J. Korean Math. Soc. **38** (2001), No. 6, pp. 1107–1116

REIDEMEISTER ZETA FUNCTION FOR GROUP EXTENSIONS

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ABSTRACT. In this paper, we study the rationality of the Reidemeister zeta function of an endomorphism of a group extension. As an application, we give sufficient conditions for the rationality of the Reidemeister and the Nielsen zeta functions of selfmaps on an exponential solvmanifold or an infra-nilmanifold or the coset space of a compact connected Lie group by a finite subgroup.

1. Introduction

A continuous selfmap $f : M \to M$ on a compact manifold M generates a discrete (semi-) dynamical system $\{f^n\}$ given by the n-th iterates of f for all natural number n. In 1965, adapting the Hasse-Weil zeta function from algebraic geometry, Artin and Mazur [1] introduced the first dynamical zeta function for a selfmap f on a compact manifold as

$$\zeta_f(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{\#\mathrm{Fix}(f^n)}{n} z^n\Big) \tag{1}$$

provided $\#\mathrm{Fix}(f^n) < \infty$ for all n. Following [1], Smale [14] introduced a homological (Lefschetz) zeta function $L_f(z)$ of a selfmap f on a topological space given by

$$L_f(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\Big) \tag{2}$$

where $L(f^n)$ denotes the Lefschetz number of the n-th iterate of f. Unlike $\zeta_f(z)$, the Lefschetz zeta function $L_f(z)$ is always rational. Other dynamical zeta functions similar to $L_f(z)$ were introduced subsequently by Franks, Fried, and Ruelle, among others. In [3], Fel'shtyn introduced,

Received June 8, 2000.
2000 Mathematics Subject Classification: Primary 58F20; Secondary 55M20.
Key words and phrases: dynamical zeta function, Nielsen fixed point theory, homogeneous spaces, solvmanifolds, infra-nilmanifolds.

from the Nielsen fixed point theory viewpoint, the Reidemeister and the Nielsen zeta functions given by

$$R_f(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\Big); \quad N_f(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\Big)$$

respectively, by replacing $L(f^n)$ in (2) with the Reidemeister number $R(f^n)$ and with the Nielsen number $N(f^n)$. While the Lefschetz zeta function is more computable and is always rational, the Nielsen zeta function is closer to the original dynamical zeta function of Artin-Mazur in the sense that the Nielsen number of f^n measures the geometric size, rather than the homological size, of the n-periodic points of f.

Recall from topological fixed point theory, the fixed point set $\mathrm{Fix} f = \{x \in X | f(x) = x\}$ is partitioned into fixed point classes each of which has an integer-valued fixed point index. The Nielsen number $N(f)$ of f is defined to be the number of (essential) fixed point classes of non-zero index. If X is a manifold of dimension $\dim X \geq 3$, then $N(f) = \min\{\#\mathrm{Fix}\, g | g \sim f\}$. On the other hand, the Reidemeister number $R(f)$, a more computable homotopy invariant defined at the fundamental group level, is an upper bound for $N(f)$. Under the so-called Jiang condition on X or for more general Jiang-type spaces such as nilmanifolds and coset spaces of the form G/K where K is a finite subgroup of a compact connected Lie group G, $N(f) = 0$ if $L(f) = 0$ and $N(f) = R(f)$ if $L(f) \neq 0$.

In a series of papers (see [4], [5], [6] and [2]), Fel'shtyn and others studied the rationality, among other properties, of $R_f(z)$ and of $N_f(z)$. It is known that $R_f(z)$ is rational if f is eventually commutative; $\pi_1(X)$ is torsion free nilpotent; or a direct product of a finite group and a free abelian group. Moreover, if R is the radius of convergence of $N_f(z)$ for a selfmap $f : X \to X$ of a finite polyhedron X, then $1 \geq R \geq \exp(-h) > 0$ where $h := \inf\{h(g) | g \sim f\}$ and $h(g)$ denotes the topological entropy of the map g. Thus, the Nielsen zeta function $N_f(z)$ provides a lower bound for the topological entropy, which is widely used as a measure of complexity in the theory of dynamical systems, in the homotopy class of f. If φ denotes the group endomorphism induced by f on the fundamental group, then $R(f) = R(\varphi)$ and $R_f(z) = R_\varphi(z)$. Let φ be an endomorphism of a group extension such that the following diagram is commutative

$$
(3)\qquad
\begin{array}{ccccccccc}
1 & \longrightarrow & H & \xrightarrow{i} & \pi & \xrightarrow{p} & Q & \longrightarrow & 1 \\
 & & \varphi' \Big\downarrow & & \varphi \Big\downarrow & & \overline{\varphi} \Big\downarrow & & \\
1 & \longrightarrow & H & \xrightarrow{i} & \pi & \xrightarrow{p} & Q & \longrightarrow & 1
\end{array}
$$

