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Nonparametric Bayesian Estimation for the Exponential Lifetime Data under the Type II Censoring

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Abstract

This paper addresses the nonparametric Bayesian estimation for the exponential populations under type II censoring. The Dirichlet process prior is used to provide nonparametric Bayesian estimates of parameters of exponential populations. In the past, there have been computational difficulties with nonparametric Bayesian problems. This paper solves these difficulties by a Gibbs sampler algorithm. This procedure is applied to a real example and is compared with a classical estimator.

Keywords : Gibbs Sampler; Mixture of Dirichlet Processes; Nonparametric Bayes; Exponential Distribution; Type II censored data.

1. Introduction

In lifetime studies, the exponential distribution has been widely used as a model in areas ranging from studies on the lifetimes of manufactured items to research involving survival or remission times in chronic diseases.

Consider I exponential populations with parameters $\theta=(\theta_1,\cdots,\theta_I)$. The observations $Y=(Y_1,Y_2,\cdots,Y_I)$ are available on these populations, where $Y_i=(Y_{i1},\cdots,Y_{in_i})$ is a $n_i\times 1$ vector of independent observations. Let the parameter θ_i be independent draws from some prior distribution $G(\cdot\mid\lambda)$, characterized by a parameter λ. This article studies a way to estimate θ_i from the observed Y_i's under type II censoring scheme by using a nonparametric Bayesian estimation which uses a Dirichlet process prior. Also the derived algorithm can be applied to type I censored data or randomly censored data.

It is well known that the asymptotic properties of the MLE (Maximum Likelihood Estimate) and the Bayes estimate are almost same. In complex situations such as likelihood function has

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many local maximum, the MLE works poorly. But in Bayesian analysis, using the MCMC methods, one can easily calculate a posterior distribution and a Bayes estimate. In Bayesian analysis with parametric models, there are almost inevitable concerns about the sensitivity of resulting inferences to assumed forms of component of distributions. These models require specification of prior distributions for parameters, about which there is usually considerable uncertainty. Hence it is of interest to combine the developments of modern Bayesian approaches to nonparametric modeling of distribution functions. That is, the strict parametric assumptions common to most standard Bayesian models can be relaxed to incorporate uncertainties about functional forms using Dirichlet process components, partly enabled by the approach to computation using Markov chain Monte Carlo (MCMC) methods.

The Dirichlet process prior (Ferguson 1973) is a prior distribution on the family of distributions that is dense in the space of distribution functions. Antoniak (1974) showed that if a Dirichlet process prior is used for G in this problem, then the posterior distribution of θ_i is sampled from a mixture of Dirichlet processes.

The Dirichlet process prior for G is determined by two parameters: a distribution function $G_0(\cdot)$ that defines the location of the Dirichlet processes prior, and a positive scalar precision parameter α. The precision parameter determines the concentration of the prior for G around the prior guess G_0, and therefore measures the strength of belief in G_0. By way of notation we write $G \sim D(G \mid G_0, \alpha)$. Therefore the Dirichlet process adds a further stage to the hierarchical model, and formally allows for the modeling of deviations away from the specific distribution G_0. So G_0 may be viewed as a baseline prior such as might be used in a typical parametric analysis and the framework enables the analysis of sensitivity to the assumptions of the baseline parametric model.

The introduction of MCMC methods in nonparametric Bayesian modeling was begun with Escobar (1988) which is published in Escobar (1994). Various applied developments in the recent year, include works on density estimation and related matters (Escobar and West 1995), mixture deconvolution (West and Turner 1994), applications in hierarchical and prior modeling (Escobar 1995, West 1997), regression and multivariate density estimation (Muller, Erkanli and West 1996, West, Muller and Escobar 1994), design (Bush and MacEachern 1996), time series (Muller, West and MacEachern 1997), mixed generalized liner model (Mukhopadhyay and Gelfand 1997) and accelerated failure time model (Kuo and Mallick 1998).

This paper is arranged as follows. In Section 2, we develop the Gibbs sampling algorithm for our model using the technique of Escobar and West (1997). In Section 3, we give a numerical result with real data analysis to illustrate our proposal.

