n \geq N. \tag{2.5}$$

Moreover, by Lemma 5 and (2.3), we have

$$\begin{aligned} &\Big\| T^j y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} \Big\| \\ &\leq k_j f^{-1}\{\max_{1\leq i,k\leq n} (||x_{i+n} - x_{k+n}|| - k_j^{-1}||T^j x_{i+n} - T^j x_{k+n}||)\} \\ &\leq k_j f^{-1}\{\max_{1\leq i,k\leq n} [||x_{i+n} - T^j x_{i+n}|| + ||T^j x_{i+n} - T^j x_{k+n}|| \\ &\quad + ||T^j x_{k+n} - x_{k+n}||] - k_j^{-1}||T^j x_{i+n} - T^j x_{k+n}||\} \\ &\leq k_j f^{-1}\{\max_{1\leq i,k\leq n} [2\epsilon + (1-k_j^{-1})k_j ||x_{i+n} - x_{k+n}||]\} \\ &\leq k_j f^{-1}(2\epsilon + 2r(1-k_j^{-1})k_j), \quad n \geq N, \end{aligned} \tag{2.6}$$

since x_{i+n} and x_{k+n} both are in C. Substituting (2.5) and (2.6) into (2.4), we have

$$||T^j y_n - y_n|| \leq k_j f^{-1}[2\epsilon + 2rk_j(1 - k_j^{-1})] + \epsilon.$$

Taking the superior limit as $n \to \infty$ in the above inequality and noting the arbitrariness of $\epsilon > 0$, we have

$$\limsup_{n\to\infty} ||T^j y_n - y_n|| \leq k_j f^{-1}[2rk_j(1 - k_j^{-1})]. \tag{2.7}$$

On the other hand, for any given $j \geq 1$, it follows from (2.2) that

$$\begin{aligned} ||T^j q - q|| &\leq ||T^j q - T^j y_n|| + ||T^j y_n - y_n|| + ||y_n - q|| \\ &\leq (k_j + 1)||y_n - q|| + ||T^j y_n - y_n|| \\ &\leq \frac{1}{n}(k_j + 1) + ||T^j y_n - y_n||. \end{aligned}$$

Taking the superior limit as $n \to \infty$ in the above inequality, it follows from (2.7) that

$$||T^j q - q|| \leq k_j f^{-1}(2rk_j(1 - k_j^{-1})).$$

Taking the superior limit as $j \to \infty$ in the above inequality, we have

$$\limsup ||T^j q - q|| \leq f^{-1}(0) = 0,$$

which implies that $||T^j q - q|| \to 0$ as $j \to \infty$. Therefore, (2.1) is proved. By the continuity of T, we have $\lim_{j\to\infty} TT^j q = Tq = q$. This completes the proof. □

Proof of Theorem 2. By the assumption, $F(T)$ is nonempty. Take $q \in F(T)$. It follows from Lemma 4 that the limit $\lim_{n\to\infty} ||x_n - q||$ exists. Therefore, $\{x_n - q\}$ is a bounded sequence in E. Let $L = \sup_{n\geq 1} k_n$. Then we have

$$\begin{aligned} &||T^n x_n - q|| \leq k_n ||x_n - q|| \leq L||x_n - q||, \\ &||y_n - q|| = ||(1 - \beta_n)(x_n - q) + \beta_n(T^n x_n - q)|| \leq L||x_n - q||, \\ &||T^n y_n - q|| \leq k_n ||y_n - q|| \leq L^2 ||x_n - q||, \end{aligned}$$

which imply that $\{x_n\}$, $\{y_n\}$, $\{T^n x_n\}$ and $\{T^n y_n\}$ all are bounded sequences in D. Therefore, there exists $r > 0$ such that

$$\{x_n\} \cup \{y_n\} \cup \{T^n y_n\} \cup \{T^n x_n\} \subset B(q,r) \cap D =: C,$$

