The Korean Communications
in Statistics Vol. 8, No. 2, 2001
pp. 395-406

Nonparametric Test for Umbrella Alternatives with the Known Peak on Ranked-Set Samples

Dong Hee Kim[1], Kyung Hee Kim[2] and Hyun Gee Kim[3]

Abstract

In this paper, we propose the test statistic for the umbrella alternatives on c-samples ranked set samples(RSS), where the peak of the umbrella is known. We obtain the asymptotic property of the proposed test statistic and the asymptotic relative efficiencies of the proposed test statistic with respect to U-statistic based on simple random samples(SRS). From the simulation work, we compare the empirical powers of the proposed test statistic with U-statistic based on SRS.

Keywords : Ranked-set samples, Umbrella alternatives, Asymptotic relative efficiency

1. Introduction

The method of ranked-set samples(RSS) is effective when the measurement of an item is difficult, destructive or expensive but the items of a set of given size are easily drawn and ranked with reasonable success by judgement.

McIntyre (1952) first suggested the RSS method for assessing yields of pasture plots without actually mowing and weighing the hay for a large number of plots. Takahasi and Wakimoto (1968) proposed the same procedure and developed a body of theory that is, however, based upon the assumption of perfect ranking. Dell and Clutter (1972) studied the sample mean as an estimate of the population mean. Stokes (1980) studied the estimation of the variance and the asymptotic efficiency in RSS. Stokes and Sager (1988) studied the empirical distribution function defined on a ranked-set sample and investigated the improvement in RSS version of the Kolmogorov-Smirnov test. This was the first study of a nonparametric procedure. Bohn and Wolfe (1992,1994) proposed the two-sample Mann-Whitney -Wilcoxon statistic and investigated the properties of test procedures based on RSS in the

1) Professor, Department of Statistics, (Research Institute of Computer, Information and Communication), Pusan National University, Pusan, 609-735, Korea.
E-mail : heekim@hyowon.cc.pusan.ac.kr

2) Department of Statistics, Pusan National University, Pusan, 609-735, Korea.

3) Department of Statistics, Pusan National University, Pusan, 609-735, Korea.

case of perfect and imperfect judgement. Hettmansperger (1995) studied the sign test on RSS. Kim, Kim and Lee (1999) studied nonparametric test for ordered alternatives on multiple RSS. Kim, Kim and Kim (1999) studied Page type test for ordered alternatives on multiple RSS. Kim, Kim and Park (2000) studied nonparametric test for ordered alternatives using Jonckheere type statistic on RSS.

In this paper, we deal with c-sample nonparametric testing problem for the umbrella alternatives on RSS, where the peak of the umbrella is known and obtain the asymptotic relative efficiencies of the proposed test statistic with respect to U-statistic based on simple random samples(SRS). From the simulation work, we compare the empirical powers of the proposed test statistic with U-statistic based on SRS, where the underlying distributions are uniform, normal, double exponential, logistic and Cauchy distribution.

In Chapter 2, we introduce the proposed test statistic. Chapter 3 deals with the asymptotic properties of the proposed test statistic. In Chapter 4, simulation design and results under underlying distributions are given. Chapter 5 provides some conclusions.

2. The Proposed Test Statistic

Let $X_{m(1)1},\cdots,X_{m(1)n_m};\cdots;X_{m(k_m)1},\cdots,X_{m(k_m)n_m}$ be a ranked-set sample of size $k_m n_m$ from a continuous distribution with cdf $F_m(x)=F(x-\theta_m)$ and pdf $f_m(x)=f(x-\theta_m)$, $m=1,\cdots,c$, where θ_m is a location parameter of the m-th population, and let k_m and n_m be the sample size and the cycle size to get $k_m n_m$ observations from $k_m^2 n_m$ preranking sample observations.

