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Improving Efficiency of the Moment Estimator of the Extreme Value Index †

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ABSTRACT

In this paper we introduce a method of improving efficiency of the moment estimator of Dekkers, Einmahl and de Haan (1989) for the extreme value index β. A new estimator of β is proposed by adding the third moment to the original moment estimator which is composed of the first two moments of the log-transformed sample data. We establish asymptotic normality of the new estimator and examine an adaptive procedure for the new estimator. The resulting adaptive estimator proves to be asymptotically better than the moment estimator particularly for $\beta < 0$.

Keywords: Extreme value index; Moment estimator; Asymptotic normality; Asymptotic relative efficiency.

1. INTRODUCTION

Suppose we are given an independent and identically distributed sample $X_1, ..., X_n$ from some unknown distribution function F. Suppose F belongs to the domain of attraction of an extreme value distribution G_β for some $\beta \in \Re$ (in short, $F \in \mathcal{D}(G_\beta)$), that is, for some constants $a_n > 0$ and $b_n \in \Re$,

$$\lim_{n\to\infty} P\{(\max\{X_1, ..., X_n\} - b_n)/a_n \le x\} = G_\beta(x)$$

for all x with $1 + \beta x > 0$, where

$$G_\beta(x) := \exp\{-(1+\beta x)^{-1/\beta}\},\ 1 + \beta x > 0.$$

Throughout the case $\beta = 0$ is interpreted as the limit when $\beta \to 0$, so that $G_0(x) = \exp(-e^{-x})$, the standard Gumbel distribution.

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The β, which is called the extreme value index, is one of the major parameters in extreme value theory and is deeply related to the theory of high quantile estimation. The sign of β is often used as a criterion classifying the types of right tail of the underlying distribution function F into three categories, namely, heavy ($\beta > 0$: Pareto, Cauchy, t distributions, etc.), medium ($\beta = 0$: normal, gamma, exponential distributions, etc.) and short tails ($\beta < 0$: uniform(0,1) distribution, etc.). There is a rich literature on the semiparametric problem of how to estimate β using the sample $X_1, ..., X_n$ (see, e.g., Hill (1975), Pickands (1975), Dekkers and de Haan (1989), Dekkers, Einmahl and de Haan (1989) and Drees (1995)).

Let $X_1^{(n)} \geq X_2^{(n)} \geq \cdots \geq X_n^{(n)}$ denote the descending order statistics of $X_1, ..., X_n$. Assuming $x_F := \sup\{x : F(x) < 1\} > 0$ which can be achieved by a simple shift if necessary, we define, for $k = 1, 2, 3$,

$$M_{n,m}^{(k)} := \frac{1}{m}\sum_{i=1}^{m}(\log X_i^{(n)} - \log X_{m+1}^{(n)})^k$$

and

$$N_{n,m}^{(k)} := (M_{n,m}^{(1)})^k / M_{n,m}^{(k)},$$

where $1 \leq m < n$. A sequence of integers $m = m(n)$ is called an intermediate sequence if $m \to \infty$ and $m/n \to 0$ as $n \to \infty$.

If the sign of β is known, some efficient estimators for β are available. For example, if one knows that $\beta > 0$, one can use the Hill (1975) estimator, $M_{n,m}^{(1)}$. For asymptotic behavior of $M_{n,m}^{(1)}$ as $n \to \infty$ with some intermediate sequence $m = m(n)$, see Mason (1982) and Haeusler and Teugels (1985) among others.

For general $\beta \in \Re$ with its sign unknown, there have also appeared several estimators in the literature. These include the Pickands (1975) estimator and some of its variants (cf. Pereira (1994), Drees (1995) and Yun (2000)), which however have poor asymptotic efficiencies particularly for nonnegative β. A competitor is the moment estimator proposed by Dekkers, Einmahl and de Haan (1989), which is defined by

$$\hat{\beta}_{n,m}^{(D)} := M_{n,m}^{(1)} + \frac{1 - 2N_{n,m}^{(2)}}{2(1 - N_{n,m}^{(2)})}. \tag{1.1}$$

They proved consistency of $\hat{\beta}_{n,m}^{(D)}$ for some intermediate sequence $m = m(n)$ and also proved that, under some extra conditions, $\sqrt{m}(\hat{\beta}_{n,m}^{(D)} - \beta)$ is asymptotically normal with mean 0 and variance

$$(\sigma_\beta^{(D)})^2 := \begin{cases} 1+\beta^2, & \text{if } \beta \geq 0, \\ (1-\beta)^2(1-2\beta) \\ \quad \times \left[4 - \dfrac{8(1-2\beta)}{1-3\beta} + \dfrac{(5-11\beta)(1-2\beta)}{(1-3\beta)(1-4\beta)}\right], & \text{if } \beta < 0. \end{cases}$$

The asymptotic performance of $\hat{\beta}_{n,m}^{(D)}$ is much better than those of the Pickands estimator and its variants for $\beta \geq 0$ but is still inferior for $\beta < 0$ (cf. Drees (1995)).

