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# A New Estimator for Seasonal Autoregressive Process †

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ABSTRACT

For estimating parameters of possibly nonlinear and/or non-stationary seasonal autoregressive(AR) processes, we introduce a new instrumental variable method which use the direction vector of the regressors in the same period as an instrument. On the basis of the new estimator, we propose new seasonal random walk tests whose limiting null distributions are standard normal regardless of the period of seasonality and types of mean adjustments. Monte-Carlo simulation shows that the powers of the proposed tests are better than those of the tests based on ordinary least squares estimator(OLSE).

*Keywords:* Instrumental variable estimation; recursive mean adjustment; seasonal random walk test

## 1. INTRODUCTION

Since the pioneering work of Dickey and Fuller(1979), the problem of the testing of the random walk hypothesis of the various time series data has been the subject of tremendous research effort in the statistics and economics literatures. In particular, motivated by the occurrence of the strong seasonal effects in most economic time series data, extensions of the tests to seasonal time series models were made by Dickey, Hasza, and Fuller(1984), Hylleberg, Engle, Granger, and Yoo(1990) and many others. See Fuller(1996, chapter 10), Hamilton(1994, chapter 17) and Stock(1994) for recent survey and extensive bibliography in this field.

However most of these results are based exclusively on the asymptotic properties of the standard OLSE-based tests for the linear processes and the types of

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null distributions are not normal and depend on specifications of model such as the period of the seasonality and type of mean adjustment. This is also true of the most of the other works which consider alternative estimators such as robust M-estimators of Lucas(1995) and Knight(1989).

In this paper, we consider a general class of possibly nonlinear and/or non-stationary seasonal time series model of the autoregressive type

$$\begin{aligned} y_t &= \mu_t + u_t \\ u_t &= f(u_{t-d}) + \beta h(u_{t-d}) + e_t, \quad t = d+1, \ldots, n \end{aligned} \tag{1.1}$$

where $n = md$, $\{y_t\}_{t=1}^{n}$ is a set of observations, $\{\mu_t\} = \{\mu_1, \mu_2, ..., \mu_d, \mu_1, \mu_2, ..., \mu_d, ..., \}$ is a repeated sequence of seasonal means $\{\mu_1, \mu_2, ..., \mu_d\}$ of the $y_t$, $\beta$ is an unknown parameter of interest, $f(u_t)$, $h(u_t)$ are known functions and $e_t$ is a sequence of independent identically distributed (*i.i.d.*) errors with zero mean and finite variance $\sigma^2$. The null hypothesis of interest is the hypothesis

$$H_0 : \beta = 0.$$

In this paper, we develop new tests whose limiting null distributions are always standard normal regardless of the period of seasonality and the type of mean adjustment.

In order to develop normal tests, So and Shin(1999a) utilized the sign of the regressor as instrumental variable. However, their tests remain to be improved because the sign is not an efficient instrument in the seasonal model. We develop new normal tests based on instrumental variable(IV) estimator with a normalized regressor in which direction of the regressors in the same period are used as a instrumental variable. When applied to the tests of random walks in the seasonal AR models with $u_t = u_{t-d} + \beta h(u_{t-d}) + e_t$, the new tests have powers better than the OLSE-based tests of Dickey , Hasza and Fuller(1984).

The rest of this paper is organized as follows. In Section 2, new IV-estimation is introduced for seasonal AR models. In Section 3, Monte-Carlo simulations compare the proposed tests favorably with OLSE-based tests .

## 2. SEASONAL AR MODELS

We first consider a simple seasonal AR model with no mean

$$y_t = f(y_{t-d}) + \beta h(y_{t-d}) + e_t, \quad t = d+1, \ldots, n \tag{2.1}$$

In order to motivate a new IV estimator, we write the model (2.1) as a vector autoregressive(VAR) process $\{x_t\}$ in $R^d$ of the form;

$$x_t = f(x_{t-1}) + \beta h(x_{t-1}) + v_t, t = 2, \ldots, m,$$

where $x_t = (y_{(t-1)d+1}, \ldots, y_{td})^T$, $f(x_t) = (f(y_{(t-1)d+1}), \ldots, f(y_{td}))^T$, $v_t = (e_{(t-1)d+1}, \ldots, e_{td})^T$.
Then OLSE of $\beta$ is given by

$$\hat{\beta}_o = \sum_{t=2}^{m} (x_t - f(x_{t-1}), h(x_{t-1})) / \sum_{t=2}^{m} |h(x_{t-1})|^2,$$