Let $f_{m(i)}(x)$ denote the common pdf for $X_{m(i)j}$, $j=1,\cdots,n_m$, $i=1,\cdots,k_m$, $m=1,\cdots,c$ and be given by

$$f_{m(i)}(x)=\frac{k_{m}!}{(i-1)!(k_m-i)!}[F(x_{(i)}-\theta_m)]^{i-1}[1-F(x_{(i)}-\theta_m)]^{k_m-i}f(x_{(i)}).$$

In this paper, the testing problem of our interest is the null hypothesis $H_0 : \theta_1=\cdots=\theta_c$ $(=\theta_0)$ against the umbrella alternative hypothesis $H_1 : \theta_1\le\cdots\le\theta_{l-1}\le\theta_l\ge\theta_{l+1}\ge\cdots\ge\theta_c$ with at least one strict inequality. Here, l is called a known peak point.

Figure 2.1 shows the c-sample ranked set sample structure.

population	cycle	cdf
1	$\begin{pmatrix} X_{1(1)1}, & X_{1(1)1}, & \cdots & ,X_{1(1)n_1} \\ \vdots & \vdots & \vdots & \vdots \\ X_{1(k_1)1}, & X_{1(1)1}, & \cdots & ,X_{1(k_1)n_1} \end{pmatrix}$	$\sim F(x-\theta_1)$

$$
\begin{array}{ccc}
2 & \begin{pmatrix} X_{2(1)1}, & X_{2(1)1}, & \cdots & , X_{2(1)n_2} \\ \vdots & \vdots & \vdots & \vdots \\ X_{2(k_2)1}, & X_{2(1)1}, & \cdots & , X_{2(k_2)n_2} \end{pmatrix} & \sim F(x-\theta_2) \\
\vdots & \vdots & \vdots \\
m & \begin{pmatrix} X_{m(1)1}, & X_{m(1)1}, & \cdots & , X_{m(1)n_m} \\ \vdots & \vdots & \vdots & \vdots \\ X_{m(k_m)1}, & X_{m(1)1}, & \cdots & , X_{m(k_m)n_m} \end{pmatrix} & \sim F(x-\theta_m) \\
\vdots & \vdots & \vdots \\
c & \begin{pmatrix} X_{c(1)1}, & X_{c(1)1}, & \cdots & , X_{c(1)n_c} \\ \vdots & \vdots & \vdots & \vdots \\ X_{c(k_c)1}, & X_{c(1)1}, & \cdots & , X_{c(k_c)n_c} \end{pmatrix} & \sim F(x-\theta_c)
\end{array}
$$

Figure 2.1 c-sample Ranked Set Sample Structure.

We use the general U-statistic theory commonly used for deriving similar properties of Mann-Whitney-Wilcoxon statistic for simple random samples.

Now, let

$$\gamma = \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} P(X_{m(i)1} \le X_{m'(i')1}) + \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} P(X_{m(i)1} \ge X_{m'(i')1})$$

where X_{ml} and $X_{m'1}$ are independently distributed as $F_m(\cdot)$ and $F_{m'}(\cdot)$, respectively,

$$X_{m1} = (X_{m(1)1}, \cdots, X_{m(k_m)1}) \ , \quad m = 1, \cdots, c, \quad \psi(t) = \begin{cases} 1 \ , & t > 0 \\ 0 \ , & t \le 0 \end{cases} .$$

Then γ is an estimable parameter of degree $(1, \cdots, 1)$ with the c-sample symmetric kernel

$$h(X_{11}; X_{21}; \cdots; X_{l1}; \cdots; X_{c1}) = \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \psi(X_{m'(i')1} - X_{m(i)1}) + \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \psi(X_{m(i)1} - X_{m'(i')1}) .$$

Our proposed test statistic corresponding to the c-sample U-statistic for γ is of the form,

$$
\begin{aligned}
& U(X_{11}, \cdots, X_{1n_1}; \cdots; X_{l1}, \cdots, X_{ln_l}; \cdots; X_{c1}, \cdots, X_{cn_c}) \\
&= \frac{1}{n_1 \cdots n_l \cdots n_c} \sum_{\beta_1=1}^{n_1} \cdots \sum_{\beta_l=1}^{n_l} \cdots \sum_{\beta_c=1}^{n_c} h(X_{1\beta_1}; \cdots; X_{l\beta_l}; \cdots; X_{c\beta_c}) \\
&= \sum_{m=1}^{c-1} \sum_{m'=m+1}^{c} \frac{V_{mm'}}{n_m n_{m'}} + \sum_{m=l}^{l-1} \sum_{m'=m+1}^{l} \frac{V_{m'm}}{n_m n_{m'}} \ , \qquad (2.1)
\end{aligned}
$$