To remedy this drawback with the good asymptotic efficiency of $\hat{\beta}_{n,m}^{(D)}$ for $\beta \geq 0$ unchanged, we introduce in this paper the third moment $M_{n,m}^{(3)}$ into (1.1) and consider estimators defined by

$$\hat{\beta}_{n,m}(p) := M_{n,m}^{(1)} + p \cdot \frac{1 - 2N_{n,m}^{(2)}}{2(1 - N_{n,m}^{(2)})} + (1-p) \cdot \frac{5 - 12N_{n,m}^{(3)} - \sqrt{48N_{n,m}^{(3)} + 1}}{12(1 - N_{n,m}^{(3)})},$$

where $-\infty < p < \infty$. Since $\hat{\beta}_{n,m}(1) = \hat{\beta}_{n,m}^{(D)}$, $\hat{\beta}_{n,m}(p)$ is a generalization of $\hat{\beta}_{n,m}^{(D)}$. We establish asymptotic normality of $\sqrt{m}(\hat{\beta}_{n,m}(p) - \beta)$ for some intermediate sequence $m = m(n)$. Further, we determine the optimal weight $p^*(\beta)$ which minimizes the asymptotic variance of $\sqrt{m}(\hat{\beta}_{n,m}(p) - \beta)$, and investigate the adaptive moment estimator $\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$, where $\tilde{\beta}_n$ is a weakly consistent estimator of β. The $\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$ proves to be asymptotically better than $\hat{\beta}_{n,m}^{(D)}$ for $\beta < 0$, while they have exactly the same asymptotic performance for $\beta \geq 0$. All proofs are collected in Section 3.

2. MAIN RESULTS

Let the function U be defined by $U(x) := F^{-1}(1 - 1/x)$, $x \geq 1$. Then a necessary and sufficient condition for $F \in \mathcal{D}(G_\beta)$ for some $\beta \in \Re$ is the existence of a function $a(t) > 0$ such that, for $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\beta - 1}{\beta} \tag{2.1}$$

(cf. de Haan (1984)). In this case the function $a(t)$ is regularly varying at infinity with index β (in short, $a(t) \in RV_\beta$), i.e., it holds that, for $x > 0$,

$$\lim_{t\to\infty} a(tx)/a(t) = x^\beta.$$

Moreover one may take $a(t) = tU'(t)$ if U has a positive derivative U'. If (2.1) holds and $x_F > 0$, then it holds that, for $x > 0$,

$$\lim_{t\to\infty} \frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = \begin{cases} \log x, & \text{if } \beta \geq 0, \\ \dfrac{x^\beta - 1}{\beta}, & \text{if } \beta < 0, \end{cases} \tag{2.2}$$

which can be easily proved by applying the well-known fact that

$$\begin{cases} U(\infty) = \infty, \ U(t) \in RV_\beta, & \text{if } \beta > 0, \\ \lim_{t\to\infty} a(t)/U(t) = \beta, & \text{if } \beta \geq 0, \\ U(\infty) < \infty, & \\ \quad a(t)/U(t) \sim -\beta(\log U(\infty) - \log U(t)) \text{ as } t \to \infty, & \text{if } \beta < 0, \end{cases} \tag{2.3}$$

where $U(\infty) := \lim_{t\to\infty} U(t) = x_F$. A nonnegative, nondecreasing function $V(t)$ defined on a semiinfinite interval (z, ∞) is said to be Π-varying with auxiliary function $a(t)$ (in short, $V(t) \in \Pi(a(t))$) if there exists a function $a(t) > 0$ such that, for $x > 0$,

$$\lim_{t\to\infty} \frac{V(tx) - V(t)}{a(t)} = \log x.$$

Thus, for $\beta \geq 0$, relation (2.2) means that $\log U(t) \in \Pi(a(t)/U(t))$.

We need the following lemma to establish asymptotic normality of $\hat{\beta}_{n,m}(p)$.

Lemma 2.1. *Let $Y_1, ..., Y_n$ be independent and identically distributed random variables with distribution function $1 - 1/x$, $x \geq 1$, and let $Y_1^{(n)} \geq Y_2^{(n)} \geq \cdots \geq Y_n^{(n)}$ be their order statistics. Let $m = m(n)$ be any sequence of integers such that $1 \leq m < n$ and $m \to \infty$ as $n \to \infty$. Then the following hold.*

(a)

$$\sqrt{m}\left(\frac{1}{m}\sum_{i=1}^m \log\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}} - 1, \frac{1}{m}\sum_{i=1}^m \log^2\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}} - 2, \frac{1}{m}\sum_{i=1}^m \log^3\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}} - 6\right)$$

is asymptotically normal with mean $(0,0,0)$ and covariance matrix $\Sigma_1 := (s_{ij})$ as $n \to \infty$, where

$$s_{11} = 1, \ s_{12} = 4, \ s_{13} = 18, \ s_{22} = 20, \ s_{23} = 108, \ s_{33} = 684.$$