where $\varphi' = \varphi|_H$. Then $R_\varphi(z)$ is rational provided the extension admits a *normal splitting* (see section 2.7 of [2]). Under such condition, the product formula $R(\varphi^n) = R(\varphi'^n) \cdot R(\overline{\varphi}^n)$ holds for each n and thus $R_\varphi(z)$ is the convolution product $R_{\varphi'}(z) * R_{\overline{\varphi}}(z)$. This approach allowed us [6] to show that $R_\varphi(z)$ is rational if the group π is finitely generated torsion free nilpotent. Topologically, the normal splitting condition is similar to the so-called *Fadell splitting* condition of an orientable fibration under which the product formula for the Nielsen numbers holds (see [11]). If $R(f^n) = N(f^n)$ for each positive integer n, then the Nielsen zeta function and the Reidemeister zeta function coincide and hence rationality of $R_f(z)$ would imply rationality of $N_f(z)$.

The objective of this paper is to give sufficient conditions for the rationality of the Reidemeister and of the Nielsen zeta functions while relaxing the normal splitting hypothesis. By making use of an addition formula for the Reidemeister numbers, we apply to the so-called Mostow fibration of an exponential solvmanifold to obtain rationality of $R_\varphi(z)$ and hence of $R_f(z)$. When the quotient group Q is finite, we apply the addition formula to maps of coset spaces of the form G/K where K is a finite subgroup of a compact connected Lie group G and to maps of infra-nilmanifolds.

Basic reference on Nielsen fixed point theory is [11] and [2] gives an updated survey on Reidemeister and Nielsen zeta functions.

2. Reidemeister numbers and rationality of $R_\varphi(z)$

In this section, we give a general addition formula for the Reidemeister number of an endomorphism of a group extension. Similar formula for the Nielsen number has been obtained for fiber-preserving maps in [10]. Sufficient conditions for $R_\varphi(z)$ to be rational are given. Our algebraic approach to Reidemeister classes and numbers follows that of [7].

Let $\varphi : \pi \to \pi$ be a group endomorphism and let π act on π via $\sigma \bullet \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}$. The orbits of this action form a set $\mathcal{R}(\varphi)$, called the set of Reidemeister classes. The Reidemeister number $R(\varphi)$ of φ is the cardinality of $\mathcal{R}(\varphi)$.

Given the commutative diagram (3), the homomorphisms i and p induce functions $\widehat{i}$ and $\widehat{p}$ respectively, yielding the following sequence of sets

$$\mathcal{R}(\varphi') \xrightarrow{\widehat{i}} \mathcal{R}(\varphi) \xrightarrow{\widehat{p}} \mathcal{R}(\overline{\varphi})$$

such that $\widehat{p}$ is surjective and $\widehat{i}(\mathcal{R}(\varphi')) = \widehat{p}^{-1}([\overline{1}])$ where $[\overline{1}]$ is the Reidemeister class of the identity $\overline{1} \in Q$.

The fixed subgroup $\mathrm{Fix}\overline{\varphi} = \{\overline{\alpha} \in Q | \overline{\varphi}(\overline{\alpha}) = \overline{\alpha}\}$ acts on $\mathcal{R}(\varphi')$ via $\overline{\theta} \bullet [\beta] \mapsto [\theta\beta\varphi(\theta^{-1})]$ where $\beta \in p^{-1}(\overline{1}), \theta \in p^{-1}(\overline{\theta}), \overline{\theta} \in \mathcal{R}(\varphi')$ so that $\mathrm{Fix}\overline{\varphi} = 1 \Rightarrow \widehat{i}$ is injective. Let $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$. Then for every $\overline{\alpha} \in Q$, the following is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \xrightarrow{i} & \pi & \xrightarrow{p} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau_\alpha\varphi'} & & \downarrow{\scriptstyle \tau_\alpha\varphi} & & \downarrow{\scriptstyle \tau_{\overline{\alpha}}\overline{\varphi}} & & \\ 1 & \longrightarrow & H & \xrightarrow{i_\alpha} & \pi & \xrightarrow{p_\alpha} & Q & \longrightarrow & 1 \end{array} \tag{4}$$

where $\alpha \in p^{-1}(\overline{\alpha}), i_\alpha = i$ and $p_\alpha = p$. Thus we have the induced maps $\widehat{i}_\alpha$ and $\widehat{p}_\alpha$ and the sequence

$$\mathcal{R}(\tau_\alpha\varphi') \xrightarrow{\widehat{i}_\alpha} \mathcal{R}(\tau_\alpha\varphi) \xrightarrow{\widehat{p}_\alpha} \mathcal{R}(\tau_{\overline{\alpha}}\overline{\varphi}).$$

It follows from [8] that there is a bijection

$$\mathcal{R}(\varphi) \longleftrightarrow \coprod_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi})} \widehat{i}_\alpha(\mathcal{R}(\tau_\alpha\varphi')).$$

The following is a special case of the addition formula for Reidemeister coincidence numbers obtained in [8].

