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GEOMETRY OF FIELD EQUATIONS ON MEX_n

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ABSTRACT. An n-dimensional ME-manifold MEX_n is a generalized Riemannian manifold connected by the ME-connection which is both Einstein and of the form (2.13). The purpose of this paper is to study the properties of the ME-curvature tensors, the contracted ME-curvature tensors and the field equations in MEX_n.

1. Introduction

In Appendix II to his last book "The meaning of relativity", Einstein [3] proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. Characterizing Einstein's unified field theory as a set of geometrical postulates for the space-time X_4, Hlavatý [4] gave the mathematical foundation for the first time. Since then the geometrical consequences of these postulates have been developed very far by a number of Mathematicians and physicists; among them Hlavatý's contributions are the most distinguished. Wrede [7] studies the Principles A and B of this theory on an n-dimensional generalized Riemannian manifold X_n. Recently, Yoo [9] introduced the concepts of n-dimensional ME-manifold, denoted by MEX_n, connected to X_n an ME-connection of the form (2.13), which is similar to Yano [8] and Imai's [5] semi-symmetric metric connection.

The purpose of the present paper is to study the properties of the ME-curvature tensors, the contracted ME-curvature tensors, and the field equations in the ME-manifold MEX_n.

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2. Preliminaries

This section is a brief collection of definitions, notations, and basic results which are needed in the present paper. The detailed proofs are given in Hlavatý [4], Mishra [6], and Yoo [9].

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys coordinate transformations $x^\nu \longrightarrow \bar{x}^\nu$ for which

$$Det\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0, \tag{2.1}$$

where, here and in the sequel, Greek indices are used for the holonomics components of tensor in X_n. They take the values $1, 2, \cdots, n$ and follow the summation convention.

The manifold X_n is assumed to be connected by a general real connection $\Gamma^\nu_{\lambda\mu}$ with the following transformation rule:

$$\bar{\Gamma}^\nu_{\lambda\mu} = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial x^\gamma}{\partial \bar{x}^\mu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\right). \tag{2.2}$$

2.1. Einstein's n-dimensional unified field theory

Einstein's n-dimensional unified field theory is based on the following three principles as indicated by Hlavatý [4]:

PRINCIPLE A. The algebraic structure is imposed on X_n by a general real tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.3}$$

where

$$\mathfrak{g} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = Det(h_{\lambda\mu}) \neq 0. \tag{2.4}$$

Hence we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.5}$$

The tensor $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of tensor in X_n in the usual manner.

PRINCIPLE B. The differential geometric structure is imposed on X_n by the tensor $g_{\lambda\mu}$ by means of the *Einstein's connection* $\Gamma^{\nu}_{\lambda\mu}$ defined by a system of Einstein's equations

$$\partial_\omega g_{\lambda\mu} - \Gamma^{\alpha}_{\lambda\omega} g_{\alpha\mu} - \Gamma^{\alpha}_{\omega\mu} g_{\lambda\alpha} = 0, \tag{2.6a}$$

or equivalently,

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}, \tag{2.6b}$$

where D_ω denotes the symbol of the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$, and

$$S_{\omega\mu}{}^{\nu} = \Gamma^{\nu}_{[\omega\mu]} = \frac{1}{2}\left(\Gamma^{\nu}_{\omega\mu} - \Gamma^{\nu}_{\mu\omega}\right) \tag{2.7}$$

is a *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$.

PRINCIPLE C. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma^{\nu}_{\lambda\mu}$, certain conditions are imposed, which may be condensed to

$$S_\lambda = S_{\lambda\alpha}{}^{\alpha} = 0, \qquad R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}, \qquad R_{(\mu\lambda)} = \frac{1}{2}\left(R_{\mu\lambda} + R_{\lambda\mu}\right) = 0, \tag{2.8}$$

where X_λ is an arbitrary vector, S_λ is *the torsion vector*, and

$$R_{\omega\mu\lambda}{}^{\nu} = 2\left(\partial_{[\mu}\Gamma^{\nu}_{|\lambda|\omega]} + \Gamma^{\nu}_{\alpha[\mu}\Gamma^{\alpha}_{|\lambda|\omega]}\right), \qquad R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha}, \tag{2.9}$$

where $R_{\omega\mu\lambda}{}^{\nu}$ is *the curvature tensor* and $R_{\mu\lambda}$ is *the contracted curvature tensor.*