(b) For any $\beta < 0$,

$$\sqrt{m}\left(\frac{-1}{\beta m}\sum_{i=1}^{m}\left\{1-\left(\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}}\right)^{\beta}\right\}-\frac{1}{1-\beta},\right.$$
$$\frac{1}{\beta^2 m}\sum_{i=1}^{m}\left\{1-\left(\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}}\right)^{\beta}\right\}^2-\frac{2}{(1-\beta)(1-2\beta)},$$
$$\left.\frac{-1}{\beta^3 m}\sum_{i=1}^{m}\left\{1-\left(\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}}\right)^{\beta}\right\}^3-\frac{6}{(1-\beta)(1-2\beta)(1-3\beta)}\right)$$

is asymptotically normal with mean $(0,0,0)$ *and covariance matrix* $\mathbf{\Sigma}_2 := (s_{ij})$ *as* $n \to \infty$, *where*

$$\begin{aligned}
s_{11} &= (1-\beta)^{-2}(1-2\beta)^{-1},\\
s_{12} &= 4(1-\beta)^{-2}(1-2\beta)^{-1}(1-3\beta)^{-1},\\
s_{13} &= 18(1-\beta)^{-2}(1-2\beta)^{-1}(1-3\beta)^{-1}(1-4\beta)^{-1},\\
s_{22} &= 4(5-11\beta)(1-\beta)^{-2}(1-2\beta)^{-2}(1-3\beta)^{-1}(1-4\beta)^{-1},\\
s_{23} &= 36(3-7\beta)(1-\beta)^{-2}(1-2\beta)^{-2}(1-3\beta)^{-1}(1-4\beta)^{-1}(1-5\beta)^{-1},\\
s_{33} &= 36(19-105\beta+146\beta^2)(1-\beta)^{-2}(1-2\beta)^{-2}(1-3\beta)^{-2}\\
&\quad \times(1-4\beta)^{-1}(1-5\beta)^{-1}(1-6\beta)^{-1}.
\end{aligned}$$

For asymptotic normality of $\hat{\beta}_{n,m}(p)$, we also need to consider the second order behavior of U. Among several second order conditions on U (cf. Dekkers and de Haan (1989), Dekkers, Einmahl and de Haan (1989), Pereira (1994) and Drees (1995)), we use the one given in Dekkers, Einmahl and de Haan (1989) since it has a fairly general form and is moreover convenient to prove the asymptotic normality of the new estimator in the present paper.

Theorem 2.1. *Suppose* $F \in \mathcal{D}(G_\beta)$ *for some* $\beta \in \Re$ *and* $x_F > 0$. *Further, suppose that there exists a function* $b(t) > 0$ *such that*

$$\begin{cases}
\pm t^{-\beta}U(t) \in \Pi(b(t)), & \text{if } \beta > 0,\\
\lim\limits_{t\to\infty}\dfrac{\log U(tx)-\log U(t)-(\log x)a(t)/U(t)}{b(t)} = \pm\dfrac{\log^2 x}{2},\ x>0, & \text{if } \beta = 0,\\
\pm t^{-\beta}(U(\infty)-U(t)) \in \Pi(b(t)), & \text{if } \beta < 0,
\end{cases}$$

with $a(t)$ in (2.1). Define

$$f(t) := \begin{cases} t^{1-2\beta}U^2(t)/b^2(t), & \text{if } \beta > 0, \\ ta^2(t)/(U(t)b(t))^2, & \text{if } \beta = 0, \\ t^{1-2\beta}(\log U(\infty) - \log U(t))^2/b^2(t), & \text{if } \beta < 0, \end{cases}$$

and $g(t) := tU^2(t)/a^2(t)$, and let f^{-1} and g^{-1} denote the asymptotic inverse functions of f and g, respectively. Let $m = m(n)$ be any intermediate sequence such that, as $n \to \infty$,

$$m = \begin{cases} o(n/f^{-1}(n)), & \text{if } \beta \neq 0, \\ o(n/(f^{-1}(n) + g^{-1}(n))), & \text{if } \beta = 0. \end{cases}$$

Then, for any $p \in (-\infty, \infty)$,

$$\sqrt{m}(\hat{\beta}_{n,m}(p) - \beta) \xrightarrow{d} N(0, \sigma_\beta^2(p)) \text{ as } n \to \infty, \tag{2.4}$$

where

$$\sigma_\beta^2(p) := \begin{cases} (p^2 - 2p + 10 + 9\beta^2)/9, & \text{if } \beta \geq 0, \\ \dfrac{(1-\beta)^2(1-2\beta)}{(1-3\beta)(1-4\beta)(1-5\beta)(1-6\beta)(3-7\beta)^2} \\ \quad \times[(1-\beta)^2(1-6\beta+35\beta^2-78\beta^3+72\beta^4)p^2 \\ \qquad -2(1-\beta)(1-3\beta) \\ \qquad \times(1+8\beta-59\beta^2+114\beta^3-144\beta^4)p \\ \qquad +2(1-3\beta)^3(5-22\beta+43\beta^2-146\beta^3)], & \text{if } \beta < 0. \end{cases}$$