where $B(q,r)$ is a closed ball of E with center q and radius r and so C is a nonempty bounded closed convex subset of D. By Lemma 1 with $p = 2$ and $\lambda = \alpha_n$ and (1.4), we have

$$\begin{aligned} ||x_{n+1} - q||^2 &= ||(1-\alpha_n)(x_n - q) + \alpha_n(T^n y_n - q)||^2 \\ &\le (1-\alpha_n)||x_n - q||^2 + \alpha_n ||T^n y_n - q||^2 \\ &\quad - \omega_2(\alpha_n) g(||x_n - T^n y_n||). \end{aligned} \tag{3.1}$$

It follows from (1.8) that

$$\omega_2(\alpha_n) = \alpha_n^2(1-\alpha_n) + \alpha_n(1-\alpha_n)^2 = \alpha_n(1-\alpha_n).$$

Substituting the above expressions into (3.1) and simplifying, we have

$$\begin{aligned} &||x_{n+1} - q||^2 \\ &\le (1-\alpha_n)||x_n - q||^2 + \alpha_n||T^n y_n - q||^2 \\ &\quad - \alpha_n(1-\alpha_n) g(||x_n - T^n y_n||) \\ &= ||x_n - q||^2 + \alpha_n\{||T^n y_n - q||^2 - ||y_n - q||^2\} \\ &\quad + \alpha_n\{||y_n - q||^2 - ||x_n - q||^2||\} \\ &\quad - \alpha_n(1-\alpha_n) g(||x_n - T^n y_n||). \end{aligned} \tag{3.2}$$

First, we consider the third term on the right side of (3.2). By Lemma 1 with $p = 2$, we have

$$\begin{aligned} &||y_n - q||^2 - ||x_n - q||^2 \\ &= ||(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - q)||^2 - ||x_n - q||^2 \\ &\le (1-\beta_n)||x_n - q||^2 + \beta_n||T^n x_n - q||^2 \\ &\quad - \omega_2(\beta_n) g(||x_n - T^n x_n||) - ||x_n - q||^2 \\ &\le (1-\beta_n)||x_n - q||^2 + \beta_n||T^n x_n - q||^2 - ||x_n - q||^2, \end{aligned} \tag{3.3}$$

which implies that

$$\begin{aligned} ||y_n - q||^2 - ||x_n - q||^2 &\le \beta_n \left\{||T^n x_n - q||^2 - ||x_n - q||^2\right\} \\ &\le \beta_n(k_n^2 - 1)||x_n - q||^2. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.2) and simplifying, we have

$$
\begin{aligned}
&||x_{n+1} - q||^2 \\
&\le ||x_n - q||^2 + \alpha_n(k_n^2 - 1)||y_n - q||^2 + \alpha_n \left\{\beta_n(k_n^2 - 1)||x_n - q||^2)\right\} \\
&\quad - \alpha_n(1 - \alpha_n)g(||x_n - T^n y_n||) \\
&\le ||x_n - q||^2 + \alpha_n(k_n^2 - 1)\{||y_n - q||^2 + ||x_n - q||^2\} \\
&\quad - \alpha_n(1 - \alpha_n)g\left(||x_n - T^n y_n||\right).
\end{aligned}
$$

Since $\{x_n\}$ and $\{y_n\}$ both belong to C, we have $||x_n - q|| \le r$ and $||y_n - q|| \le r$. Besides, by the condition (1.5), $0 < \epsilon \le \alpha_n$ and $\epsilon \le 1 - \alpha_n$ for all $n \ge n_0$. Hence we have

$$
\begin{aligned}
||x_{n+1} - q||^2 \le &||x_n - q||^2 + 2\alpha_n(k_n^2 - 1)r^2 \\
&- \epsilon^2 g\left(||x_n - T^n y_n||\right), \quad n \ge n_0.
\end{aligned} \tag{3.5}
$$