where $(a, b) = \sum_{i=1}^{d} a_i b_i$ for $a, b \in R^d$, $|a|^2 = (a, a)$ is a Euclidean norm of $a \in R^d$ . For the special linear model with $y_t = \rho y_{t-d} + e_t$ and $\rho = 1 + \beta$, Dickey, Hasza and Fuller(1984) constructed tests for $H_0$ as given by $n(\hat{\beta}_o - 0)$ and $\hat{\tau}_o = (\hat{\beta}_o - 0)/se(\hat{\beta}_o)$. Here and in the sequel, $se(\hat{\beta})$ denotes the standard error of an estimator $\hat{\beta}$. The limiting null distributions of the test statistics are not normal and are given by

$$n(\hat{\beta}_o - 0) \Rightarrow (\int_0^1 |W(t)|^2 dt)^{-1} (\int_0^1 (W(t), dW(t))),$$

$$\hat{\tau}_o \Rightarrow (\int_0^1 |W(t)|^2 dt)^{-1/2} (\int_0^1 (W(t), dW(t))),$$

where $W(t) = (W_1(t), \ldots, W_d(t))$ is a vector of independent standard Brownian motions on [0, 1] and $\Rightarrow$ denotes convergence in distribution. The reason for nonnormality of the test statistics is that the distribution $\int_0^1 (W(t), dW(t)) = \{|W(1)|^2 - d\}/2$ is not normal, which is the weak limit of $n^{-1} \sum e_t y_{t-d}$.

As a simple trick of attaining normality, we normalize a regressor vector $h(x_{t-1})$ and obtain a new vector of instrumental variables $h^*(x_{t-1}) = h(x_{t-1}) / |h(x_{t-1})|$ of unit length. Then the corresponding IV estimator is defined by;

$$\hat{\beta}_c = \sum_{t=2}^{m} (x_t - f(y_{t-1}), h^*(x_{t-1})) / \sum_{t=2}^{m} |h(x_{t-1})|.$$

Now, the weak limit is

$$m^{-1/2} \sum (v_t, h^*(x_{t-1})) \Rightarrow N(0, \sigma^2)$$

by the martingale central limit theorem(CLT) as is to be shown below in Lemma 1.

Using $\hat{\beta}_c$, we can construct the new tests

$$\hat{\tau}_c = (\hat{\beta}_c - 0) / \, se(\hat{\beta}_c), \tag{2.2}$$

where $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2$ and

$$se(\hat{\beta}_c) = \hat{\sigma} m^{1/2} / \sum_{t=2}^{m} |h(x_{t-1})|.$$

A simple consistent estimator of $\sigma^2$ is the OLSE

$$\hat{\sigma}^2 = (n-1)^{-1} \sum_{t=2}^{m} |x_t - \hat{\beta}_o h(x_{t-1})|^2.$$

Limiting normality of the test statistics are given in Theorem 1. Proofs of theorems are provided in the Appendix.

**Lemma 1.** Consider model (2.1). For any fixed $\beta$, we have

$$m^{-1/2} \sum_{t=2}^{m} (v_t, h^*(x_{t-1})) \Rightarrow \sigma Z,$$

where $Z$ is a standard normal random variable.

**Proof.** Since $\sum v_t, h^*(x_{t-1}))$ is a martingale for all fixed $\beta$, the result follows immediately from the martingale central limit theorem (Fuller, 1996, $p$. 235).

Lemma 1 is a consequence of the fact that $\sum_{t=2}^{m} (v_t, h^*(x_{t-1}))$ is a martingale for all real $\beta$ and the martingale central limit theorem (Fuller, 1996, $p$.235).

**Theorem 1.** Consider model (2.1) with $\beta \in R$. Then, for any fixed $\beta \in R$,

$$\hat{\tau}_c \Rightarrow Z.$$

**Proof.** Note that

$$\hat{\tau}_c = \{\hat{\sigma}^{-1} m^{-1/2} \sum_{t=2}^{m} (v_t, h^*(x_{t-1})).$$

Since $\sum_{t=2}^{m} (v_t, h^*(x_{t-1}))$ is a martingale, it is asymptotically normal if normalized by $m^{1/2}\hat{\sigma}$ by a martingale central limit theorem (Fuller, 1996, $p$. 235) and the result follows from Lemma 1.

Because of asymptotic normality, the statistic $\hat{\tau}_c$ can be used for testing $H_0$ by comparing $-z_\alpha$, the left $\alpha$-th percentile of the standard normal distribution. Unlike the pivotal statistic $\hat{\tau}_c$, the other statistic $\hat{\tau}_o$, based on OLSE has nonnormal limiting null distribution for non-stationary processes.