where $V_{mm'} = \sum_{i=1}^{k_m} \sum_{j=1}^{n_m} \sum_{i'=1}^{k_{m'}} \sum_{j'=1}^{n_{m'}} \psi(X_{m'(i')j'} - X_{m(i)j})$ is the Mann-Whitney-Wilcoxon statistic for the m-th and m'-th ranked-set sample. From now we will denote (2.1) by U_{RSS}, briefly.

Under the umbrella alternatives H_1, the proposed test statistic tends to have large values , so we reject H_0 in favor of H_1 for large values of U_{RSS}.

3. Asymptotic Properties

To obtain the asymptotic properties of the proposed statistic U, we calculate the mean and variance of U-statistic, under H_0.

$$\begin{aligned} E\,(U_{RSS}) = & \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} \Big\{ \sum_{a=i}^{k_m} \sum_{b=i'}^{k_{m'}} \binom{k_m}{a} \binom{k_{m'}}{b} [F(t+\theta_{m'}-\theta_m)]^a \\ & \times\ [1-F(t+\theta_{m'}-\theta_m)]^{k_m-a} F(t)^{b-1} [1-F(t)]^{k_{m'}-b-1} [b-k_{m'}F(t)] f(t) \Big\} dt \\ & + \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} \Big\{ \sum_{a=i}^{k_m} \sum_{b=i'}^{k_{m'}} \binom{k_m}{a} \binom{k_{m'}}{b} [F(s+\theta_m-\theta_{m'})]^b \\ & \times\ [1-F(s+\theta_m-\theta_{m'})]^{k_{m'}-b} F(t)^{a-1} [1-F(t)]^{k_m-a-1} [a-k_m F(s)] f(s) \Big\} ds \end{aligned}$$

where $t = x-\theta_{m'}$, $s = x-\theta_m$.

Now, we will compute the variance for the proposed test statistic.

Generally, for the variance expressions of k-sample U-statistics with an estimable parameter γ of degree $(r_1, \cdots, r_k)$ and symmetric kernel h(·), let $\zeta_{c_1,\cdots,c_k}$ denote the covariance between two kernel random variables where the integer $c_1, \cdots, c_k$ such that $0 \le c_i \le r_i$, $i=1,\cdots,k$.

That is, define

$$\begin{aligned} \zeta_{c_1,\cdots,c_k} = Cov(& h(X_{11},\cdots, X_{1c_1}, X_{1,c_1+1},\cdots, X_{1r_1};\cdots; X_{k1},\cdots, X_{kc_k}, X_{k,c_k+1},\cdots, X_{kr_k}), \\ & h(X_{11},\cdots, X_{1c_1}, X_{1,r_1+1},\cdots, X_{1,2r_1-c_1};\cdots; X_{k1},\cdots, X_{kc_k}, X_{kr_{k+1}},\cdots, X_{k,2r_k-c_k})). \end{aligned}$$

Here, we deal with the case of degree $(1,\cdots,1)$ in above notation of general covariance. As shown in the below representation, $\zeta_{0,\cdots,0,1,0,\cdots,0}$ has three forms as the peak point l is less than or greater than or equal to m, where '1' in the subscripts appears at the mth position.