Therefore, we have

$$
\epsilon^2 g\left(||x_n - T^n y_n||\right) \le ||x_n - q||^2 - ||x_{n+1} - q||^2 + 2(k_n^2 - 1)r^2, \quad n \ge n_0.
$$

For any $m > n_0$, we have

$$
\begin{aligned}
&\epsilon^2 \sum_{n=n_0}^{m} g\left(||x_n - T^n y_n||\right) \\
&\le ||x_{n_0} - q||^2 - ||x_{m+1} - q||^2 + 2r^2 \sum_{n=n_0}^{m} (k_n^2 - 1) \\
&\le ||x_{n_0} - q||^2 + 2r^2 \sum_{n=n_0}^{m} (k_n^2 - 1).
\end{aligned}
$$

Letting $m \to \infty$, by the condition (ii), we have

$$
\epsilon^2 \sum_{n=n_0}^{\infty} g\left(||x_n - T^n y_n||\right) \le ||x_{n_0} - q||^2 + 2r^2 \sum_{n=n_0}^{\infty} (k_n^2 - 1) < \infty, \tag{3.6}
$$

which implies that, as $n \to \infty$,

$$
g\left(||x_n - T^n y_n||\right) \to 0. \tag{3.7}
$$

Since $g : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing with $g(0) = 0$, it follows from (3.7) that, as $n \to \infty$,

$$||x_n - T^n y_n|| \to g^{-1}(0) = 0. \tag{3.8}$$

From (1.4), we have

$$\begin{aligned} ||x_n - y_n|| &= ||\beta_n(x_n - T^n x_n)|| \\ &\leq \beta_n\{||x_n - T^n y_n|| + ||T^n y_n - T^n x_n||\} \\ &\leq \beta_n\{||x_n - T^n y_n|| + L||y_n - x_n||\}, \end{aligned} \tag{3.9}$$

i.e., it follows that

$$(1 - L\beta_n)||x_n - y_n|| \leq \beta_n||x_n - T^n y_n|| \leq ||x_n - T^n y_n||.$$

By the condition (ii), we have $1 - L \cdot \beta_n > 0$ for all $n \geq n_1$. Therefore, from (3.8), we have

$$\lim_{n\to\infty} ||x_n - y_n|| = 0 \tag{3.10}$$

and so it follows from (3.10) and (3.8) that, as $n \to \infty$,

$$\begin{aligned} ||T^n x_n - x_n|| &\leq ||T^n x_n - T^n y_n|| + ||T^n y_n - x_n|| \\ &\leq L||x_n - y_n|| + ||T^n y_n - x_n|| \to 0. \end{aligned} \tag{3.11}$$

Therefore, by Lemma 2, as $n \to \infty$,

$$||Tx_n - x_n|| \to 0. \tag{3.12}$$

Next, we prove that $\omega_w(x_n)$ is a nonempty set and

$$\omega_w(x_n) \subset F(T), \tag{3.13}$$

where $\omega_w(x_n)$ is the weak $\omega-$limit set of $\{x_n\}$ defined by

$$\omega_w(x_n) = \{y \in E : y = w - \lim_{k\to\infty} x_{n_k} \text{ for some } n_k \to \infty\}.$$

Indeed, since E is uniformly convex and C is a nonempty bounded closed convex subset of D and so C is a weakly compact and weakly closed subset of D. This implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $p \in \omega_w(x_n)$, which shows that $\omega_w(x_n)$ is nonempty. For any $p \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ weakly. Again, by (3.12), we have $\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$. Therefore, it follows from Theorem 1 that $p - Tp = 0$, i.e., $p \in F(T)$. The conclusion (3.13) is proved. Taking any $p \in \omega_w(x_n) \subset F(T)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that, as $n_i \to \infty$,

$$x_{n_i} \to \ p \text{ weakly.} \tag{3.14}$$

Hence, from (3.11) and (3.14), it follows that, as $n_i \to \infty$,

$$T^{n_i} x_{n_i} = (T^{n_i} x_{n_i} - x_{n_i}) + x_{n_i} \to \ p \text{ weakly.}$$