When $d = 1$, $h^*(y_{t-1}) = sign(y_{t-1})$ and the estimator $\hat{\beta}_c$ becomes the Cauchy estimator $\hat{\beta}_{ss}$ proposed by So and Shin(1999a). They showed that the limiting null distribution of $\hat{\tau}_{ss} = (\hat{\beta}_{ss} - 0)/\ se(\hat{\beta}_{ss})$ is standard normal. This is a special case of Theorem 1 for $d = 1$. However, for stationary seasonal AR processes, the estimator $\hat{\beta}_{ss}$ is less efficient than the new IV estimator $\hat{\beta}_c$ because the efficiency of $\hat{\beta}_c$ with respect to the LSE estimator is

$$\begin{aligned} ARE(\hat{\beta}_c; \hat{\beta}_o) &= (E|x|)^2/E|x|^2 \\ &= 2\Gamma((d+1)/2)^2/d\Gamma(d/2)^2 > ARE(\hat{\beta}_{ss}; \hat{\beta}_o) = 2/\pi = .637 \end{aligned}$$

for $d > 1$ as is to be shown in Table 1 below.

**Table 1.** The asymptotic relative efficiency(ARE) of Cauchy estimator $\hat{\beta}_c$ with respect to LSE $\hat{\beta}_o$ for model : $y_t = y_{t-d} + \beta y_{t-d} + e_t$.

| $d$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $ARE(\hat{\beta}_c; \hat{\beta}_o)$ | $2/\pi = .637$ | $\pi/4 = .785$ | $8/3\pi = .849$ | $9\pi/32 = .884$ |

We next consider general seasonal AR model (1.1) with seasonal mean $\mu = (\mu_1, \ldots, \mu_d)^T$. Recall that for asymptotic null normality of $\hat{\tau}_c$, independence of the regressor $x_{t-1}$ and the error $v_t$ is essential to preserve a martingale structure of $\hat{\tau}_c$. Similarly, in the mean model (1.1), we need a sequence of estimator $\tilde{\mu}_t$ such that the regressor $(x_{t-1} - \tilde{\mu}_{t-1})$ is independent of $v_t$ for normality of the random walk test. Thus, we consider the mean adjusted VAR(1) model

$$x_t - \bar{x}_{t-1} = f(x^o_{t-1}) + \beta h(x^o_{t-1}) + \hat{v}_t, \tag{2.3}$$

where $x^o_t = x_t - \bar{x}_t$, $\bar{x}_t = t^{-1}\sum_{i=1}^t x_i$ is the sample mean of the observations $x_1, ..., x_t$. Now the IV-estimator of $\beta$ is

$$\hat{\beta}_c = \sum_{t=2}^m (x_t - \bar{x}_{t-1} - f(x^o_{t-1}), h^*(x^o_{t-1}))/ \sum_{t=2}^m |h(x^o_{t-1})|.$$

The pivotal statistic for $\beta$ is

$$\hat{\tau}_c = (\hat{\beta}_c - \beta_0)/\ se(\hat{\beta}_c) \tag{2.4}$$

where

$$se(\hat{\beta}_c) = \{\hat{\sigma} m^{1/2}/ \sum_{t=2}^m |h(x^o_{t-1})|.$$

In (2.3), we have adjusted $x_t$ for mean $\mu$ by $\bar{x}_{t-1}$ in order to have

$$x_t - \bar{x}_{t-1} = x_{t-1} - \bar{x}_{t-1} + \beta h(x_{t-1} - \bar{x}_{t-1}) = v_t$$

when $f(x_t) = x_t, \beta = 0$ and $x_t = x_{t-1} + v_t$. This method provides us with limiting normality of $\hat{\tau}_c$ under the null hypothesis of the seasonal random walk.

**Theorem 2.** Consider model (1.1) with $f(u_t) = u_t$. Let the true model of $y_t$ be a random walk $y_t = y_{t-d} + e_t$. Then $\hat{\tau}_c$ converges in distribution to the standard normal distribution.
**Proof.** Limiting normality of

$$\hat{\tau}_c = (\hat{\sigma}^{-1} m^{-1/2} \sum_{t=2}^{m} (v_t, h^*(x^o_{t-1}))$$

again follows immediately from that of $m^{-1/2} \sum (v_t, h^*(x_{t-1} - \bar{x}_{t-1}))$, which is a martingale because of independence of $v_t$ and $x_{t-1} - \bar{x}_{t-1}$.