By $\xrightarrow{d}$ we denote convergence in distribution. Note that (2.4) includes the asymptotic normality of $\hat{\beta}_{n,m}^{(D)} = \hat{\beta}_{n,m}(1)$ and thus that $(\sigma_\beta^{(D)})^2 = \sigma_\beta^2(1)$. For each real value of β, $\sigma_\beta^2(p)$ is convex as a function of p and the optimal choice of p minimizing $\sigma_\beta^2(p)$ is

$$p^*(\beta) := \begin{cases} 1, & \text{if } \beta \geq 0, \\ \dfrac{(1-3\beta)(1+8\beta-59\beta^2+114\beta^3-144\beta^4)}{(1-\beta)(1-6\beta+35\beta^2-78\beta^3+72\beta^4)}, & \text{if } \beta < 0, \end{cases}$$

in which case $\sigma_\beta^2(p)$ becomes

$$\sigma_\beta^2(p^*(\beta)) = \begin{cases} 1+\beta^2, & \text{if } \beta \geq 0, \\ \dfrac{(1-\beta)^2(1-2\beta)^2(1-3\beta)}{(1-4\beta)(1-5\beta)} \\ \quad \times\dfrac{(1+\beta+6\beta^2)(1-5\beta+12\beta^2)}{(1-6\beta+35\beta^2-78\beta^3+72\beta^4)}, & \text{if } \beta < 0. \end{cases}$$

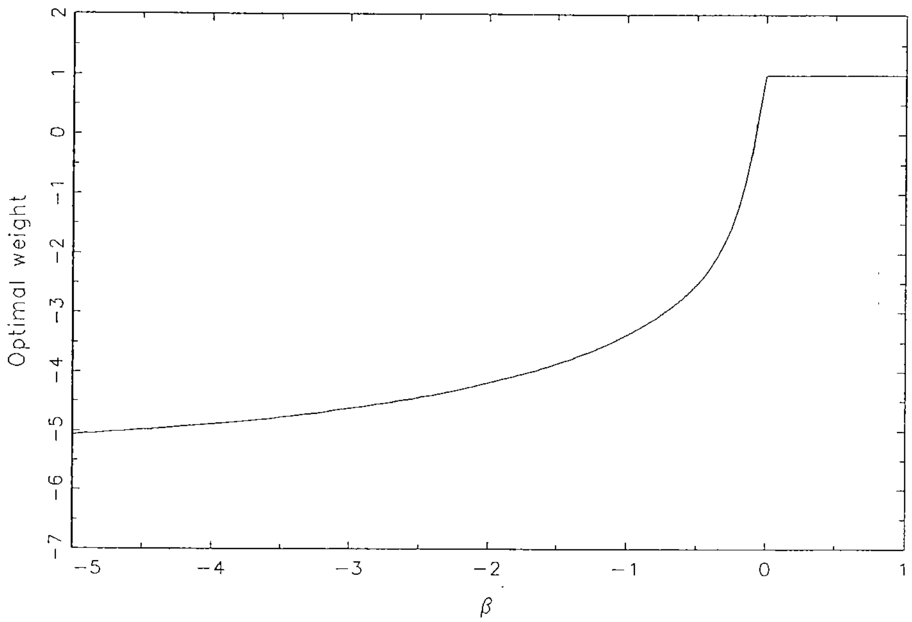

Figure 2.1: Optimal weights $p^*(\beta)$.

Figure 2.1 displays the function of optimal weights $p^*(\beta)$ which range between -6 and 1. In fact, $p^*(\beta) \to -6$ as $\beta \to -\infty$. The fact that $p^*(\beta) = 1$ for $\beta \geq 0$ implies that $\hat{\beta}^{(D)}_{n,m}$ is still optimal for $\beta \geq 0$. Since the optimal weight p^* depends on the unknown parameter β, it is reasonable to utilize the adaptive estimator $p^*(\tilde{\beta}_n)$, where $\tilde{\beta}_n$ is any initial estimator of β which is weakly consistent. The following result then implies that the adaptive estimator $\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$ has the same asymptotic performance as the optimal estimator $\hat{\beta}_{n,m}(p^*(\beta))$ with underlying β.

Theorem 2.2. *Suppose the assumptions of Theorem 2.1 hold for some $\beta \in \Re$. Then,*

$$\sqrt{m}(\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n)) - \beta) \xrightarrow{d} N(0, \sigma^2_\beta(p^*(\beta))) \text{ as } n \to \infty$$

for any initial estimator $\tilde{\beta}_n$ of β which is weakly consistent.