PROPOSITION 1. *Given the commutative diagram* (3), *if the fixed subgroup* $\mathrm{Fix}\,\tau_{\overline{\alpha}}\overline{\varphi}$ *is trivial for all* $\overline{\alpha} \in Q$, *then*

$$R(\varphi) = \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi})} R(\tau_\alpha\varphi'). \tag{5}$$

Now we present the main result of the section, proving rationality of $R_\varphi(z)$ by assuming the addition formula (5).

THEOREM 2. *Given the commutative diagram* (3), *suppose that for all* $n \geq 1$, $R(\varphi^n) < \infty$ *and the addition formula* (5) *holds. If* $\bigcup_n \mathcal{R}(\overline{\varphi}^n)$ *is represented by a finite subset of* Q *or* $R((\tau_\alpha\varphi')^n)$ *is independent of* α, *and if* $R_{\tau_\alpha\varphi'}(z)$ *is rational for all* $\alpha \in p^{-1}(\overline{\alpha}), [\overline{\alpha}] \in \bigcup_n \mathcal{R}(\overline{\varphi}^n)$, *then* $R_\varphi(z)$ *is rational.*

Proof. Since $R(\varphi^n) < \infty, R((\tau_{\overline{\alpha}}\overline{\varphi})^n) < \infty$. Thus, by the rationality of $R_{\tau_\alpha \varphi'}(z)$, for every $[\alpha_i] \in \mathcal{R}(\tau_{\overline{\alpha}}\overline{\varphi})$, there exist a finite set of complex numbers $\mu_{\alpha_i,1}, \ldots, \mu_{\alpha_i,p_i}$ and $\nu_{\alpha_i,1}, \ldots, \nu_{\alpha_i,q_i}$ such that

$$R((\tau_{\alpha_i}\varphi')^n) = \sum_{j=1}^{q_i} (\nu_{\alpha_i,j})^n - \sum_{j=1}^{p_i} (\mu_{\alpha_i,j})^n.$$

Under the assumption on $\bigcup_n \mathcal{R}(\overline{\varphi}^n)$ and on $R((\tau_\alpha \varphi')^n)$, the addition formula (5) implies that $R(\varphi^n)$ can be written in the form

$$R(\varphi^n) = \sum {b_s}^n - \sum {a_r}^n$$

for a finite set of complex numbers $\{a_r, b_s\}$. Here, $\{a_r\} = \{\mu_{\alpha_i,j}\}$ and $\{b_s\} = \{\nu_{\alpha_i,j}\}$. Taking the logarithmic derivative of $R_\varphi(z)$ yields

$$\frac{1}{R_\varphi(z)} \cdot (R_\varphi(z))' = \sum_n R(\varphi^n) z^{n-1}. \tag{6}$$

On the other hand, the logarithmic derivative of

$$\frac{\prod_r (1 - a_r z)}{\prod_s (1 - b_s z)}$$

is equal to

$$\begin{aligned}
&\frac{d}{dz}\left(\sum_r \log(1 - a_r z) - \sum_s \log(1 - b_s z)\right) \\
=&\sum_r \left(\frac{-a_r}{1 - a_r z}\right) - \sum_s \left(\frac{-b_s}{1 - b_s z}\right) \\
=&\sum_s b_s (\sum_n {b_s}^n z^n) - \sum_r a_r (\sum_n {a_r}^n z^n) \\
=&\sum_n (\sum_s {b_s}^n - \sum_r {a_r}^n) z^{n-1} \\
=&\sum_n R(\varphi^n) z^{n-1}
\end{aligned}$$

which coincides with the right hand side of (6). It follows that

$$R_\varphi(z) = \frac{\prod_r (1 - a_r z)}{\prod_s (1 - b_s z)}$$

and hence is a rational function. □

3. Applications

In this section, we apply Theorem 2 to prove rationality of $R_f(z)$ and of $N_f(z)$ for selfmaps of exponential solvmanifolds, of infra-nilmanifolds and of certain coset spaces of compact connected Lie groups. For the rest of this paper, we assume that $R_f(z)$ is well defined, i.e., $R(f^n) < \infty$ for all positive integer n.