2. The Estimation of Parameter for Exponential Populations

We interest in the estimation of parameter of exponential lifetime data under type II censoring. The exponential model with parameter θ is given by

$$f(X \mid \theta)=\frac{1}{\theta}\exp(-\frac{X}{\theta}), \tag{1}$$

where $X>0$ and $\theta>0$.

There is a situation where observations for the failure times of all the items is neither possible nor desirable. Since life testing experiments usually destroy items at the end of the study, items can not be used any more. Another reason is that the life testing experiment requires a much time to obtain all lifetimes being tested. Because of time and money, censoring is popular in life testing or reliability area. An experimenter may often have to terminate the experiment after a certain number of units to fail instead of waiting for all the units to fail. Samples observed in this manner are called type II censored samples. And there is a strong necessity for analyzing these censored sample.

Consider I exponential populations with parameter $\theta=(\theta_1,\cdots,\theta_I)$. The observation $X=(X_1,\cdots,X_I)$ are available on these populations, where $X_i=(X_{i1},\cdots,X_{in_i})$ is $n_i\times 1$ vector of independent observations. Under type II censoring, the observed sample consists of the ordered failure times $Y_{i1}\le Y_{i2}\le\cdots\le Y_{ir_i}$ and (n_i-r_i) survivors.

Let $Y_i=(Y_{i1},\cdots,Y_{ir_i})$, $i=1,\cdots,I$ and $Y=(Y_1,\cdots,Y_I)$. Then the probability density function of Y_i is

$$f(Y_i \mid \theta_i)=\frac{n_i!}{(n_i-r_i)!}\left(\frac{1}{\theta_i}\right)^{r_i}\exp\left[-\left(\frac{\sum_{j=1}^{r_i}Y_{ij}+(n_i-r_i)Y_{ir_i}}{\theta_i}\right)\right]. \tag{2}$$

To develop the estimation of parameter of the exponential lifetime data under type II censoring scheme using the Dirichlet process prior, we adapt the improved algorithm of Escobar and West (1997).

Consider the distribution of $(Y_i \mid \theta_i,\lambda)$. We assume that the θ_i's come from G, and that $G\sim D(G \mid G_0,\alpha)$. This structure results in a posterior distribution which is a mixture of Dirichlet processes (Antoniak 1974). Using the Polya urn representation of the Dirichlet process (Blackwell and MacQueen 1973), the joint posterior distribution of $[\theta \mid Y,\lambda]$ has the form

$$[d\theta \mid Y,\lambda]\propto\prod_{i=1}^{I}f(Y_i \mid \theta_i)\frac{\alpha G_0(d\theta_i \mid \lambda)+\sum_{k<i}\delta(d\theta_i \mid \theta_k)}{\alpha+i-1}, \tag{3}$$

where $f(Y_i \mid \theta_i)$ is the density or the probability function of $F(Y_i \mid \theta_i)$ at θ_i, and where $\delta(d\theta_i \mid \theta_k)$ is the distribution which is a point mass on θ_k.

The above equation clarifies the effect of the precision parameter α. In the limiting case

$\alpha\rightarrow\infty$, we have

$$[d\theta \mid Y,\lambda] \propto \prod_{i=1}^{I} G_b(d\theta_i \mid Y_i,\lambda) \propto \prod_{i=1}^{I} f(Y_i \mid \theta_i) G_0(d\theta_i \mid \lambda), \tag{4}$$

where $G_b(d\theta_i \mid Y_i,\lambda) \propto f(Y_i \mid \theta_i) G_0(d\theta_i \mid \lambda)$ is the baseline posterior, that is, the posterior assuming θ_i to come from the baseline prior G_0. As α gets very small, the estimation of θ_i is a little more complicated to understand. The posterior for θ_i is based largely on the other θ_k's which are near Y_i, so that inference for θ_i heavily depends on Y_i and nearest neighboring Y_k's.