To compare the asymptotic performance of the adaptive estimator $\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$ with that of the Dekkers, Einmahl and de Haan estimator $\hat{\beta}^{(D)}_{n,m}$, we now consider the asymptotic relative efficiency (ARE) of $\hat{\beta}^{(D)}_{n,m}$ with respect to

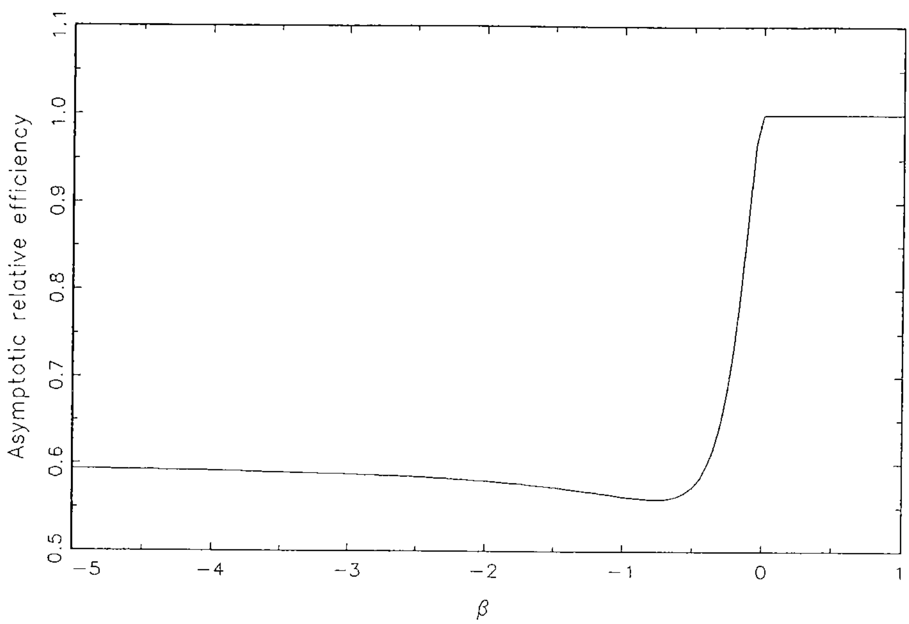

Figure 2.2: ARE of $\hat{\beta}^{(D)}_{n,m}$ with respect to $\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$.

$\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$ which is given by

$$\begin{aligned}
&\mathrm{ARE}_\beta(\hat{\beta}^{(D)}_{n,m}|\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))) \\
&:= \sigma^2_\beta(p^*(\beta))/(\sigma^{(D)}_\beta)^2 \\
&= \begin{cases} 1, & \text{if } \beta \geq 0, \\ \dfrac{(1-2\beta)(1-3\beta)^2(1+\beta+6\beta^2)(1-5\beta+12\beta^2)}{(1-5\beta)(1-\beta+6\beta^2)(1-6\beta+35\beta^2-78\beta^3+72\beta^4)}, & \text{if } \beta < 0. \end{cases}
\end{aligned}$$

Figure 2.2 displays the function $\beta \mapsto \mathrm{ARE}_\beta(\hat{\beta}^{(D)}_{n,m}|\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n)))$. Note that the ARE function has a minimum 0.56 (approx.) at $\beta = -0.76$ (approx.) and

$$\mathrm{ARE}_\beta(\hat{\beta}^{(D)}_{n,m}|\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))) \to 0.6 \text{ as } \beta \to -\infty.$$

The plot clearly exhibits that $\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n))$ is asymptotically better than $\hat{\beta}^{(D)}_{n,m}$ for $\beta < 0$, while they have exactly the same asymptotic performance for $\beta \geq 0$. For a reasonable initial estimator $\tilde{\beta}_n$ of β, we propose to use $\tilde{\beta}_n = \hat{\beta}_{n,m}(p^*(0)) = \hat{\beta}_{n,m}(1) = \hat{\beta}^{(D)}_{n,m}$, the optimal estimator for $\beta = 0$, since $\beta = 0$ is the critical boundary value classifying the types of right tail of the underlying distribution function F.

3. PROOFS

Proof of Lemma 2.1

Let $Z_1, ..., Z_m$ be independent and identically distributed random variables with distribution function $1 - 1/x$, $x \geq 1$, and let $Z_1^{(m)} \geq Z_2^{(m)} \geq \cdots \geq Z_m^{(m)}$ be their order statistics. Then

$$(Y_1^{(n)}/Y_{m+1}^{(n)}, ..., Y_m^{(n)}/Y_{m+1}^{(n)}) \stackrel{d}{=} (Z_1^{(m)}, ..., Z_m^{(m)}).$$

(a) The random vector in (a) is equal in distribution to

$$\sqrt{m}\left(\frac{1}{m}\sum_{i=1}^{m}\log Z_i^{(m)} - 1, \frac{1}{m}\sum_{i=1}^{m}\log^2 Z_i^{(m)} - 2, \frac{1}{m}\sum_{i=1}^{m}\log^3 Z_i^{(m)} - 6\right)$$
$$= \sqrt{m}\left(\frac{1}{m}\sum_{i=1}^{m}\log Z_i - 1, \frac{1}{m}\sum_{i=1}^{m}\log^2 Z_i - 2, \frac{1}{m}\sum_{i=1}^{m}\log^3 Z_i - 6\right).$$

Since $(\log Z_i, \log^2 Z_i, \log^3 Z_i)$, $i = 1, ..., m$, are independent and identically distributed random vectors with mean $(1, 2, 6)$ and covariance matrix $\mathbf{\Sigma}_1$, the assertion follows by the multivariate central limit theorem (cf. Serfling (1980), Theorem B, page 28).