Recall that a solvmanifold is simply the homogeneous coset space of the form $M = G/\Gamma$ where G is a connected and simply connected solvable Lie group and Γ is a closed subgroup (not necessarily discrete). We only consider compact solvmanifolds so we assume that Γ is cocompact. A solvmanifold M is called *exponential* if the exponential map $\exp : \mathfrak{g} \to G$ is surjective, where $\mathfrak{g}$ denotes the Lie algebra of G. It is well-known that every solvmanifold M fibers over a torus T with a nilmanifold N as typical fiber. Such a fibration $N \hookrightarrow M \to T$ is called a *Mostow fibration* of M. Furthermore, given such a fibration $p : M \to T$, every selfmap $f : M \to M$ is homotopic to a fiber preserving map $g : p \to p$ with induced map $\overline{g}$. The fiber-preserving map g induces the commutative diagram (3) from the homotopy exact sequence of fibration. Although solvmanifolds are not Jiang-type spaces in general, the equality $N(f) = R(f)$ holds when $R(f) < \infty$ (in fact, it holds for coincidences of selfmaps of a solvmanifold [8]).

It is shown in [12] that for $b \in \text{Fix}\overline{g}$, the Lefschetz number $L(g_b)$ of $g_b = g|_{p^{-1}(b)}$ is independent of b. Since the fiber is a nilmanifold, we have $|L(g_b)| = N(g_b) = R(g_b)$. Since $R(g) < \infty$, $R(\overline{g})$ must also be finite. The base T is a torus so $R(\overline{g}) < \infty \Rightarrow \text{Fix}(\overline{g}_\#) = 1$ and hence by Proposition 1, the addition formula holds. Now the fact that the Reidemeister number on the fiber is independent of the basepoint b is equivalent to $R((\tau_\alpha\varphi'))$ being independent of α so Theorem 2 applies. Thus, we have just proved the following

THEOREM 3. *Let M be a compact exponential solvmanifold. For any selfmap $f : M \to M$, $R_f(z)$ is rational and coincides with $N_f(z)$.*

REMARK 1. Exponential solvmanifolds include the class of all nilmanifolds. Thus, Theorem 3 enlarges the class of manifolds for which $R_f(z)$ and $N_f(z)$ are always rational for any selfmap f.

In the case where Q is finite, the commutative diagram (3) may also arise from considering the exact sequence of homotopy groups of finite covers and lifts.

In [5], Fel'shtyn and Hill asked whether some power of $R_f(z)$ is rational when f is any selfmap of an infra-nilmanifold, or equivalently an almost flat manifold. The fundamental group of an infra-nilmanifold is virtually (or almost) nilpotent which has polynomial growth in the sense of Gromov. Next, we give a partial answer to their conjecture.

It is well-known that an infra-nilmanifold M is finitely covered by a nilmanifold. By extending Bieberbach's rigidity theorem to infra-nilmanifolds, Lee [13] applied his rigidity result to fixed point theory. In particular, he showed (see Theorem 2.2 of [13]) that every selfmap $f : M \to M$ can be lifted to a *selfmap* $\widehat{f} : \widehat{M} \to \widehat{M}$ on a finite cover $\widehat{M}$ of M where $\widehat{M}$ is a nilmanifold. This allows us to obtain, at the fundamental group level, the commutative diagram (3).

THEOREM 4. *Let $f : M \to M$ be a selfmap on an infra-nilmanifold M. There exist a finite cover $\widehat{M}$ of M and a lift $\widehat{f} : \widehat{M} \to \widehat{M}$ of f with $\widehat{M}$ a nilmanifold. Let H denote the fundamental group of $\widehat{M}$, π denote that of M and Q the quotient group. If* $\mathrm{Fix}\tau_{\overline{\alpha}}\overline{\varphi} = 1$ *for all $\overline{\alpha} \in Q$ where φ denotes the homomorphism induced by f, then both $R_f(z)$ and $N_f(z)$ are rational.*

Proof. The rationality of $R_f(z)$ follows from Proposition 1 and Theorem 2. Using the addition formula for Reidemeister numbers and the fact that the Reidemeister classes in the fibers, which are nilmanifolds, must be essential, we conclude that the Reidemeister classes of f^n must also be essential. Thus $N(f^n) = R(f^n)$ and hence $R_f(z) = N_f(z)$. □

The final application of Theorem 2 is for selfmaps of the coset space $M = G/K$ of a compact connected Lie group G modulo a finite subgroup K. It was shown in [15] that such a space is of Jiang-type so that for any selfmap $f : M \to M$, $L(f) \neq 0 \Rightarrow N(f) = R(f)$. Hence, $R_f(z)$ is well-defined and is equal to $N_f(z)$ provided $L(f^n) \neq 0$ for all n.

