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On Stationarity of TARMA(p,q) Process†

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ABSTRACT

We consider the threshold autoregressive moving average (TARMA) process and find a sufficient condition for strict stationarity of the process. Given region for stationarity of TARMA(p,q) model is the same as that of TAR(p) model given by Chan and Tong(1985), which shows that the moving average part of TARMA(p,q) process does not affect the stationarity of the process. We find also a sufficient condition for the existence of kth moments($k \geq 1$) of the process with respect to the stationary distribution.

Keywords: threshold ARMA(p,q) model; stationarity; ergodicity; moment

1. INTRODUCTION

The processs $\{y_t\}$ is said to be a threshold autoregressive moving average process of order (p,q) and with delay $d \in \{1, 2, \cdots\}$, denoted by TARMA(p,q), if

$$y_t = \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \sum_{i=1}^{q} \theta_i^{(j)} e_{t-i} + e_t, \quad a_{j-1} < y_{t-d} \leq a_j, \tag{1}$$

where $j = 1, 2, \cdots, l$ and $-\infty = a_0 < a_1 < a_2 < \cdots < a_l = \infty$. The coefficients $\phi_i^{(j)}$ and $\theta_i^{(j)}$ are constants, and $\{e_t\}$ is a series of independent and identically distributed random variables with $E|e_t| < \infty$. Threshold model, first proposed by Tong(1978) is one of the important class of nonlinear time series model and probabilistic properties such as stationarity, ergodicity, geometric ergodicity, existence of moments of these models are studied by many authors, for example, Petrucelli and Woolford(1984), Chan and Tong(1985), Tong(1990), Li and Li(1996), Ling(1999) etc. The process (1) reduces to the threshold AR model, denoted by

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TAR(p) if $q = 0$. Chan and Tong(1985) proved that a sufficient condition for the stationarity and ergodicity of the TAR(p) model is

$$\max_j \sum_{i=1}^{p} |\phi_i^{(j)}| < 1. \tag{2}$$

Our aim in this paper is to give a sufficient conditon for strict stationarity and existence of moments of y_t obtained by (1). To do this, we first rephrase y_t as a Markov chain (see, e.g. Tjøstheim(1990), Bhattacharya and Lee(1995)) and then find a proper test function to use the following results due to Tweedie(1988).

For terminologies and relevant results in Markov chain theory, we refer to Meyn and Tweedie(1993).

Theorem 1. *Suppose $\{X_t\}$ is a Feller chain with transition probability function $P(x, dy)$. If there exists, for some compact set $A \in \mathcal{B}$, a nonnegative function g and an $\epsilon > 0$ such that*

$$\int_{A^c} P(x, dy) g(y) \leq g(x) - \epsilon, \quad x \in A^c, \tag{3}$$

then there exists a $\sigma-$finite invariant measure μ for P with $0 < \mu(A) < \infty$. Further, if

$$\int_A \mu(dx) [\int_{A^c} P(x, dy) g(y)] < \infty, \tag{4}$$

then μ is finite. Further, if

$$\int_{A^c} P(x, dy) g(y) \leq g(x) - f(x), \quad x \in A^c, \tag{5}$$

then μ admits a finite $f-$ moment.

We may assume, without loss of generality, $d \in \{1, 2, \cdots, p\}$.
Let

$$Y_t = (y_t, y_{t-1}, \cdots, y_{t-p+1}, e_t, \cdots, e_{t-q+1})'_{(p+q)\times 1},$$

and let

$$\Phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} & \phi_2^{(j)} & \cdots & \phi_p^{(j)} & \theta_1^{(j)} & \theta_2^{(j)} & \cdots & \theta_q^{(j)} \\ & I_{(p-1)\times(p-1)} & & 0 & & \mathbf{O}_{(p-1)\times q} & & \\ & & & & 0 & \cdots & & 0 \\ & & & & & & & \vdots \\ & \mathbf{O}_{q\times p} & & & & I_{(q-1)\times(q-1)} & & 0 \end{pmatrix}$$