(b) The random vector in (b) is equal in distribution to

$$\sqrt{m}\left(\frac{-1}{\beta m}\sum_{i=1}^{m}(1 - Z_i^{\beta}) - \frac{1}{1-\beta},\right.$$
$$\frac{1}{\beta^2 m}\sum_{i=1}^{m}(1 - Z_i^{\beta})^2 - \frac{2}{(1-\beta)(1-2\beta)},$$
$$\left.\frac{-1}{\beta^3 m}\sum_{i=1}^{m}(1 - Z_i^{\beta})^3 - \frac{6}{(1-\beta)(1-2\beta)(1-3\beta)}\right).$$

Since

$$\left(\frac{-(1 - Z_i^{\beta})}{\beta}, \frac{(1 - Z_i^{\beta})^2}{\beta^2}, \frac{-(1 - Z_i^{\beta})^3}{\beta^3}\right), \ i = 1, ..., m,$$

are independent and identically distributed random vectors with mean

$$\left(\frac{1}{1-\beta}, \frac{2}{(1-\beta)(1-2\beta)}, \frac{6}{(1-\beta)(1-2\beta)(1-3\beta)}\right)$$

and covariance matrix $\mathbf{\Sigma}_2$, the assertion follows as before. □

Proof of Theorem 2.1

Let $Y_1, ..., Y_n$ be independent and identically distributed random variables with distribution function $1 - 1/x$, $x \geq 1$, and let $Y_1^{(n)} \geq Y_2^{(n)} \geq \cdots \geq Y_n^{(n)}$ be their order statistics. Then $(X_1, ..., X_n) \stackrel{d}{=} (U(Y_1), ..., U(Y_n))$, so that $(X_1^{(n)}, ..., X_n^{(n)}) \stackrel{d}{=} (U(Y_1^{(n)}), ..., U(Y_n^{(n)}))$. Using (2.2) and applying an argument similar to the proof of Theorem 2.1 of Dekkers, Einmahl and de Haan (1989), we can show that for $k = 1, 2, 3$, as $n \to \infty$,

$$N_{n,m}^{(k)} \xrightarrow{p} \begin{cases} 1/k!, & \text{if } \beta \geq 0, \\ \dfrac{1}{k!(1-\beta)^k} \displaystyle\prod_{j=1}^{k} (1 - j\beta), & \text{if } \beta < 0. \end{cases} \tag{3.1}$$

By $\xrightarrow{p}$ we denote convergence in probability. Define

$$h(t) := \frac{a(1/(1-F(t)))}{U(1/(1-F(t)))}.$$

Again, using (2.3) and applying an argument similar to the proof of Theorem 3.1 of Dekkers, Einmahl and de Haan (1989), we can show that for $k = 1, 2, 3$, as $n \to \infty$,

$$\sqrt{m} \left(\frac{M_{n,m}^{(k)}}{h^k(X_{m+1}^{(n)})} - \frac{1}{m} \sum_{i=1}^{m} \log^k \frac{Y_i^{(n)}}{Y_{m+1}^{(n)}} \right) = o_p(1) \quad \text{for } \beta \geq 0,$$

and

$$\sqrt{m} \left[\frac{M_{n,m}^{(k)}}{h^k(X_{m+1}^{(n)})} - \frac{1}{(-\beta)^k m} \sum_{i=1}^{m} \left\{ 1 - \left(\frac{Y_i^{(n)}}{Y_{m+1}^{(n)}} \right)^{\beta} \right\}^k \right] = o_p(1) \quad \text{for } \beta < 0.$$

Thus, Lemma 2.1 implies that for $\beta \geq 0$,

$$\sqrt{m} \left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} - 1, \frac{M_{n,m}^{(2)}}{h^2(X_{m+1}^{(n)})} - 2, \frac{M_{n,m}^{(3)}}{h^3(X_{m+1}^{(n)})} - 6 \right) \tag{3.2}$$

is asymptotically normal with mean $(0, 0, 0)$ and covariance matrix $\mathbf{\Sigma}_1$ as $n \to \infty$

and for $\beta < 0$,

$$\sqrt{m}\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} - \frac{1}{1-\beta}, \frac{M_{n,m}^{(2)}}{h^2(X_{m+1}^{(n)})} - \frac{2}{(1-\beta)(1-2\beta)}, \frac{M_{n,m}^{(3)}}{h^3(X_{m+1}^{(n)})} - \frac{6}{(1-\beta)(1-2\beta)(1-3\beta)}\right) \tag{3.3}$$

is asymptotically normal with mean $(0,0,0)$ and covariance matrix $\mathbf{\Sigma}_2$ as $n \to \infty$. Let (Q_1, Q_2, Q_3) and (R_1, R_2, R_3) denote the limiting normal random vectors in (3.2) and (3.3), respectively, and write

$$A_{n,m} := \frac{1 - 2N_{n,m}^{(2)}}{2(1 - N_{n,m}^{(2)})}, \quad B_{n,m} := \frac{5 - 12N_{n,m}^{(3)} - \sqrt{48N_{n,m}^{(3)} + 1}}{12(1 - N_{n,m}^{(3)})}.$$