The following quantities will be used in our further considerations:

$${}^{(0)}k_\lambda{}^{\nu} = \delta^{\nu}_{\lambda}, \quad {}^{(p)}k_\lambda{}^{\nu} = {}^{(p-1)}k_\lambda{}^{\alpha} k_\alpha{}^{\nu} \quad (p = 1, 2, \cdots), \tag{2.10a}$$

$${}^{(p)}X_\lambda = {}^{(p)}k_\lambda{}^{\alpha} X_\alpha \quad (p = 0, 1, 2, \cdots), \tag{2.10b}$$

(2.10c) $$^{(p)}X^\nu = (-1)^{p(p)}k_\alpha{}^\nu X^\alpha \quad (p = 0, 1, 2, \cdots),$$

(2.10d) $$X = X_\alpha X^\alpha.$$

It has been shown Hlavatý [4] that if the equations (2.6) admit a solution $\Gamma^\nu_{\lambda\mu}$, it must be of the form

(2.11) $$\Gamma^\nu_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu},$$

where

(2.12) $$U^\nu{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^\beta k_{\mu)\beta}$$

and $\left\{ {\nu \atop \lambda\mu} \right\}$ are the Christoffel symbol defined by $h_{\lambda\mu}$.

2.2 n-dimensional ME-manifold MEX_n

The Einstein's connection $\Gamma^\nu_{\lambda\mu}$ which takes the form

(2.13) $$\Gamma^\nu_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + 2\delta_\lambda{}^\nu X_\mu - 2g_{\lambda\mu}X^\nu,$$

for a non-null vector X^ν, is called *an ME-connection* and a generalized Riemannian manifold X_n connected by this connection is called *an n-dimensional ME manifold*, denoted by MEX_n.

In our further considerations, we use the word "present condition" to describe the situations that Einstein's connection, given by (2.11), take the form (2.13). It has been also shown Yoo [9] that for a non-null vector X^ν, the present condition holds if and only if

(a) the torsion tensor $S_{\lambda\mu}{}^\nu$ is given by

(2.14) $$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda}X_{\mu]} - 2k_{\lambda\mu}X^\nu,$$

(b) the tensor field $g_{\lambda\mu}$ satisfies

(2.15a) $$\delta^\nu_{(\lambda}X_{\mu)} - h_{\lambda\mu}X^\nu = k_{(\lambda}{}^\nu X_{\mu)} + 2k_{(\lambda}{}^\nu k_{\mu)}{}^\alpha X_\alpha,$$

or equivalently

(2.15b) $$g_{\nu(\lambda}X_{\mu)} + 2k_{\nu(\lambda}k_{\mu)}{}^\alpha X_\alpha - h_{\lambda\mu}X_\nu = 0.$$

As a direct consequence of (2.12) and (2.14), we have

(2.16) $$\begin{aligned} U^\nu{}_{\lambda\mu} &= 2k_{(\lambda}{}^\nu X_{\mu)} + 4k_{(\lambda}{}^\nu k_{\nu)}{}^\alpha X_\alpha \\ &= 2\delta^\nu_{(\lambda}X_{\mu)} - 2h_{\lambda\mu}X^\nu. \end{aligned}$$

3. The ME-curvature tensors and the contracted ME-curvature tensors in MEX_n

This section is devoted to the study of the n-dimensional ME-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ defined by the ME-connection $\Gamma^{\nu}_{\lambda\mu}$, the first and second contracted ME-curvature tensors defined by (3.9), and some identities involving the tensor $R_{\omega\mu\lambda}{}^{\nu}$ and $R_{\mu\lambda}$.

LEMMA 3.1. *The ME-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ in MEX_n defined by the connection (2.11) is given by*

$$\begin{aligned} R_{\omega\mu\lambda}{}^{\nu} = H_{\omega\mu\lambda}{}^{\nu} &+ 2\nabla_{[\mu}\left(S_{|\lambda|\omega]}{}^{\nu} + U^{\nu}{}_{|\lambda|\omega]}\right) \\ &+ 2\Big(S_{\alpha[\mu}{}^{\nu}S_{|\lambda|\omega]}{}^{\alpha} + S_{\alpha[\mu}{}^{\nu}U^{\alpha}{}_{|\lambda|\omega]} \\ &\qquad + U^{\nu}{}_{\alpha[\mu}S_{|\lambda|\omega]}{}^{\alpha} + U^{\nu}{}_{\alpha[\mu}U^{\alpha}{}_{|\lambda|\omega]}\Big), \end{aligned} \tag{3.1}$$

where

$$H_{\omega\mu\lambda}{}^{\nu} = 2\left(\partial_{[\mu}\left\{ {\nu \atop \omega]\lambda} \right\} + \left\{ {\nu \atop \alpha[\mu} \right\}\left\{ {\alpha \atop \omega]\lambda} \right\}\right) \tag{3.2}$$

is the curvature tensor defined by $\left\{ {\nu \atop \lambda\mu} \right\}$ and ∇_ω is the symbol of the covariant derivative with respect to the Christoffel symbol $\left\{ {\nu \atop \lambda\mu} \right\}$ defined by $h_{\lambda\mu}$.