where $I_{r\times r}$ is the $r \times r$ identity matrix. Then the equation (1) can be rewritten as the following equation

$$Y_t = \sum_{j=1}^{l}(\Phi_0^{(j)} + \Phi^{(j)}Y_{t-1})I_{\{Y_{t-1}\in R_j\}} + \eta e_t, \tag{6}$$

where $\Phi_0^{(j)} = (\phi_0^{(j)}, 0, \cdots, 0)'_{(p+q)\times 1}$, $\eta = (1, 0, 0\cdots, 0, 1, 0, \cdots, 0)'$ is a $(p+q)\times 1$ vector all of whose components are zero except for the first and $(p+1)$th which are both equal to 1 and $R_j = \mathbf{R}^{d-1} \times (a_{j-1}, a_j] \times \mathbf{R}^{p-d} \times \mathbf{R}^q$. Y_t defined by the equation (6) is a Markov chain with the state space $\mathbf{R}^{p+q}$.

Liu and Susko(1992) proved that $\phi_1^{(1)} < 1, \phi_1^{(l)} < 1$ and $\phi_1^{(1)}\phi_1^{(l)} < 1$ is a sufficient condition for the stationarity of the TARMA(1,q) model. Under the assumption that $\theta_i^{(j)} = \theta_i$ for all j, Brockwell, Liu and Tweedie(1992) proved that a condition

$$\rho(\max_j |\Phi^{(j)}|) < 1 \tag{7}$$

is sufficient for stationarity and ergodicity of the TARMA(p,q) model, where for a series of matrices $A^{(k)} = (a_{ij}^{(k)}), \max_k |A^{(k)}| = (\max_k |a_{ij}^{(k)}|)$. Ling(1999) considered the TARMA(p,q) model as a special case of double threshold ARMA conditional heteroskedastic model and proved that under the assumption (7), (1) has a strictly stationary solution.

2. MAIN RESULTS

We now make the assumptions:
Assumption A : The function F on $\mathbf{R}^{p+q}$ defined by

$$F(x) = \sum_{j=1}^{l}(\Phi_0^{(j)} + \Phi^{(j)}x)I_{\{x\in R_j\}}$$

is continuous in x.
Assumption B :

$$\max_j \sum_{i=1}^{p} |\phi_i^{(j)}| < 1. \tag{8}$$

Recall that Assumption A ensures that the process $\{Y_t\}$ given in (6) is a Feller chain.

Next theorem is one of our main theorems which extends the earlier results of, for example, Ling(1999), Liu and Susko(1992), Brockwell and *et al.*(1992) and Chan and Tong(1985) etc.

Theorem 2. *Suppose that Assumption A and Assumption B hold. Then there exists a strictly stationary solution $\{y_t\}$ satisfying the equation (1) and $E_\pi|y_t| < \infty$, where π is the stationary distribution of $\{y_t\}$.*

Proof. Suppose $\max_j \sum_{i=1}^p |\phi_i^{(j)}| < 1$. For simplicity of notation, we first define, for $x = (x_1, \cdots, x_p, z_1, \cdots, z_q)'$,

$$h(x) = \sum_{j=1}^{l} (\phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} x_i + \sum_{i=1}^{q} \theta_i^{(j)} z_i) I_{(x \in R_j)},$$

where $R_j = \mathbf{R}^{d-1} \times (a_{j-1}, a_j] \times \mathbf{R}^{p-d} \times \mathbf{R}^q$.
Then Y_t given $Y_{t-1} = x$ can be written by

$$Y_t = (h(x) + e_t, x_1, \cdots, x_{p-1}, e_t, z_1, \cdots, z_{q-1})'. \tag{9}$$

We now define a test function $g : \mathbf{R}^{p+q} \to \mathbf{R}$ by

$$g(x) = \max_{1 \le i \le p} \{\alpha_i |x_i|\} + \sum_{j=1}^{q} \beta_j |z_j|, \quad x = (x_1, \cdots, x_p, z_1, \cdots, z_q)', \tag{10}$$

where positive $\alpha_i (1 \le i \le p)$, and $\beta_j (1 \le j \le q)$ are to be given later.