Again, from (1.4), (3.11) and (3.14), it follows that, as $n_i \to \infty$,

$$y_{n_i} = x_{n_i} - \beta_{n_i}(x_{n_i} - T^{n_i} x_{n_i}) \to \ p \text{ weakly.}$$

On the other hand, from (3.8) and (3.14), we have, as $n_i \to \infty$,

$$T^{n_i} y_{n_i} = (T^{n_i} y_{n_i} - x_{n_i}) + x_{n_i} \to \ p \text{ weakly.} \tag{3.15}$$

By the same way, from (1.4), (3.8) and (3.14), as $n_i \to \infty$,

$$x_{n_i+1} = x_{n_i} - \alpha_{n_i}(x_{n_i} - T^{n_i} y_{n_i}) \to \ p \text{ weakly.} \tag{3.16}$$

Therefore, it follows from (3.11) and (3.16) that, as $n_i \to \infty$,

$$T^{n_i+1} x_{n_i+1} = (T^{n_i+1} x_{n_i+1} - x_{n_i+1}) + x_{n_i+1} \to \ p \text{ weakly.} \tag{3.17}$$

By (1.4), (3.11) and (3.16), we have, as $n_i \to \infty$,

$$y_{n_i+1} = x_{n_i+1} - \beta_{n_i+1}\left(T^{n_i+1} x_{n_i+1} - x_{n_i+1}\right) \to \ p \text{ weakly.} \tag{3.18}$$

Therefore, from (3.8) and (3.16), it follows that, as $n_i \to \infty$,

$$T^{n_i+1} y_{n_i+1} = (T^{n_i+1} y_{n_i+1} - x_{n_i+1}) + x_{n_i+1} \to \ p \text{ weakly.} \tag{3.19}$$

Continuing in this way, by induction, we can prove that, for any $m \geq 0$,

$$x_{n_i+m} \to \ p \text{ weakly}, \quad y_{n_i+m} \to \ p \text{ weakly} \quad (n_i \to \infty),$$
$$T^{n_i+m} x_{n_i+m} \to \ p \text{ weakly}, \quad T^{n_i+m} y_{n_i+m} \to \ p \text{ weakly} \quad (n_i \to \infty).$$

Next, we prove that $x_n \to p$ weakly as $n \to \infty$. In fact, it is easy to see that

$$\{x_n\}_{n=n_1}^{\infty} = \lim_{k\to\infty} \bigcup_{m=0}^{k} \{x_{n_i+m}\}_{i=1}^{\infty}. \tag{3.20}$$

Since the sequences $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ both converge weakly to p as $n_i \to \infty$, the sequence $\bigcup_{m=0}^{1}\{x_{n_i+m}\}_{i=1}^{\infty}$ converges weakly to p. By induction, we can prove that, for any positive integer k,

$$\bigcup_{m=0}^{k} \{x_{n_i+m}\}_{i=1}^{\infty} \to \ p \text{ weakly}.$$

Letting $k \to \infty$, it follows from (3.20) that the sequence $\{x_n\}$ converges weakly to p. Similarly, we can also prove that $y_n \to p$ weakly. This completes the proof. □

Proof of Theorem 3. In Theorem 2, taking $\beta_n = 0$ for all $n \geq 0$, then the conclusion of Theorem 3 can be obtained from Theorem 2 immediately. □

Proof of Theorem 4. It follows from Remark 1.1 that, if $T : D \to D$ is a nonexpansive mapping, then T is asymptotically nonexpansive with a constant sequence $\{1\}$. Therefore, the conclusion of Theorem 4 follows from Theorem 2 immediately. □

REMARK 2.1. Theorems 2-4 improve and extend the corresponding results of Bose [1], Gornicki [8], Passty [9], Reich [10], Schu [12], Tan and Xu [13], [15], [16], [18].

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