PROOF. Substituting (2.11) into (2.9), we obtain the relation (3.1) by a straightforward computation. □

THEOREM 3.2. *The ME-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ in MEX_n may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = H_{\omega\mu\lambda}{}^{\nu} + \overset{\circ}{R}_{\omega\mu\lambda}{}^{\nu} + \overset{\dagger}{R}_{\omega\mu\lambda}{}^{\nu}, \tag{3.3}$$

where

$$\overset{\circ}{R}_{\omega\mu\lambda}{}^{\nu} = 4\delta^{\nu}_{\lambda}\nabla_{[\mu}X_{\omega]} + 4\left(h_{\lambda[\mu}\nabla_{\omega]} + k_{\lambda[\mu}\nabla_{\omega]}\right)X^{\nu} - 4X^{\nu}\nabla_{[\omega}k_{\mu]\lambda}, \tag{3.4a}$$

$$\overset{\dagger}{R}_{\omega\mu\lambda}{}^{\nu} = 8\left(h_{\lambda[\omega}X_{\mu]} - h_{\lambda[\omega}{}^{(1)}X_{\mu]} + k_{\lambda[\omega}X_{\mu]} - k_{\lambda[\omega}{}^{(1)}X_{\mu]}\right)X^{\nu}. \tag{3.4b}$$

PROOF. The relation (3.3) may be obtained by substituting (2.14) and (2.16) into (3.1) and making use of (2.3), (2.10), (3.2), (3.4a), and (3.4b) by a long computation. □

LEMMA 3.3. *The ME-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ in MEX_n satisfies the following identities:*

(3.5) $$R_{\omega\mu\lambda}{}^{\nu} = R_{[\omega\mu]\lambda}{}^{\nu},$$

(3.6) $$\begin{aligned} R_{[\omega\mu\lambda]}{}^{\nu} &= 4\left(\delta^{\nu}_{\lambda}\nabla_{[\mu}X_{\omega]} - X^{\nu}\nabla_{[\omega}k_{\mu\lambda]}\right) \\ &\quad + 4\left(k_{[\lambda\mu}\nabla_{\omega]} + 2k_{[\lambda\omega}X_{\mu]} - 2k_{\lambda\omega}{}^{(1)}X_{\mu]}\right)X^{\nu}. \end{aligned}$$

PROOF. Equation (3.5) follows immediately from (2.9). The relation (3.6) may be obtained by using (3.3) and (3.4) as in the following way:
(3.7)
$$\begin{aligned} R_{[\omega\mu\lambda]}{}^{\nu} &= H_{[\omega\mu\lambda]}{}^{\nu} + \overset{\circ}{R}_{[\omega\mu\lambda]}{}^{\nu} + \overset{\dagger}{R}_{[\omega\mu\lambda]}{}^{\nu} \\ &= H_{[\omega\mu\lambda]}{}^{\nu} + 4\delta^{\nu}_{[\lambda}\nabla_{\mu}X_{\omega]} \\ &\quad + 4\left(h_{[\lambda\mu}\nabla_{\omega]} + k_{[\lambda\mu}\nabla_{\omega]}\right)X^{\nu} - 4X^{\nu}\nabla_{[\omega}k_{\mu\lambda]} \\ &\quad + 8\left(h_{[\lambda\omega}X_{\mu]} - h_{\lambda[\omega}{}^{(1)}X_{\mu]} + k_{\lambda[\omega}X_{\mu]} - k_{[\lambda\omega}{}^{(1)}X_{\mu]}\right)X^{\nu} \\ &= \text{The right-hand side of (3.6).} \end{aligned}$$
□

LEMMA 3.4. *The torsion vector S_{λ} and the vector U_{λ} in MEX_n may be given by*

(3.8a) $$S_{\lambda} = (1-n)X_{\lambda} - 2^{(1)}X_{\lambda},$$

(3.8b) $$U_{\lambda} = U^{\alpha}_{\lambda\alpha} = (n-1)X_{\lambda}.$$

PROOF. The relations (3.8) follow from (2.14) and (2.16), putting $\mu = \nu = \alpha$ and making use of (2.10). □

The tensors

(3.9) $$R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha}, \qquad V_{\omega\mu} = R_{\omega\mu\alpha}{}^{\alpha}$$

are called *the first and second contracted ME-curvature tensors* of the ME-connection $\Gamma^{\nu}_{\lambda\mu}$, respectively. They also appear as functions of $g_{\lambda\mu}$ and its first two derivatives.