For $x = (x_1, \cdots, x_p, z_1, \cdots, z_q)'$, from (9) and (10),

$$\begin{aligned} E[g(Y_t)|Y_{t-1} = x] &= E[g(h(x) + e_t, x_1, \cdots, x_{p-1}, e_t, z_1, \cdots, z_{q-1})'] \\ &\le E[\max\{\alpha_1 |h(x) + e_t|, \alpha_2 |x_1|, \cdots, \alpha_p |x_{p-1}|\} \\ &\quad + \beta_1 |e_t| + \beta_2 |z_1| + \cdots + \beta_q |z_{q-1}|] \\ &= \max\{\alpha_1 |h(x)|, \alpha_2 |x_1|, \cdots, \alpha_p |x_{p-1}|\} \\ &\quad + \beta_2 |z_1| + \cdots + \beta_q |z_{q-1}| + (\alpha_1 + \beta_1) E|e_t| \\ &\le \max\{\alpha_1 (\sum_{j=1}^{l} (\sum_{i=1}^{p} |\phi_i^{(j)}| |x_i|) I_{(x \in R_j)}), \alpha_2 |x_1|, \cdots, \alpha_p |x_{p-1}|\} \\ &\quad + \alpha_1 \sum_{j=1}^{l} (|\phi_0^{(j)}| + \sum_{i=1}^{q} |\theta_i^{(j)}| |z_i|) I_{(x \in R_j)} \\ &\quad + \beta_2 |z_1| + \cdots + \beta_q |z_{q-1}| + (\alpha_1 + \beta_1) E|e_t| \qquad (11) \\ &= I + II, \end{aligned}$$

where $I = \max\{\alpha_1(\sum_{j=1}^{l}(\sum_{i=1}^{p}|\phi_i^{(j)}||x_i|)I_{(x\in R_j)}), \alpha_2|x_1|, \cdots, \alpha_p|x_{p-1}|\}$ and II is the remaining part of the equation (11).

From assumption $\max_j \sum_{i=1}^{p}|\phi_i^{(j)}| < 1$, we may choose $\rho > 0$ and $\delta > 0$ such that

$$\max_j \sum_{i=1}^{p} |\phi_i^{(j)}| < \rho < \rho^{\frac{1}{p}} < \delta < 1 \tag{12}$$

and choose $\alpha_1 > 0$ arbitrarily and fix. Define

$$\alpha_k = \rho^{\frac{1}{p}}\alpha_{k-1}, \quad k = 2, 3, \cdots, p.$$

Then we have that

$$\frac{\alpha_{i+1}}{\alpha_i} < \delta \quad \text{and} \quad \max_j \sum_{i=1}^{p} |\phi_i^{(j)}|\frac{\alpha_1}{\alpha_i} < \delta < 1. \tag{13}$$