$$\zeta_{0,\cdots,0,1,0,\cdots,0} = \begin{cases} \zeta^1, & m < l \\ \zeta^2, & m > l \\ \zeta^3, & m = l \end{cases} \quad ,\text{for } m = 1,\cdots, c$$

$$
\begin{aligned}
\zeta^{1} &= Cov(h(X_{11};\cdots;X_{m-1,1};X_{m1};X_{m+1,1};\cdots;X_{l1};\cdots,X_{c1}),\\
&\qquad h(X_{12};\cdots;X_{m-1,2};X_{m1};X_{m+1,2};\cdots;X_{l2};\cdots;X_{c2}))\\
&= \sum_{m'=1}^{m-1}\sum_{i=1}^{k_m}\sum_{i'=1}^{k_{m'}}[P(\max(X_{m'(i')1},X_{m'(i')2})<X_{m(i)1})-[P(X_{m'(i')1}<X_{m(i)1})]^{2}]\\
&+ \sum_{m'=1}^{m-1}\sum_{i=1}^{k_m}\sum_{a\neq b}^{k_{m'}}[P(\max(X_{m'(a)1},X_{m'(b)1})<X_{m(i)1})\\
&\qquad\qquad -P(X_{m'(a)1}<X_{m(i)1})P(X_{m'(b)1}<X_{m(i)1})]\\
&+ \sum_{m'=m+1}^{l}\sum_{i=1}^{k_m}\sum_{i'=1}^{k_{m'}}[P(\min(X_{m'(i')1},X_{m'(i')2})>X_{m(i)1})-[P(X_{m(i)1}<X_{m'(i')1})]^{2}]\\
&+ \sum_{m'=m+1}^{l}\sum_{i=1}^{k_m}\sum_{a\neq b}^{k_{m'}}[P(\min(X_{m'(a)1},X_{m'(b)1})>X_{m(i)1})\\
&\qquad\qquad -P(X_{m(i)1}<X_{m'(a)1})P(X_{m(i)1}<X_{m'(b)1})]\ ,
\end{aligned}
$$

$$
\begin{aligned}
\zeta^{2} &= Cov(h(X_{11};\cdots;X_{l1};\cdots;X_{m-1,1};X_{m1};X_{m+1,1};\cdots,X_{c1}),\\
&\qquad h(X_{12};\cdots;X_{l2};\cdots;X_{m-1,2};X_{m1};X_{m+1,2};\cdots,X_{c2}))\\
&= \sum_{m'=m+1}^{c}\sum_{i=1}^{k_m}\sum_{i'=1}^{k_{m'}}[P(\max(X_{m'(i')1},X_{m'(i')2})<X_{m(i)1})-[P(X_{m'(i')1}<X_{m(i)1})]^{2}]\\
&+ \sum_{m'=m+1}^{c}\sum_{i=1}^{k_m}\sum_{a\neq b}^{k_{m'}}[P(\max(X_{m'(a)1},X_{m'(b)1})<X_{m(i)1})\\
&\qquad\qquad -P(X_{m'(a)1}<X_{m(i)1})P(X_{m'(b)1}<X_{m(i)1})]\\
&+ \sum_{m'=l}^{m-1}\sum_{i=1}^{k_m}\sum_{i'=1}^{k_{m'}}[P(\min(X_{m'(i')1},X_{m'(i')2})>X_{m(i)1})-[P(X_{m(i)1}<X_{m'(i')1})]^{2}]\\
&+ \sum_{m'=l}^{m-1}\sum_{i=1}^{k_m}\sum_{a\neq b}^{k_{m'}}[P(\min(X_{m'(a)1},X_{m'(b)1})>X_{m(i)1})\\
&\qquad\qquad -P(X_{m(i)1}<X_{m'(a)1})P(X_{m(i)1}<X_{m'(b)1})]\ ,
\end{aligned}
$$

$$
\begin{aligned}
\zeta^{3} &= Cov(h(X_{11};\cdots;X_{l1};\cdots;X_{c1}),h(X_{12};\cdots;X_{l1};\cdots;X_{c2}))\\
&= \sum_{m'=1}^{m-1}\sum_{i=1}^{k_m}\sum_{i'=1}^{k_{m'}}[P(\max(X_{m'(i')1},X_{m'(i')2})<X_{m(i)1})-[P(X_{m'(i')1}<X_{m(i)1})]^{2}]\\
&+ \sum_{m'=1}^{m-1}\sum_{i=1}^{k_m}\sum_{a\neq b}^{k_{m'}}[P(\max(X_{m'(a)1},X_{m'(b)1})<X_{m(i)1})\\
&\qquad\qquad -P(X_{m'(a)1}<X_{m(i)1})P(X_{m'(b)1}<X_{m(i)1})]\\
&+ \sum_{m'=m+1}^{l}\sum_{i=1}^{k_m}\sum_{i'=1}^{k_{m'}}[P(\min(X_{m'(i')1},X_{m'(i')2})>X_{m(i)1})-[P(X_{m(i)1}<X_{m'(i')1})]^{2}]
\end{aligned}
$$