THEOREM 3.5. *In MEX_n, the following relations hold:*

$$S_{\lambda\mu}{}^{\alpha}X_{\alpha} = -2k_{\lambda\mu}X, \tag{3.10a}$$

$$S_{\lambda\mu}{}^{\alpha}S_{\alpha} = 4X_{[\lambda}{}^{(1)}X_{\mu]} + 2(n-1)k_{\lambda\mu}X + 4k_{\lambda\mu}X^{\alpha(1)}X_{\alpha}, \tag{3.10b}$$

$$S_{\lambda\mu}{}^{\alpha}U_{\alpha} = 2(1-n)k_{\lambda\mu}X. \tag{3.10c}$$

PROOF. The relations (3.10) follow from (2.14) and making use of (2.10) and (3.8). □

THEOREM 3.6. *In MEX_n, the following relations hold:*

$$U^{\alpha}_{\lambda\mu}X_{\alpha} = 2X_{\lambda}X_{\mu} - 2h_{\lambda\mu}X, \tag{3.11a}$$

$$\begin{aligned} U^{\alpha}_{\lambda\mu}S_{\alpha} = {} & 2(1-n)X_{\lambda}X_{\mu} - 4X_{(\lambda}{}^{(1)}X_{\mu)} \\ & + 2(n-1)h_{\lambda\mu}X + 4h_{\lambda\mu}X^{\alpha(1)}X_{\alpha}, \end{aligned} \tag{3.11b}$$

$$U^{\alpha}_{\lambda\mu}U_{\alpha} = 2(n-1)\left(X_{\lambda}X_{\mu} - h_{\lambda\mu}X\right). \tag{3.11c}$$

PROOF. In virtue of (2.10), (2.16), and (3.8) we have the relations (3.11). □

THEOREM 3.7. *The contracted ME-curvature tensor $V_{\omega\mu}$ in MEX_n may be given by*

$$V_{\omega\mu} = 4\nabla_{[\omega}\left(X_{\mu]} - {}^{(1)}X_{\mu]}\right) \tag{3.12}$$

PROOF. Putting $\lambda = \nu = \alpha$ in (3.3), we have

$$V_{\omega\mu} = H_{\omega\mu\alpha}{}^{\alpha} + \overset{\circ}{R}_{\omega\mu\alpha}{}^{\alpha} + \overset{\dagger}{R}_{\omega\mu\alpha}{}^{\alpha}. \tag{3.13}$$

In virtue of (2.10) and (3.2) we obtain

$$H_{\omega\mu\alpha}{}^{\alpha} = \overset{\dagger}{R}_{\omega\mu\alpha}{}^{\alpha} = 0, \qquad \overset{\circ}{R}_{\omega\mu\alpha}{}^{\alpha} = 4\nabla_{[\omega}\left(X_{\mu]} - {}^{(1)}X_{\mu]}\right).$$

Hence we have the relation (3.12). □

THEOREM 3.8. *Under the present condition the contracted ME-curvature tensor $R_{\mu\lambda}$ may be given by*

$$(3.14)\qquad \begin{aligned} R_{\mu\lambda} = H_{\mu\lambda} &+ 4\left(K_{\lambda\mu} - k_{\lambda\mu}X + 2X_{[\mu}{}^{(1)}X_{\lambda]} + g_{\lambda\mu}X^{\alpha(1)}X_{\alpha}\right) \\ &+ 4\left(h_{\mu[\lambda}\nabla_{\alpha]}X^{\alpha} + \nabla_{[\alpha}(k_{|\lambda|\mu]}X^{\alpha})\right), \end{aligned}$$

where

$$(3.15a)\qquad H_{\mu\lambda} = H_{\alpha\mu\lambda}{}^{\alpha},$$

$$(3.15b)\qquad K_{\lambda\mu} = X_{\lambda}X_{\mu} - {}^{(1)}X_{\lambda}{}^{(1)}X_{\mu} - h_{\lambda\mu}X.$$