Therefore we have, using the inequality in (13), that

$$\begin{aligned} I &= \max\{\alpha_1 \sum_{j=1}^{l}(\sum_{i=1}^{p}|\phi_i^{(j)}||x_i|)I_{(x\in R_j)}, \alpha_2|x_1|, \cdots, \alpha_p|x_{p-1}|\} \\ &\le \max\{\sum_{j=1}^{l}(\sum_{i=1}^{p}|\phi_i^{(j)}|\frac{\alpha_1}{\alpha_i}\alpha_i|x_i|)I_{(x\in R_j)}, \alpha_1\delta|x_1|, \cdots, \alpha_{p-1}\delta|x_{p-1}|\} \\ &\le \max\{\sum_{j=1}^{l}(\max_{1\le i\le p}\{\alpha_i|x_i|\}\sum_{i=1}^{p}|\phi_i^{(j)}|\frac{\alpha_1}{\alpha_i})I_{(x\in R_j)}, \alpha_1\delta|x_1|, \cdots, \alpha_{p-1}\delta|x_{p-1}|\} \\ &\le \max\{\max_{1\le i\le p}\{\alpha_i|x_i|\}\delta, \alpha_1\delta|x_1|, \cdots, \alpha_{p-1}\delta|x_{p-1}|\} \\ &\le \delta \max_{1\le i\le p}\{\alpha_i|x_i|\}. \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned} II &= \alpha_1 \sum_{j=1}^{l}(|\phi_0^{(j)}| + \sum_{i=1}^{q}|\theta_i^{(j)}||z_i|)I_{(x\in R_j)} + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}| \\ &\quad +(\alpha_1+\beta_1)E|e_t| \\ &\le \alpha_1 \sum_{j=1}^{l}(\sum_{i=1}^{q}|\theta_i^{(j)}||z_i|)I_{(x\in R_j)} + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}| \\ &\quad +\alpha_1 \max_j |\phi_0^{(j)}| + (\alpha_1+\beta_1)E|e_t| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_1(\sum_{i=1}^{q}\theta_i|z_i|) + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}| + K \\
&\leq (\alpha_1\theta_1 + \beta_2)|z_1| + (\alpha_1\theta_2 + \beta_3)|z_2| + \cdots + (\alpha_1\theta_{q-1} + \beta_q)|z_{q-1}| \\
&\quad + \alpha_1\theta_q|z_q| + K, \qquad (15)
\end{aligned}$$

where

$$\theta_i = \max_j |\theta_i^{(j)}| \ (1 \leq i \leq q) \text{ and } K = \alpha_1 \max_j |\phi_0^{(j)}| + (\alpha_1 + \beta_1)E|e_1|. \qquad (16)$$

Now define $\beta_1, \cdots, \beta_q$ such that, for α_1, δ, and θ_i given in (13) and (16),

$$\begin{aligned}
\beta_1 &= \alpha_1(\frac{\theta_1}{\delta} + \frac{\theta_2}{\delta^2} + \cdots + \frac{\theta_q}{\delta^q}) \\
\beta_2 &= \alpha_1(\frac{\theta_2}{\delta} + \frac{\theta_3}{\delta^2} + \cdots + \frac{\theta_q}{\delta^{q-1}}) \\
&\vdots \\
\beta_{q-2} &= \alpha_1(\frac{\theta_{q-2}}{\delta} + \frac{\theta_{q-1}}{\delta^2} + \frac{\theta_q}{\delta^3}) \\
\beta_{q-1} &= \alpha_1(\frac{\theta_{q-1}}{\delta} + \frac{\theta_q}{\delta^2}) \\
\beta_q &= \alpha_1\frac{\theta_q}{\delta}.
\end{aligned}$$

Then we have

$$\delta\beta_i = \alpha_1\theta_i + \beta_{i+1}, \quad i = 1, 2, \cdots, q-1 \qquad (17)$$

and

$$\delta\beta_q = \alpha_1\theta_q, \qquad (18)$$

and hence, from (15), (17) and (18), we conclude that

$$\begin{aligned}
II &= \delta\beta_1|z_1| + \delta\beta_2|z_2| + \cdots + \delta\beta_{q-1}|z_{q-1}| + \delta\beta_q|z_q| + K \\
&= \delta\sum_{i=1}^{q}\beta_i|z_i| + K.
\end{aligned}$$

Therefore, for $\delta < 1$,

$$\begin{aligned}
I + II &\leq \delta(\max_{1\leq i\leq p}\{\alpha_i|x_i|\} + \sum_{i=1}^{q}\beta_i|z_i|) + K \\
&= \delta g(x) + K. \qquad (19)
\end{aligned}$$

Note that for fixed α_i, $(1 \le i \le p)$ and β_i, $(1 \le i \le q)$ given above, if we define $\|x\| = g(x)$, then $\|.\| : \mathbf{R}^{p+q} \to \mathbf{R}$ is indeed a norm and $B_M = \{x \in \mathbf{R}^{p+q} : \|x\| \le M$ for some $M > 0\}$ is a compact subset of $\mathbf{R}^{p+q}$. Therefore for any given $\epsilon > 0$, there exist $\delta' > 0$ with $\delta < \delta' < 1$ and M' so large that