First, suppose $\beta \geq 0$. Then $\sqrt{m}(M_{n,m}^{(1)} - \beta) \xrightarrow{d} \beta Q_1$ as $n \to \infty$ since $m = o(n/g^{-1}(n))$ for $\beta = 0$ (cf. the proof of Corollary 3.2 of Dekkers, Einmahl and de Haan (1989)), and

$$\begin{aligned}
&\sqrt{m}A_{n,m}\\
&= (N_{n,m}^{(2)} - 1)^{-1}\frac{h^2(X_{m+1}^{(n)})}{M_{n,m}^{(2)}}\\
&\quad \times \left[\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} + 1\right)\sqrt{m}\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} - 1\right) - \frac{1}{2}\sqrt{m}\left(\frac{M_{n,m}^{(2)}}{h^2(X_{m+1}^{(n)})} - 2\right)\right]\\
&\xrightarrow{d} (2^{-1} - 1)^{-1}2^{-1}(2Q_1 - 2^{-1}Q_2) = -2Q_1 + 2^{-1}Q_2 \text{ as } n \to \infty
\end{aligned} \tag{3.4}$$

by (3.1) and (3.2). Also, using the Taylor expansion $\sqrt{48x+1} = 3 + 8(x - 1/6) + o(x - 1/6)$ as $x \to 1/6$, we have

$$\begin{aligned}
&\sqrt{m}B_{n,m}\\
&= (5/3)(N_{n,m}^{(3)} - 1)^{-1}\sqrt{m}[N_{n,m}^{(3)} - 1/6 + o_p(N_{n,m}^{(3)} - 1/6)]\\
&= (5/3)(N_{n,m}^{(3)} - 1)^{-1}\\
&\quad \times \left[\frac{h^3(X_{m+1}^{(n)})}{M_{n,m}^{(3)}}\left\{\left(\frac{(M_{n,m}^{(1)})^2}{h^2(X_{m+1}^{(n)})} + \frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} + 1\right)\sqrt{m}\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} - 1\right)\right.\right.\\
&\qquad \left.\left. -\frac{1}{6}\sqrt{m}\left(\frac{M_{n,m}^{(3)}}{h^3(X_{m+1}^{(n)})} - 6\right)\right\} + \sqrt{m}o_p\left(N_{n,m}^{(3)} - \frac{1}{6}\right)\right]
\end{aligned}$$

$$\xrightarrow{d} \quad (5/3)(6^{-1}-1)^{-1}6^{-1}(3Q_1 - 6^{-1}Q_3) = -Q_1 + 18^{-1}Q_3 \text{ as } n \to \infty. \quad (3.5)$$

Thus,

$$\begin{aligned}\sqrt{m}(\hat{\beta}_{n,m}(p) - \beta) &= \sqrt{m}(M_{n,m}^{(1)} - \beta) + p\sqrt{m}A_{n,m} + (1-p)\sqrt{m}B_{n,m} \\ &\xrightarrow{d} (\beta - p - 1)Q_1 + 2^{-1}pQ_2 + 18^{-1}(1-p)Q_3 \text{ as } n \to \infty.\end{aligned}$$

Next, suppose $\beta < 0$. Then $\sqrt{m}M_{n,m}^{(1)} \xrightarrow{p} 0$ as $n \to \infty$ and

$$\begin{aligned}&\sqrt{m}(A_{n,m} - \beta) \\ &= (1-\beta)(N_{n,m}^{(2)} - 1)^{-1}\frac{h^2(X_{m+1}^{(n)})}{M_{n,m}^{(2)}} \\ &\quad \times \left[\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} + \frac{1}{1-\beta}\right)\sqrt{m}\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})} - \frac{1}{1-\beta}\right)\right. \\ &\quad \left. - \frac{1-2\beta}{2(1-\beta)}\sqrt{m}\left(\frac{M_{n,m}^{(2)}}{h^2(X_{m+1}^{(n)})} - \frac{2}{(1-\beta)(1-2\beta)}\right)\right] \\ &\xrightarrow{d} (1-\beta)\left(\frac{1-2\beta}{2(1-\beta)} - 1\right)^{-1}\frac{(1-\beta)(1-2\beta)}{2}\left(\frac{2}{1-\beta}R_1 - \frac{1-2\beta}{2(1-\beta)}R_2\right) \\ &= -2(1-\beta)^2(1-2\beta)R_1 + 2^{-1}(1-\beta)^2(1-2\beta)^2R_2 \text{ as } n \to \infty \end{aligned} \quad (3.6)$$