$$+ \sum_{m'=m+1}^{l} \sum_{i=1}^{k_m} \sum_{a \neq b}^{k_{m'}} [P(\min (X_{m'(a)1}, X_{m'(b)1}) > X_{m(i)1})$$
$$- P(X_{m(i)1} < X_{m'(a)1}) P(X_{m(i)1} < X_{m'(b)1})]$$
$$+ \sum_{m'=m+1}^{c} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} [P(\max (X_{m'(i')1}, X_{m'(i')2}) < X_{m(i)1}) - [P(X_{m'(i')1} < X_{m(i)1})]^2]$$
$$+ \sum_{m'=m+1}^{c} \sum_{i=1}^{k_m} \sum_{a \neq b}^{k_{m'}} [P(\max (X_{m'(a)1}, X_{m'(b)1}) < X_{m(i)1})$$
$$- P(X_{m'(a)1} < X_{)1}) P(X_{m'(b)1} < X_{m(i)1})]$$
$$+ \sum_{m'=l}^{m-1} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} [P(\min (X_{m'(i')1}, X_{m'(i')2}) > X_{m(i)1}) - [P(X_{m(i)1} < X_{m'(i')1})]^2]$$
$$+ \sum_{m'=l}^{m-1} \sum_{i=1}^{k_m} \sum_{a \neq b}^{k_{m'}} [P(\min (X_{m'(a)1}, X_{m'(b)1}) > X_{m(i)1})$$
$$- P(X_{m(i)1} < X_{m'(a)1}) P(X_{m(i)1} < X_{m'(b)1})]$$
$$+ \sum_{m=1}^{l-1} \sum_{m'=l+1}^{c} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} [P(\max (X_{m(i')1}, X_{m'(i')2}) < X_{l(i)1})$$
$$- P(X_{m(i')1} < X_{l(i)1}) P(X_{m'(i')1} < X_{l(i)1})]$$
$$+ \sum_{m=1}^{l-1} \sum_{m'=l+1}^{c} \sum_{i=1}^{k_m} \sum_{a \neq b}^{k_{m'}} [P(\max (X_{m(a)1}, X_{m'(b)1} < X_{l(i)1})$$
$$- P(X_{m'(a)1} < X_{l(i)1}) P(X_{m'(b)1} < X_{l(i)1})] \ .$$

Note that $\zeta_{0,\cdots,0} = 0$.

Taking $N = \sum_{m=1}^{c} n_m$, we have the asymptotic variance of the c-sample U-statistic in the following Lemma 3.1.

Lemma 3.1 (Theorem 3.4.8 of Randles and Wolfe(1979))

If $E\ [h^2(X_{11};\ \cdots\ ;X_{c1})] < \infty$ and if $\lim_{N \to \infty} \frac{n_m}{N} = \lambda_m$, $N = \sum_{m=1}^{c} n_m$, $0 < \lambda_m < 1$, for $m = 1, \cdots, c$, then

$$\lim_{N \to \infty} N\ Var[\ U(X_{11}, \cdots, X_{1n_1};\ \cdots\ ; X_{c1}, \cdots, X_{cn_c})] = \sum_{m=1}^{c} \frac{\zeta_{0,\ \cdots\ ,0,1,0,\ \cdots\ ,0}}{\lambda_m} .$$

By Theorem 3.6.6 of Randles and Wolfe(1979), we have the asymptotic normality of the proposed U-statistic in the following Theorem 3.1.