$$I + II \le \delta' g(x) - \epsilon, \quad if \quad x \in B^c_{M'}. \tag{20}$$

Moreover, from (19), for any x,

$$E[g(Y_t)|Y_{t-1} = x] \le \delta g(x) + K,$$

and hence

$$E[g(Y_t)|Y_{t-1} = x] \le \delta M + K, \quad \forall x \in B_M. \tag{21}$$

(20) and (21) show that the conditions (3) and (4) in Theorem 1 hold, from which the existence of a strictly stationary solution of (1) follows. Moreover, since $E[g(Y_t)|Y_{t-1} = x] \le \delta' g(x) = g(x) - (1-\delta')g(x)$, $x \in B^c_{M'}$, we may take $f(x) = (1-\delta')g(x)$ in (5) and hence by Theorem 1, $E_\pi|y_t| < \infty$ for stationary distribution π.

Remark 1. Theorem 2 shows that moving average part does not affect the stationarity of the TARMA(p,q) model.

We now consider the nonlinear ARMA(p,q) model y_t which is given by

$$y_t = \phi_0(x) + \sum_{i=1}^{p} \phi_i(x) y_{t-i} + \sum_{i=1}^{q} \theta_i(x) e_{t-i} + e_t, \tag{22}$$

where $x = (y_{t-1}, y_{t-2}, \cdots, y_{t-p})$. Threshold ARMA(p,q) given by (1) is a special case of the process obtained by (22), where the coefficient functions $\phi_0, \cdots, \phi_p$ and $\theta_1, \cdots, \theta_q$ are finitely piecewise continuous on $\mathbf{R}^p$.

We extend the results in Theorem 2 to the following corollary, since stationarity is a problem of how the process behaves when it becomes large.

Corollary 1. *Assume that y_t is given by (22), where $\phi_0, \theta_1, \cdots, \theta_q$ are bounded and for $x \in R_j$, $|\phi_i(x)| \le |\phi_i^{(j)}|$, $\|x\| > M_j$ for some $M_j > 0$, where $\|\cdot\|$ is any norm on $\mathbf{R}^p$. If Assumption A and Assumption B hold, then the results in Theorem 2 hold.*

Theorem 3. *Assume y_t is given by (22), where $\phi_0, \theta_1, \cdots, \theta_q$ are bounded. If Y_t is a Feller chain and*

$$\sup_{x \in R^p} \sum_{i=1}^{p} |\phi_i(x)| < 1, \tag{23}$$

then the conclusions of Theorem 2 hold.

Proof. Since the proof is essentially the same line as that of Theorem 2, we give $\alpha_i, (1 \le i \le p)$ and the test function $g(x)$ to complete the proof. From equation (23), we can choose $\rho > 0$ and $\delta > 0$ such that

$$\sup_{x \in R^p} \sum_{i=1}^{p} |\phi_i(x)| < \rho < \rho^{\frac{1}{p}} < \delta < 1. \tag{24}$$

Now let $\alpha_i = \rho^{\frac{i}{p}}, i = 1, 2, \cdots, p$ and define a test function $g(x)$ by, for $x = (x_1, \cdots, x_p, z_1, \cdots, z_q)$,

$$g(x) = \max_{1 \le i \le p} \{\rho^{\frac{i}{p}} |x_i|\} + \sum_{i=1}^{q} \beta_j |z_j|, \tag{25}$$

where $\beta_j's$ are defined by the same manner as those in the proof of Theorem 2, with α_i and δ given above and $\theta_i = \max_{x \in R^p} |\theta_i(x)|$.

Corollary 2. *Suppose that the process $\{Y_t\}$ is φ-irreducible and aperiodic. Then under the Assumption B, the process has a unique stationary solution which is geometrically ergodic.*

Proof. When the process is φ-irreducible and aperiodic, (20) and (21) are sufficient for geometric ergodicity.(see, Tweedie(1983)). In this case Assumption A is not necessary.