THEOREM 5. *Let $M = G/K$ be the coset space of a compact connected Lie group G and K be a finite subgroup. For any map $f : M \to M$ such that $f_\#(p_\#(\pi_1(G))) \subseteq p_\#(\pi_1(G))$, if* Fix $\tau_{\overline{\alpha}}\overline{f}_\# = 1$ *for all $\overline{\alpha} \in K$ where $\overline{f}_\# : K \to K$ denotes the homomorphism induced by $\overline{f}$, then $R_f(z)$ is rational and is equal to $N_f(z)$.*

Proof. The condition $f_\#(p_\#(\pi_1(G))) \subseteq p_\#(\pi_1(G))$ guarantees that f can be lifted to a map $\widehat{f} : G \to G$. Now, we proceed as in the proof of Theorem 4. □

REMARK 2. In both Theorem 4 and Theorem 5, the hypothesis on the fix-subgroup can be satisfied, for example, when $\varphi(\pi) \subseteq H$ in the diagram (3).

4. The Klein bottle

In this final section, we illustrate that our main results in section 3 give sufficient but not necessary conditions for the rationality of the Reidemeister zeta function. In particular, we show that the Nielsen zeta function on the Klein bottle is always rational without any additional hypotheses on the map. We shall examine the Klein bottle both as a solvmanifold and as an infra-nilmanifold.

Let K be the Klein bottle and $\pi = \pi_1(K)$ be the fundamental group of K. It is well-known that π has the following group presentation

$$\pi = \langle a, b | aba = b \rangle.$$

Let $\mathcal{M}_2$ be the group of all rigid motions of $\mathbb{R}^2$. Every $\sigma \in \mathcal{M}_2$ can be represented by an ordered pair (m, s) with $m \in O_2$ (rotation) and $s \in \mathbb{R}^2$ (translation). Equivalently, σ can be represented by a 3×3 matrix of the form

$$\sigma = \begin{pmatrix} m & s \\ 0 & 1 \end{pmatrix}$$

so that the composition of rigid motions will simply be the usual matrix multiplication. With this representation, we can consider π as a subgroup of $\mathcal{M}_2$ generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

The rotational part of π is given by $r : \pi \to O_2$ via

$$r \begin{pmatrix} m & s \\ 0 & 1 \end{pmatrix} = m.$$

For the Klein bottle,

$$r(\pi) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}_2.$$

Thus, π is the torsion free extension given by

$$\mathbb{Z}^2 \to \pi \xrightarrow{r} \mathbb{Z}_2$$

with the nontrivial action of $\mathbb{Z}_2$ on $\mathbb{Z}^2$ represented by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

For any endomorphism $\varphi : \pi \to \pi$, there exist integers u, v, w such that $\varphi(a) = a^u; \varphi(b) = a^v b^w$. Let $f : K \to K$ be a map with induced homomorphism $\varphi : \pi \to \pi$. Halpern [9] showed that the Nielsen number of f^n is given by

$$N(f^n) = \begin{cases} |u^n(w^n - 1)| & \text{if } u \neq 0; \\ |w^n - 1| & \text{otherwise.} \end{cases}$$

It is easy to see that $N_f(z)$ is *always* a rational function. Moreover, if $R_f(z)$ is well-defined then it coincides with $N_f(z)$ since K is a solvmanifold.

Suppose f is a map with induced homomorphism φ such that $\varphi(a) = 1$ and $\varphi(b) = b^3$. Since the kernel $\ker r$ is easily seen to be the free abelian group generated by a^{-2} and b^2, the endomorphism φ maps $\ker r$ to itself so it induces $\overline{\varphi}$ on the quotient $r(\pi) \cong \mathbb{Z}_2$. Note that $r(b) = -1$ and $\overline{\varphi}(-1) = -1$ so that $\operatorname{Fix} \overline{\varphi} \neq 1$. Thus, the hypothesis of Theorem 4 is not satisfied. On the other hand, it is easy to see the Reidemeister numbers on $\ker r$ are all finite and thus $R(\varphi) < \infty$. It follows that $R_f(z) = N_f(z)$ is rational. This shows that the conditions for rationality of $R_f(z)$ and therefore for $N_f(z)$ given in Theorem 4 are sufficient but not necessary. Furthermore, since K is not an exponential solvmanifold, Theorem 3 is not applicable.

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