The mean-adjusted tests of Dickey, Hasza and Fuller(1984) and others are based on the mean adjusted observations $x_t - \bar{x}_n$. In our scheme, the mean $\mu$ is adjusted using $\bar{x}_t$ recursively. The adjustment scheme $(x_{t-1} - \bar{x}_{t-1})$ is called a recursive adjustment because $\mu$ is adjusted recursively by $\bar{x}_{t-1}$. We have adopted the recursive method in order to preserve independence of $v_t$ and $h(x^o_{t-1})$ under the null hypothesis and, hence, the martingale structure of $\sum_{t=2}^{m} (h(x^o_{t-1}), v_t)$.

The results of Theorems 1, 2, together with asymptotic normality of the t-type quantity $(\hat{\beta}_c - \beta)/se(\hat{\beta}_c)$ under stationary process $x_t$, enable us to construct a confidence interval of $\beta$ as given by

$$\hat{\beta}_c \pm z_{\alpha/2} se(\hat{\beta}_c) \tag{2.5}$$

which is asymptotically valid for any $\beta \in R$, where $z_\alpha$ is the $\alpha$-th right percentile of the standard normal distribution. On the other hand, the OLSE $\hat{\beta}_o$ cannot be used to construct a confidence interval in a manner of (2.5) because the limiting distribution of $(\hat{\beta}_o - \beta)/se(\hat{\beta}_o)$ is not normal if $f(x_t) = 0, \beta = 0$. Moreover, even in the stationary case , if $\beta$ is close to zero, finite sample distribution of $(\hat{\beta}_o - \beta)/se(\hat{\beta}_o)$ is very skewed and is far from being normal. This implies that the confidence interval $\hat{\beta}_o \pm z_{\alpha/2} se(\hat{\beta}_o)$ has a severe coverage probability distortion if $\beta$ is close to zero as is to be shown in Table 2, 3 below.

## 3. A MONTE CARLO STUDY

We first compare finite sample size and power of the proposed tests with those of the existing tests based on the OLSE of Dickey, Hasza and Fuller(1984) because they are the most widely used in practice. We consider a linear VAR(1) model

$$x_t = x_{t-1} + \beta(x_{t-1} - \mu) + v_t. \tag{3.1}$$

Series length is set to $m = 50$. Nominal level is 5%. Model (3.1) is simulated with $\mu = 0$ and $x_1 \sim N_d(0, I_d/(1-\rho^2))$ , $\rho = 1 + \beta$. The $e_t$ are independent standard normal errors and are generated by RANDN, a MATLAB subroutine. Observations $\{y_t\}_{t=1}^{n}$ are used for computing test statistics. We consider the seasonal mean adjusted $t$-type test statistics $\hat{\tau}_c$, $\hat{\tau}_o$, based on the Cauchy-estimator, the OLSE respectively. Number of replications is 10,000. In Table 2, 3 below, a.v., s.d., mad and Pr. stand for $E[\hat{\beta}]$, standard deviation, median absolute deviation ( median$[\hat{\beta} - \beta]$) of the estimators and coverage probabilities of the 90%-confidence intervals respectively.

**Table 2.** The empirical powers(%) of the seasonal mean adjusted tests for model : $y_t - y_{t-d} = \beta(y_{t-1} - \mu_t) + e_t$, $m = 50$, $d = 2$, $\sigma = 1$

| | LSE | | | | | Cauchy | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta$ | power | a.v. | s.d. | mad | Pr. | power | a.v. | s.d. | mad | Pr. |
| 0.0 | .049 | -.08 | .053 | .072 | .358 | .048 | -.011 | .050 | .029 | .905 |
| -.05 | .097 | -.119 | .057 | .061 | .647 | .140 | -.050 | .059 | .038 | .894 |
| -0.1 | .219 | -.163 | .063 | .055 | .753 | .328 | -.094 | .066 | .047 | .886 |
| -0.15 | .445 | -.209 | .067 | .054 | .796 | .568 | -.141 | .074 | .052 | .891 |
| -0.2 | .691 | -.256 | .072 | .054 | .830 | .779 | -.189 | .081 | .057 | .891 |
| -0.25 | .885 | -.304 | .076 | .056 | .844 | .913 | -.238 | .087 | .062 | .892 |

We first investigate empirical sizes and powers of the tests. In Table 2, we report percentages of the test statistics smaller than the left 5% null percentiles. The left 5% percentiles $-3.38$ of $\hat{\tau}_o$ are from Dickey , Hasza and Fuller(1984, Table 7). The left 5% percentile of $\hat{\tau}_c$ is -1.645, the left 5% percentile of the standard normal distribution. The test statistics are simulated under $\beta = 0$. We can summarize the results as follows. The powers of the tests based on Cauchy estimator are uniformly higher than those of the LSE-based tests in the range considered and there is a considerable distortion in the empirical coverage probabilities of the LSE-based confidence intervals near $\beta = 0$ ($\beta = 0, -0.05, -0.1, -0.15$ ). As for the relative efficiency of the estimators based on the sizes of the mad, Cauchy estimator is more efficient in the near non-stationary cases but is less efficient for strictly stationary processes ( $\beta = -2.5, -2.0$ ) than LSE.