Theorem 3.1

Under $H_0 : \theta_1 = \cdots = \theta_c (= \theta_0)$,

$\sqrt{N} (U_{RSS} - \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} \frac{k_m k_{m'}}{2} - \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} \frac{k_m k_{m'}}{2})$ has a limiting normal distribution with mean 0 and variance $\sigma^2_0 = \sum_{m=1}^{c} \left[\frac{\zeta^{(0)}_{0,\cdots,0,1,0,\cdots,0}}{\lambda_m} \right]$

where

$$\zeta^{(0)}_{0,\cdots,0,1,0,\cdots,0} = \begin{cases} \zeta^{1(0)}, & m < l \\ \zeta^{2(0)}, & m > l \\ \zeta^{3(0)}, & m = l \end{cases},$$

$\zeta^{(0)}_{0,\cdots,0,1,0,\cdots,0}$ represented the null specification for $\zeta_{0,\cdots,0,1,0,\cdots,0}$.

To obtain the asymptotic relative efficiency (ARE), we consider a sequence of translation alternatives of the form

$$H_{1N}: \theta_m = \begin{cases} \theta_0 + m\frac{\theta}{\sqrt{N}} & , \ m = 1, \cdots, l \\ \theta_0 + (2\ l - m)\frac{\theta}{\sqrt{N}} & , \ m = l, \cdots, c\ . \end{cases}$$

We compare the asymptotic relative efficiencies (ARE) of the proposed c-sample U-statistic with respect to U-statistic based on SRS. Under the sequence of translation alternatives, the efficacy of the proposed test statistic is obtained as follows.

$$\begin{aligned} E_\theta(U_{RSS}) = & \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} \left\{ \sum_{a=i}^{k_m} \sum_{b=i'}^{k_{m'}} \binom{k_m}{a} \binom{k_{m'}}{b} [F(t + \frac{(m' - m)}{\sqrt{N}}\theta)]^a \right. \\ & \left. \times [1 - F(t + \frac{(m' - m)}{\sqrt{N}}\theta)]^{k_m - a} F(t)^{b-1} (1 - F(t))^{k_{m'} - b - 1} (b - k_{m'} F(t)) f(t) \right\} dt \\ & + \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} \left\{ \sum_{a=i}^{k_m} \sum_{b=i'}^{k_{m'}} \binom{k_m}{a} \binom{k_{m'}}{b} [F(s + \frac{(m' - m)}{\sqrt{N}}\theta)]^b \right. \\ & \left. \times [1 - F(s + \frac{(m' - m)}{\sqrt{N}}\theta)]^{k_{m'} - b} F(s)^{a-1} (1 - F(s))^{k_m - a - 1} (a - k_m F(s)) f(s) \right\} ds. \end{aligned}$$

Let τ_{RSS} denote the derivative of $E_\theta(U_{RSS})$ at $\theta = 0$ and be given by

$$\begin{aligned} \tau_{RSS} = & \frac{1}{\sqrt{N}} \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} (m' - m) \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} \left\{ \sum_{a=i}^{k_m} \sum_{b=i'}^{k_{m'}} \binom{k_m}{a} \binom{k_{m'}}{b} [F(t)]^{a+b-2} \right. \\ & \left. \times [1 - F(t)]^{k_m + k_{m'} - a - b - 2} [b - k_{m'} F(t)][a - k_m F(t)] f(t)^2 \right\} dt \\ & + \frac{1}{\sqrt{N}} \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} (m' - m) \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} \left\{ \sum_{a=i}^{k_m} \sum_{b=i'}^{k_{m'}} \binom{k_m}{a} \binom{k_{m'}}{b} [F(s)]^{a+b-2} \right. \\ & \left. \times [1 - F(s)]^{k_m + k_{m'} - a - b - 2} [b - k_{m'} F(s)][a - k_m F(s)] f(s)^2 \right\} ds. \end{aligned}$$

By the Definition 5.2.14 of Randles and Wolfe(1979), we have the following efficacy of U_{RSS}.

$$eff(U_{RSS}) = \lim_{N\to\infty} \frac{[\frac{\partial}{\partial\theta} E_\theta(U_{RSS})|_{\theta=0}]}{\sqrt{NVar_0(U_{RSS})}} = \frac{\tau_{RSS}}{\left[\sum_{m=1}^{c} \frac{1}{\lambda_m} \zeta^{(0)}_{0,\cdots,0,1,0,\cdots,0}\right]^{1/2}}.$$