Gibbs sampling exploits the structure of the conditional posteriors for the elements of θ, resulting in the following conditional distribution. For each $i=1,\cdots,I$,

$$[d\theta_i \mid \theta_k, k\neq i, Y,\lambda] \propto q_0 G_b(d\theta_i \mid Y_i,\lambda) + \sum_{k\neq i} q_k \delta(d\theta_i \mid \theta_k), \tag{5}$$

where $G_b(\theta_i \mid Y_i,\lambda)$ is the baseline posterior distribution, $q_0 \propto \alpha \int f(Y_i \mid \theta_i) dG_0(\theta_i \mid \lambda)$, just α times the density of the marginal distribution of Y_i under the baseline prior $G_0(\cdot \mid \lambda)$, $q_k \propto f(Y_i \mid \theta_k)$, the density of the marginal distribution of Y_i conditional on $\theta_i=\theta_k$, and the quantities q_i are standardized to unit sum, $1=q_0+\sum_{k\neq i} q_k$.

In our model, we consider a baseline inverted gamma prior G_0, under which the θ_i's are independent inverted gamma with specified shape λ_1 and scale λ_2, that is, $\theta_i \sim IG(\lambda_1,\lambda_2)$, and set $\lambda=\{\lambda_1,\lambda_2\}$. Then the posterior distribution for known λ has $[\theta_i \mid Y_i,\lambda] \sim IG(r_i+\lambda_1, T_i+\lambda_2)$, where $T_i=\sum_{j=1}^{r_i} Y_{ij}+(n_i-r_i)Y_{ir_i}$. Thus

$$[\theta_i \mid Y, \theta_k, k\neq i, \alpha, \lambda] \sim q_0 IG(r_i+\lambda_1, T_i+\lambda_2) + \sum_{k\neq i} q_k \delta(d\theta_i \mid \theta_k), \tag{6}$$

where

$$q_0 \propto \alpha \frac{n_i!}{(n_i-r_i)!} \frac{\lambda_2^{\lambda_1}}{\Gamma(\lambda_1)} \frac{\Gamma(r_i+\lambda_1)}{(T_i+\lambda_2)^{r_i+\lambda_1}} \text{ and } q_k \propto \frac{n_i!}{(n_i-r_i)!} \left(\frac{1}{\theta_k}\right)^{r_i} \exp\left(-\frac{T_i}{\theta_k}\right).$$

Next, the required computations are reduced by the fact that the distinct θ_i's typically reduce to fewer than I due to the clustering of the θ_i's inherent in the Dirichlet process (Antoniak 1974). Using the superscript '*' to denote distinct values, and suppose that the conditioning quantities θ_k's concentrate on $I^* \leq I-1$ distinct values θ_k^*, with some n_k taking this common value. Then, the above formula can be rewritten as:

$$[d\theta \mid \theta_k, k\neq i, Y,\lambda] \propto q_0 G_b(d\theta_i \mid Y_i,\lambda) + \sum_{k=1}^{I^*} n_k q_k^* \delta(d\theta_i \mid \theta_k^*), \tag{7}$$

with $q_k^* \propto f(Y_i \mid \theta_k^*)$ and $1 = q_0 + \sum_k n_k q_k^*$. Besides simplifying notation, the cluster structure of the θ_i can also be used to improve the efficiency of the algorithm.

When using the above conditional distribution, (7), in a Gibbs sampling algorithm, there may occur problems if the sum of the q_k's becomes very large relative to q_0 on any iteration. This occurs when the Markov chain has stabilized on a small number of clusters, and it is then unlikely to generate a new value of θ_k^*. In order to prevent the algorithm from getting stuck on a small set of θ_j^*'s in this way, it is helpful to remix the θ_j^*'s after every step. This improvement is used in Bush and MacEachern (1996) and West, Muller, Escobar (1994), and it is a combination of the above algorithm with the algorithm developed in MacEachern (1994). The combined algorithm mixes better than the Escobar algorithm alone because the θ_k^* 's are resampled at each step providing more movement in the MCMC sampler's which in turn improves convergence (See Escobar and West (1997)).