We next consider nonlinear and possibly non-stationary seasonal time series model given by ;

$$y_t = y_{t-d}+\beta(y_{t-d}-\mu_t)/(1+|y_{t-d}-\mu_t|)+e_t, \tag{3.2}$$

and investigate empirical powers of the test statistics based on Cauchy estimator and OLSE. Since the size of $\hat{\tau}_o$ is not known for finite $n$, we use the empirical 5% percentile of a test statistic simulated under $\beta = 0$ as the critical value. For example, the empirical 5% percentile of $\hat{\tau}_o$ computed under ($\beta = 0$; $d = 4$; $m = 50$; 10,000 replications) is used as the critical value of $\hat{\tau}_o$ for computing power for ($\beta < 0$; $d = 2$; $m = 50$; 10,000 replications). In Table 3 below, the size-adjusted powers of the test statistics are reported. We note that the power of $\hat{\tau}_c$ increases as $\beta$ decreases and are uniformly more powerful than the test $\hat{\tau}_o$ based on OLSE in the range from 0 to $-0.25$. We also note a severe distortion of coverage probability of the 90%-confidence intervals based on LSE in contrast to the stable behavior of the intervals based on Cauchy estimator. Furthermore, we note that Cauchy estimator with recursive demeaning has a uniformly smaller mad than LSE with a global mean adjustment.

**Table 3.** $m = 50$, $d = 4$, $\sigma = 1$; $y_t - y_{t-d} = \beta y_{t-d}/(1 + |y_{t-d} - \mu_t|) + e_t$, $(y_1, \ldots, y_4) = 0$

| | LSE | | | | | Cauchy | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta$ | power | a.v. | s.d. | mad | Pr. | power | a.v. | s.d. | mad | Pr. |
| 0.0 | .050 | -.275 | .105 | .267 | .181 | .049 | -.008 | .122 | .080 | .906 |
| -.05 | .067 | -.296 | .107 | .239 | .300 | .093 | -.052 | .123 | .082 | .909 |
| -0.1 | .084 | -.318 | .107 | .212 | .421 | .153 | -.096 | .125 | .084 | .907 |
| -0.15 | .115 | -.340 | .109 | .183 | .554 | .243 | -.137 | .127 | .086 | .903 |
| -0.2 | .158 | -.367 | .114 | .160 | .650 | .353 | -.178 | .132 | .092 | .890 |
| -0.25 | .222 | -.397 | .116 | .141 | .716 | .475 | -.221 | .133 | .094 | .889 |

## REFERENCES

Dickey, D. A., Hasza, D. P. and Fuller, W. A. (1984). Testing for unit roots in seasonal time series with a unit root, *J. Journal of American Statistical Association* , 79, 355-367.

Dickey, D. A. and Fuller, W. A. (1979). Distribution of the estimators for autoregressive time series with a unit root, *Journal of American Statistical Association*, 74, 427-431.

Fuller, W. A. (1996). *Introduction to statistical time series*(2nd ed.), John Wiley & Sons, New York.

Hamilton, J. D. (1994). *Time Series Analysis*, Princeton university press, New Jersey.

Hylleberg, S., Engle, R. F., Granger, C. W. J. and Yoo, B. S. (1990). Seasonal integration and cointegration, *Journal of Econometrics*, 44, 215-238.

Knight, K.(1989). Limit theory for autoregressive-parameter estimates in an infinite-variance random walk, *The Canadian Journal of Statistics*, 17-3, 261-278.

Lucas, A. (1995). Unit root tests based on M-estimators, *Econometric Theory*, 12, 129-153.

So, B. S. and Shin, D. W. (1999a). Cauchy estimation for autoregressive process with applications to unit root tests and confidence interval, *Econometric Theory*, 15, 165-176.

So, B. S. and Shin, D. W. (1999b). Recursive mean adjustment in time series inferences, *Statistics & Probability Letters*, 43, 65-73.

Stock, J. H.(1994). Unit roots, structural breaks and trend, *Handbook of Econometrics*, IV, Ed. Engle and Mcfadden, D. L., Amsterdam, Elsevier.