Remark 2. It is not an easy task to prove the $\varphi-$irreducibility of the general TARMA(p,q) model. But if we assume that $\theta_1^{(j)} = \theta_2^{(j)} = \cdots = \theta_p^{(j)} = 0, \ \forall j$, then the $\varphi-$irreducibility of $\{Y_t\}$ can be obtained by the same way as that of TAR(p) model. In this case we take $Y_t = (y_t, \cdots, y_{t-p+1}, e_{t-p}, \cdots, e_{t-q+1})'$ and the first row of $q \times q$ matrix $\Phi_i^{(j)}$ is $(\phi_1^{(j)}, \cdots, \phi_p^{(j)}, \theta_{p+1}^{(j)}, \cdots, \theta_q^{(j)})$. Then $\{Y_t\}$ is $\varphi-$irreducible and aperiodic provided the distribution of $\{e_t\}$ is absolutely continuous with positive probability density function.

Following theorem gives us a sufficient condition under which the kth moment of $\{y_t\}$ in (1) is finite.

Theorem 4. *Suppose that Assumption A holds and* $E|e_t|^k < \infty$, *for some* $k \geq 1$. *If* $\max_j \sum_{i=1}^p |\phi_i^{(j)}| < \frac{1}{2}$, *then* $E_{\pi_1}|y_t|^k < \infty$, *where* π_1 *is the stationary distribution of* $\{y_t\}$.

Proof. Since we assume $\max_j \sum_{i=1}^p |\phi_i^{(j)}| < \frac{1}{2}$, there exist $\alpha_1 > \alpha_2 > \cdots > \alpha_p > 0$ and $\delta > 0$ such that

$$\frac{\alpha_{i+1}}{\alpha_i} < \delta \quad \text{and} \quad \max_j \sum_{i=1}^p |\phi_i^{(j)}| < \delta < \frac{1}{2}.$$

Define $v(x) = (g(x))^k, \quad k \geq 1$ where $g(x)$ is the test function given in (10). Then we have that, by the same argument used in the proof of Theorem 2, for $x = (x_1, \cdots, x_p, z_1, \cdots, z_q)'$,

$$\begin{aligned} E[v(Y_t)|Y_{t-1} = x] &\leq E[(\delta g(x) + \alpha_1 \max_{1 \leq j \leq l} |\phi_0^{(j)}| + (\alpha_1 + \beta_1)|e_t|)^k] \\ &\leq (2\delta)^{k-1} \cdot \delta \cdot (g(x))^k + K_1 \\ &\leq \delta(g(x))^k + K_1, \end{aligned} \tag{26}$$

where

$$K_1 = 2^{k-1} E[(\alpha_1 \max_{1 \leq j \leq l} |\phi_0^{(j)}| + (\alpha_1 + \beta_1)|e_1|)^k] < \infty, \quad \text{if} \quad E|e_1|^k < \infty.$$

If we choose $\delta'' < 1$ and $M'' > 0$ sufficiently large, then

$$\begin{aligned} E[v(Y_t)|Y_{t-1} = x] &\leq \delta v(x) + K_1 \\ &\leq v(x) - (1 - \delta'')v(x), \quad \text{for } x \in B_{M''}^c, \end{aligned}$$

and therefore, by Theorem 1, we have, for a stationary distribution π of Y_t,

$$E_\pi[v(Y_t)] < \infty, \tag{27}$$

Now if we take for any Borel subset B of $\mathbf{R}$, $\pi_1(B) = \pi(B \times \mathbf{R}^{p+q-1})$, then π_1 is the stationary distribution of $\{y_t\}$ and hence from (27), $E_{\pi_1}|y_t|^k < \infty$ is obtained.

Remark 3. Under the Assumption A and $E|e_t|^k < \infty$, it is shown that $\sum_{i=1}^p \max_j |\phi_i^{(j)}| < 1$ is sufficient for finiteness of $E_\pi|y_t|^k$ (Ling(1999)). It depends on the values $\{\phi_i^{(j)}\}$ that which one is stronger than the other between $\max_j \sum_{i=1}^p |\phi_i^{(j)}| < \frac{1}{2}$ and $\sum_{i=1}^p \max_j |\phi_i^{(j)}| < 1$.

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