Some notation is introduced to define the remixing algorithm. Conditioning on I^*, introduce indicators $S_i = j$ if $\theta_i \equiv \theta_j^*$ so that, given $S_i = j$ and θ^*, $y_i \sim F_i(\cdot \mid \theta_j^*)$. The cluster structure, which is called a configuration, is defined by the set of indices $S = \{S_1, \cdots, S_n\}$. The set S determines a one-way classification of the data $Y = \{y_1, \cdots, y_n\}$ into I^* distinct groups or clusters; the $n_j = \#\{S_i = j\}$ observations in group j share the common parameter value θ_j^*. Now, define J_j as the set of indices of observations in group j; i.e., $J_j = \{i : S_i = j\}$. Let $Y_{(j)} = \{Y_i : S_i = j\}$ be the corresponding group of observations. Once the set S is known, the posterior analysis of the θ_j^*'s devolves into a collection of I^* independent analyses. Specifically, the θ_j^*'s are conditionally independent with posterior densities

$$p(\theta_j^* \mid Y, S, I^*, \lambda) \equiv p(\theta_j^* \mid Y_{(j)}, S, I^*, \lambda) \propto \prod_{i \in J_j} f_i(y_i \mid \theta_j^*) dG_0(\theta_j^*, \lambda) \tag{8}$$

for $j = 1, \cdots, I^*$. Note that this is just the posterior of θ_j^* given several Y_i's sampled from the $F(\cdot \mid \theta_j^*)$.

Thus, in our model, the conditional posterior distributions of θ_j^*'s are given by

$$[\theta_j^* \mid Y, S, \lambda] \sim IG\Big(\sum_{i \in J_j} r_i + \lambda_1, \sum_{i \in J_j} T_i + \lambda_2\Big), \tag{9}$$

for $j = 1, \cdots, I^*$.

The precision parameter, α, of the Dirichlet process is extremely important for the model. When α is small, then G tends to concentrate on a few atoms of probability. When α is large, then G is a distribution with many support points and the nonparametric model is closer to G_0, the baseline model. These features are to be borne in mind when considering priors for α. If the prior for α is a gamma distribution, we have access to a neat data augmentation device for sampling α (Escobar and West 1995). Let α have a gamma prior with parameters a and b, that is, $\alpha \sim Ga(a, b)$.

At each Gibbs iteration, the currently sampled values of I^* and α allow us to draw a new value of α by first sampling, conditionally on α and I^* fixed at their most recent values, a latent variable η from the simple beta distribution $(\eta \mid \alpha, I^*) \sim B(\alpha+1, I)$, a beta distribution with mean $(\alpha+1)/(\alpha+I+1)$. Then a new α value is sampled from a mixture of two gamma distributions based on the same I^* and the new η, that is,

$$[\alpha \mid \eta, I^*] \sim \pi_\eta Ga(a+I^*, b-\log(\eta)) + (1-\pi_\eta) Ga(a+I^*-1, b-\log(\eta)), \tag{10}$$

where the weights π_η are defined in odds form by

$$\frac{\pi_\eta}{(1-\pi_\eta)} = \frac{a+I^*-1}{I(b-\log(\eta))}. \tag{11}$$

Therefore the Gibbs sampling analysis is implemented using the following conditional posterior distributions:

$$[\theta_i \mid Y, \theta_k, k \neq i, \alpha, \lambda] \sim q_0 IG(r_i+\lambda_1, T_i+\lambda_2) + \sum_{k\neq i} q_k \delta(d\theta_i \mid \theta_k) \tag{12}$$

$$[\theta_j^* \mid Y, S, \lambda] \sim IG(\sum_{i\in J_j} r_i + \lambda_1, \sum_{i\in J_j} T_i + \lambda_2) \tag{13}$$

$$[\alpha \mid \eta, I^*] \sim \pi_\eta Ga(a+I^*, b-\log(\eta)) + (1-\pi_\eta) Ga(a+I^*-1, b-\log(\eta)) \tag{14}$$

$$[\eta \mid \alpha, I^*] \sim B(\alpha+1, I) \tag{15}$$

where

$$q_0 \propto \alpha \frac{n_i!}{(n_i-r_i)!} \frac{\lambda_2^{\lambda_1}}{\Gamma(\lambda_1)} \frac{\Gamma(r_i+\lambda_1)}{(T_i+\lambda_2)^{r_i+\lambda_1}} \text{ and } q_k \propto \frac{n_i!}{(n_i-r_i)!} \left(\frac{1}{\theta_k}\right)^{r_i} \exp\left(-\frac{T_i}{\theta_k}\right).$$

Gibbs sampling proceeds by simply iterating through (12) - (15) in order, sampling at each stage based on current values of all the conditioning variates.