PROOF. Putting $\omega = \nu = \alpha$ in (3.3) and making use of (2.3), (2.10), (3.15a), and the fact $\nabla_{\omega}h_{\lambda\mu} = 0$, we have

$$(3.16)\qquad R_{\mu\lambda} = H_{\mu\lambda} + \overset{\circ}{R}_{\alpha\mu\lambda}{}^{\alpha} + \overset{\dagger}{R}_{\alpha\mu\lambda}{}^{\alpha}.$$

The relation (3.4a) gives

$$(3.17a)\qquad \begin{aligned} \overset{\circ}{R}_{\alpha\mu\lambda}{}^{\alpha} &= 4\left(h_{\lambda[\mu}\nabla_{\alpha]} + k_{\lambda[\mu}\nabla_{\alpha]} - \nabla_{[\alpha}k_{\mu]\lambda}\right)X^{\alpha} \\ &= 4h_{\lambda[\mu}\nabla_{\alpha]} + 4\nabla_{[\alpha}\left(k_{|\lambda|\mu}X^{\alpha}\right). \end{aligned}$$

On the other hand, the relation (3.4b) gives in virtue of (3.15b)

$$(3.17b)\qquad \begin{aligned} \overset{\dagger}{R}_{\alpha\mu\lambda}{}^{\alpha} &= 8\left(h_{\lambda[\alpha}X_{\mu]} - k_{\lambda[\alpha}{}^{(1)}X_{\mu]}\right)X^{\alpha} \\ &= 4K_{\lambda\mu} + 4k_{\lambda\mu}X^{\alpha(1)}X_{\alpha}. \end{aligned}$$

Our assertion follows immediately from (3.16), (3.17a), and (3.17b). □

THEOREM 3.9. *In MEX_n, the contracted ME-curvature tensors are related by*

(3.18)

$$\begin{aligned} 2R_{[\mu\lambda]} + V_{\mu\lambda} = 12&\left(4X_{[\mu}{}^{(1)}X_{\lambda]} + (2-n)\nabla_{[\mu}X_{\lambda]} + 2\nabla_{[\lambda}{}^{(1)}X_{\mu]}\right) \\ &+ 12\left(\nabla_{\alpha}(k_{\lambda\mu}X^{\alpha}) + 2k_{\lambda\mu}X + 2k_{\lambda\mu}X^{\alpha(1)}X^{\alpha}\right). \end{aligned}$$

PROOF. Summing for $\omega = \nu$ in (3.6) and making use of (3.9) we obtain the left-hand side of (3.18) as in the following way:

$$\begin{aligned} 3R_{[\omega\mu\lambda]}{}^{\omega} &= R_{[\omega\mu]\lambda}{}^{\omega} + R_{[\mu\lambda]\omega}{}^{\omega} + R_{[\lambda\omega]\mu}{}^{\omega} \\ &= R_{\omega\mu\lambda}{}^{\omega} + R_{\mu\lambda\omega}{}^{\omega} + R_{\lambda\omega\mu}{}^{\omega} \\ &= R_{\mu\lambda} + V_{\mu\lambda} - R_{\lambda\mu} \\ &= 2R_{[\mu\lambda]} + V_{\mu\lambda}. \end{aligned}$$

Similarly, we obtain the right-hand side of (3.18). □

4. Field equations in ME-manifold MEX_n

In this section we mean a set of partial differential equations for $g_{\lambda\mu}$ by field equations. In what follows, we are concerned with the geometry of field equations in MEX_n and not with their physical applications.

The following theorem is given in Hlavatý [4].

THEOREM 4.1. *Put*

$$P_{\lambda\mu} = \partial_\alpha \Gamma^\alpha_{\lambda\mu} - \Gamma^\alpha_{\lambda\beta}\Gamma^\beta_{\alpha\mu} + \Gamma^\alpha_{\lambda\mu}\Gamma^\beta_{(\alpha\beta)} - \frac{1}{2}\left(\partial_\mu \Gamma^\alpha_{(\lambda\alpha)} + \partial_\lambda \Gamma^\alpha_{(\mu\alpha)}\right). \tag{4.1a}$$

Then

$$R_{\mu\lambda} + P_{\lambda\mu} = D_\mu S_\lambda. \tag{4.1b}$$

Einstein first proposed the following set of twenty field equations [4].