by (3.1) and (3.3). Applying the Taylor expansion

$$\begin{aligned}\sqrt{48x+1} &= \frac{3-7\beta}{1-\beta} + \frac{24(1-\beta)}{3-7\beta}\left(x - \frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}\right) \\ &\quad + o\left(x - \frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}\right) \text{ as } x \to \frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2},\end{aligned}$$

we also have

$$\begin{aligned}&\sqrt{m}(B_{n,m} - \beta) \\ &= \frac{(1-\beta)(5-7\beta)}{3-7\beta}(N_{n,m}^{(3)} - 1)^{-1} \\ &\quad \times \sqrt{m}\left[N_{n,m}^{(3)} - \frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2} + o_p\left(N_{n,m}^{(3)} - \frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}\right)\right] \\ &= \frac{(1-\beta)(5-7\beta)}{3-7\beta}(N_{n,m}^{(3)} - 1)^{-1}\end{aligned}$$

$$
\begin{aligned}
&\times\left[\frac{h^3(X_{m+1}^{(n)})}{M_{n,m}^{(3)}}\left\{\left(\frac{(M_{n,m}^{(1)})^2}{h^2(X_{m+1}^{(n)})}+\frac{M_{n,m}^{(1)}}{(1-\beta)h(X_{m+1}^{(n)})}+\frac{1}{(1-\beta)^2}\right)\right.\right.\\
&\quad\times\sqrt{m}\left(\frac{M_{n,m}^{(1)}}{h(X_{m+1}^{(n)})}-\frac{1}{1-\beta}\right)\\
&\quad\left.-\frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}\sqrt{m}\left(\frac{M_{n,m}^{(3)}}{h^3(X_{m+1}^{(n)})}-\frac{6}{(1-\beta)(1-2\beta)(1-3\beta)}\right)\right\}\\
&\quad\left.+\sqrt{m}o_p\left(N_{n,m}^{(3)}-\frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}\right)\right]\\
\stackrel{d}{\to}\;&\frac{(1-\beta)(5-7\beta)}{3-7\beta}\left(\frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}-1\right)^{-1}\frac{(1-\beta)(1-2\beta)(1-3\beta)}{6}\\
&\times\left(\frac{3}{(1-\beta)^2}R_1-\frac{(1-2\beta)(1-3\beta)}{6(1-\beta)^2}R_3\right)\\
=\;&\frac{-3(1-\beta)^2(1-2\beta)(1-3\beta)}{3-7\beta}R_1+\frac{(1-\beta)^2(1-2\beta)^2(1-3\beta)^2}{6(3-7\beta)}R_3\\
&\text{as } n\to\infty.
\end{aligned}
\tag{3.7}
$$

Thus,

$$
\begin{aligned}
&\sqrt{m}(\hat{\beta}_{n,m}(p)-\beta)\\
=\;&\sqrt{m}M_{n,m}^{(1)}+p\sqrt{m}(A_{n,m}-\beta)+(1-p)\sqrt{m}(B_{n,m}-\beta)\\
\stackrel{d}{\to}\;&\frac{(1-\beta)^2(1-2\beta)}{6(3-7\beta)}[-6(3+3p-9\beta-5p\beta)R_1+3p(1-2\beta)(3-7\beta)R_2\\
&\qquad+(1-p)(1-2\beta)(1-3\beta)^2R_3]
\end{aligned}
$$

as $n\to\infty$. Elementary computation now completes the proof. □

Proof of Theorem 2.2

Since $p^*(\cdot)$ is a continuous function, $p^*(\tilde{\beta}_n)\stackrel{p}{\to}p^*(\beta)$ as $n\to\infty$. We use the same notations as in the proof of Theorem 2.1. Then (3.4), (3.5), (3.6) and (3.7) imply that as $n\to\infty$,

$$
\begin{aligned}
&\sqrt{m}(A_{n,m}-B_{n,m})\\
\stackrel{d}{\to}\;&\begin{cases}
-Q_1+2^{-1}Q_2-18^{-1}Q_3, & \text{if } \beta\ge 0,\\
-\dfrac{(1-\beta)^2(1-2\beta)(3-5\beta)}{3-7\beta}R_1+\dfrac{(1-\beta)^2(1-2\beta)^2}{2}R_2 & \\
-\dfrac{(1-\beta)^2(1-2\beta)^2(1-3\beta)^2}{6(3-7\beta)}R_3, & \text{if } \beta<0.
\end{cases}
\end{aligned}
$$

Thus,

$$\begin{aligned}\sqrt{m}(\hat{\beta}_{n,m}(p^*(\tilde{\beta}_n)) - \hat{\beta}_{n,m}(p^*(\beta))) &= (p^*(\tilde{\beta}_n) - p^*(\beta))\sqrt{m}(A_{n,m} - B_{n,m}) \\ &= o_p(1) \text{ as } n \to \infty,\end{aligned}$$

and therefore the assertion is immediate from (2.4). □

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