To estimate θ_i, for some large m, we draw samples of $\theta_{(l)} \equiv \theta_{(l)}^{(m)}$, for $l = 1, \cdots, L$ and estimate $E[\theta_i \mid Y]$ by

$$\hat{\theta}_i = \frac{1}{L} \sum_{l=1}^{L} E[\theta_{i(l)} \mid Y, \theta_{k(l)}, k \neq i], \tag{16}$$

where $E[\theta_{i(l)} \mid Y, \theta_{k(l)}, k \neq i]$ is calculated from the distribution defined in Equation (12). Or the alternative formula could be used:

$$\hat{\theta}_i = \frac{1}{L} \sum_{l=1}^{L} \theta_{i(l)}. \tag{17}$$

Note finally that extensions to include learning about λ are straightforward. For example of such a baseline model, see George, Makov, and Smith (1994), who showed how to construct conditional samplers to perform the analysis. Once this is done, the Dirichlet process enhancement may use the same algorithm to sample λ as the baseline model, but the vector of θ_j^*'s would be substituted in for the vector of θ_i's.

3. Illustrative Example

The following data are time intervals of successive failures of the air conditioning system in Boeing 720 jet airplanes(Proschan (1963)). We assume that the time between successive failures for each plane is independent and exponentially distributed. Using this data, we apply the type II censoring at $(r_1, r_2, r_3, r_4, r_5, r_6) = (21, 14, 24, 27, 23, 12)$ and $(n_1, n_2, n_3, n_4, n_5, n_6) = (24, 16, 29, 30, 27, 15)$.

Plane 1	3,5,5,13,14,15,22,22,23,30,36,39,44,46,50,72,79,88,97,102,139,188,197,210
Plane 2	14,14,27,32,34,54,57,59,61,66,67,102,134,152,209,230
Plane 3	10,14,20,23,24,25,26,29,44,44,49,56,59,60,61,62,70,76,79,84,90,101,118,130,156,186,208,208,310
Plane 4	1,3,5,7,11,11,11,12,14,14,14,16,16,20,21,23,42,47,52,62,71,71,87,90,95,120,120,225,246,261
Plane 5	1,4,11,16,18,18,18,24,31,39,46,51,54,63,68,77,80,82,97,106,111,141,142,163,191,206,216
Plane 6	12,21,26,27,29,29,48,57,59,70,74,153,326,386,502

The analysis is illustrated, based on baseline inverted gamma prior with $\lambda = (\lambda_1, \lambda_2)$ set at $(3, 200)$. The value $(3, 200)$ is fairly noninformative. For the precision parameter, α, we consider two priors: $Ga(0.01, 0.01)$ and $Ga(1, 1)$. The former prior is fairly noninformative, giving reasonable mass to both high and low values of α. The $Ga(1, 1)$ prior favors relatively low values of α.

Table 1 : Estimates of the θ

Plane	MLE	BE	NBE^1	NBE^2
1	64.810	67.870	68.919	67.961
2	84.071	86.062	86.195	86.011
3	83.500	84.770	85.054	84.820
4	52.444	55.724	56.344	55.857
5	81.130	82.640	83.093	83.167
6	88.667	90.286	89.283	90.038

(BE - Baseline Model, NBE^1- $\alpha \sim Ga(1, 1)$, NBE^2- $\alpha \sim Ga(0.01, 0.01)$)

Table 1 and Figure 1 list the Bayes estimates and marginal posteriors for θ_i approximated by the Gibbs sampling algorithm using 10,000 iterations with 5,000 burn-in iterations and 5 replications. After considering sample autocorrelations, cross-correlations and the monitoring statistic of Gelman and Rubin (1992), we were satisfied with the convergence of our algorithm. Figure 1 displays the estimated marginal posteriors under the baseline model as the solid line, nonparametric Bayesian model under the gamma prior, $Ga(1,1)$ as the dotted line and nonparametric Bayesian model under the gamma prior, $Ga(0.01,0.01)$ as the dashed line for θ_i, respectively.

Also, in Table 1, MLE denotes maximum likelihood estimate, BE denotes Bayes estimator under the baseline model, and NBE^1 and NBE^2 denote nonparametric Bayesian estimator under two gamma priors.