$$S_\lambda = 0, \tag{4.2a}$$

$$P_{\lambda\mu} = 0 \tag{4.2b}$$

for sixteen unknown $g_{\lambda\mu}$. From Theorem (4.1) we see that the second set reduces by virtue of the first set to

$$R_{\lambda\mu} = 0. \tag{4.2c}$$

Later, Einstein proposed a weaker set consisting of eighteen field equation (4.2a) and

$$R_{(\mu\lambda)} = 0, \tag{4.3a}$$

$$\partial_{[\omega} R_{\mu\lambda]} = 0. \tag{4.3b}$$

The set (4.3b) is obviously equivalent to

$$R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}, \tag{4.3c}$$

where X_λ is an arbitrary vector. The set consisting of (4.3a) and (4.3c) also includes (4.2c). One obtains by choosing a gradient for S_λ.

REMARK 4.2. Einstein proposed several different sets of field equations in his four-dimensional unified field theory. His final suggestion consisting of three sets of tensorial differential equations:

$$S_\lambda = 0, \tag{4.4a}$$

$$R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}, \tag{4.4b}$$

$$R_{(\mu\lambda)} = 0. \tag{4.4c}$$

Hlavatý formulated Einstein's idea mathematically by giving sixty-four equations (2.6) determining the connection $\Gamma^{\nu}_{\lambda\mu}$ and twenty field equations (4.4) for twenty unknown $g_{\lambda\mu}$ and X_λ. Therefore, it would seem natural to follow the analogy of Einstein's field equations (2.8) in MEX_n, too.

THEOREM 4.3. *Under the present condition the following relations hold:*

$$\begin{aligned} R_{[\mu\lambda]} = 2\left(\partial_{[\mu} X_{\lambda]} - \nabla_{[\mu}{}^{(1)} X_{\lambda]} - \nabla_\alpha (k_{\mu\lambda} X^\alpha)\right) \\ + 4k_{\mu\lambda}(X - X^{\alpha(1)} X_\alpha) + 8 X_{[\mu}{}^{(1)} X_{\lambda]}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} R_{(\mu\lambda)} = H_{\mu\lambda} + 2\left(\partial_{(\mu} X_{\lambda)} - \nabla_{(\mu}{}^{(1)} X_{\lambda)} + h_{\mu\lambda} \nabla_\alpha X^\alpha\right) \\ + 4\left(K_{\mu\lambda} + h_{\mu\lambda} X^{\alpha(1)} X_\alpha\right). \end{aligned} \tag{4.6}$$

PROOF. The relations (4.5) and (4.6) follow from (3.14) and making use of (2.3) and (2.10). □

THEOREM 4.4. *The field equation* (4.4b) *in* MEX_n *is equivalent to*

$$\begin{aligned} & -\partial_{[\mu}X_{\lambda]} + 2\nabla_{[\mu}{}^{(1)}X_{\lambda]} + 2\nabla_\alpha(k_{\mu\lambda}X^\alpha) \\ & = 4\left(k_{\mu\lambda}(X - X^{\alpha(1)}X_\alpha) + 2X_{[\mu}{}^{(1)}X_{\lambda]}\right). \end{aligned} \tag{4.7}$$

PROOF. The relation (4.7) follows from (4.5) in virtue of (4.4b). □

THEOREM 4.5. *In* MEX_n, *the field equation* (4.4c) *is equivalent to*

$$\begin{aligned} & H_{\mu\lambda} + 2\left(\partial_{[\mu}X_{\lambda)} - \nabla_{(\mu}{}^{(1)}X_{\lambda)} + h_{\mu\lambda}\nabla_\alpha X^\alpha\right) \\ & + 4\left(K_{\mu\lambda} + h_{\mu\lambda}X^{\alpha(1)}X_\alpha\right) = 0. \end{aligned} \tag{4.8}$$

PROOF. Our assertion (4.8) is an immediate consequence of (4.4c) and (4.6). □

REMARK 4.6. The condition (4.4a) implies $X_\lambda = 0$ and hence $\Gamma^\nu_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\}$ in virtue of (2.13) and (3.7). Hence $R_{\mu\lambda} = H_{\mu\lambda}$. Therefore in our further considerations we restrict ourselves to the conditions $X_\lambda \neq 0$.

REMARK 4.7. The relation (4.4a) is too strong in the field theory in MEX_n, so we shall not adopt (4.4a) as a starting point and impose the field equations as given in (4.7) and (4.8) in MEX_n.

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