Under this prior, the inclusion of the Dirichlet process does not change the posterior distribution very much. The baseline model produces posterior distributions which are very close to the posteriors obtained under the two Dirichlet process models. The use of the Dirichlet process sometimes does lead to marginally increased spread in posteriors, reflecting additional uncertainty implied by use of the Dirichlet structure. In part, the concordance arises as the $IG(3,200)$ prior for λ is not in conflict with the observed data.

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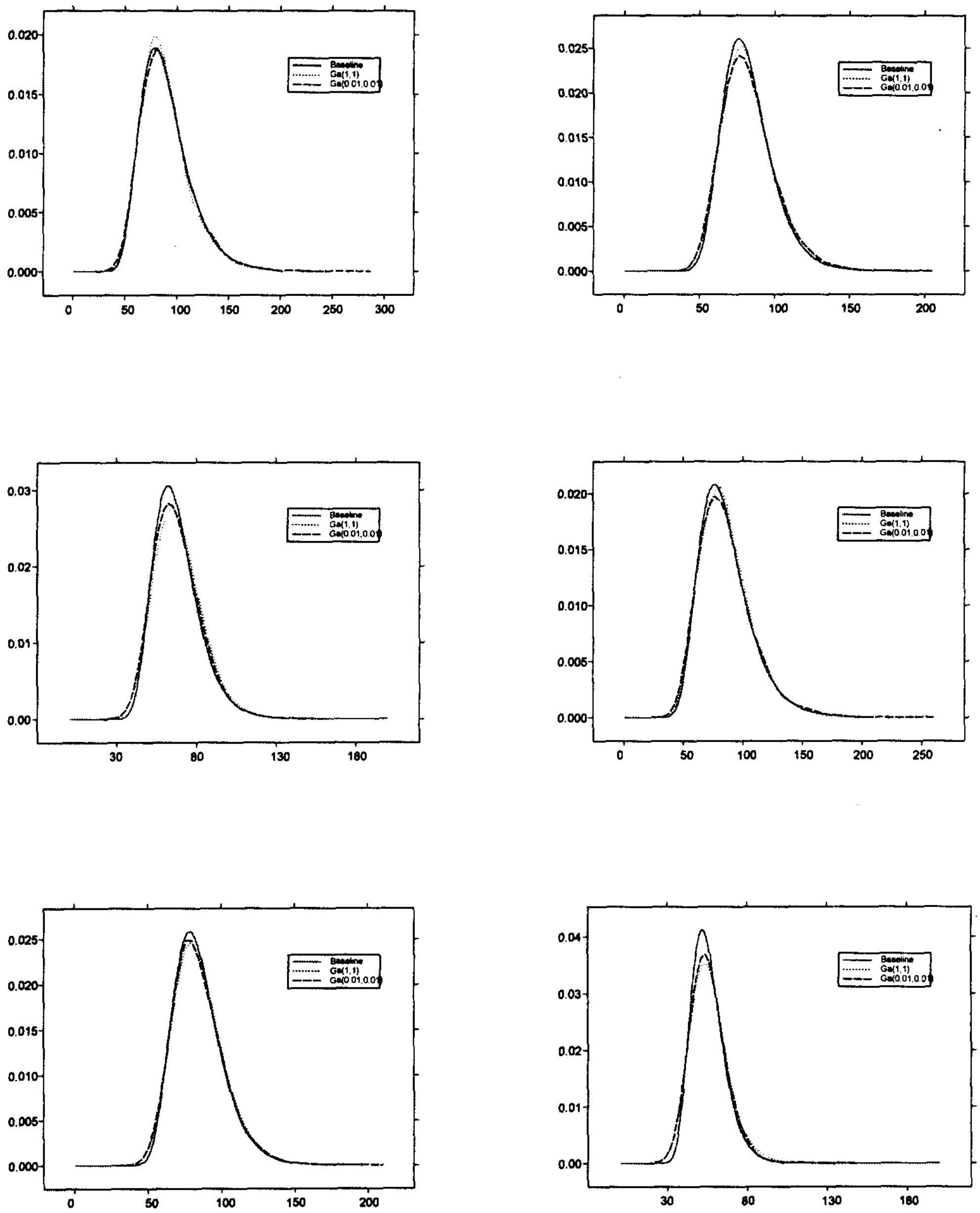

그림 1 The Posterior Distributions of θ_i's