U-statistic U_{SRS} based on SRS is

$$U_{SRS} = \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} \frac{V_{mm'}}{n_m n_{m'}} + \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} \frac{V_{m'm}}{n_m n_{m'}},$$

where $V_{mm'} = \sum_{i=1}^{k_m} \sum_{j=1}^{n_m} \sum_{i'=1}^{k_{m'}} \sum_{j'=1}^{n_{m'}} \psi(X_{m'i'j'} - X_{mij})$.

For SRS test of $H_0 : \theta_m = \theta_0,\ m=1,\cdots,c$, the efficacy is

$$eff(U_{SRS}) = \frac{\tau_{SRS}}{\left[\sum_{m=1}^{c} \frac{1}{\lambda_m} \zeta^{(0)}_{0,\cdots,0,1,0,\cdots,0(SRS)}\right]^{1/2}},$$

where

$$\tau_{SRS} = \frac{1}{\sqrt{N}} \sum_{m=1}^{l-1} \sum_{m'=m+1}^{l} (m'-m) \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} f(t)^2 dt$$
$$+ \frac{1}{\sqrt{N}} \sum_{m=l}^{c-1} \sum_{m'=m+1}^{c} (m'-m) \sum_{i=1}^{k_m} \sum_{i'=1}^{k_{m'}} \int_{-\infty}^{\infty} f(s)^2 ds.$$

Then, the ARE of U_{RSS} with respect to U_{SRS} is $ARE(U_{RSS}, U_{SRS}) = \frac{eff^2(U_{RSS})}{eff^2(U_{SRS})}$.

When the underlying distributions are uniform, logistic, in the case of c=3, n_m=n=3 and k_m=k=3, l=2, $ARE(U_{RSS}, U_{SRS})$=2.00, after a bit of calculation.

4. Power Comparison

To examine the empirical powers of U_{RSS}, U_{SRS} for a umbrella alternative, we conduct a small-sample Monte Carlo study. In our power study, we consider the various underlying distributions such as the uniform(0,1), normal(with variance 1), double exponential(with scale parameter 1), logistic(with scale parameter 1), Cauchy(with scale parameter 1) distributions. The location parameter θ has 0.0 (0.2) 1.0. Also, we consider the design with the sample size k_m=k=3, 4, 5, the cycle n_m=n=3, 4, 5, $m=1,\cdots,c$ and c=3 and l(peak point)=2. In each case, we use 1,000 replications in obtaining the various power estimates. For the power study we use the limiting normal distribution of the statistics to determine the critical values at level α=0.05.

The IMSL(International Mathematics and Statistical Library) procedures RNUN, RNNOA, and RNCHY are used to generate the required uniform, normal and cauchy pseudo-random variates, respectively. And the double exponential and logistic random variates are generated by using the probability integral transformation.

The results of the empirical powers of tests are presented in Table 4.1 - Table 4.2. From the simulation results, the empirical powers of U_{RSS} and U_{SRS} increase as k and n increase.

From Table 4.1 to Table 4.2, in each k and n, under the underlying distributions, we conclude that the empirical powers of U_{RSS} are superior to U_{SRS}, because it is thought that U_{RSS} uses more informations than U_{SRS}.

5. Conclusions

In this paper, we consider c-sample nonparametric testing problem for the umbrella alternatives using the RSS method. We propose a test statistic using the Mann-Whitney-Wilcoxon statistic for testing umbrella alternatives and obtain the ARE of the proposed test statistic, U_{RSS} with respect to U_{SRS}(the test statistic based on SRS).

From the simulation results, in each k, n, and c, we know that the empirical powers of U_{RSS} are superior to U_{SRS}. Therefore, we clearly know that an RSS approach to data collection can lead to a more powerful version than SRS, especially in setting where the actual measurements of sample observations are much more expensive than judgement rankings of them.

Table 4.1 Empirical powers of Tests (c=3, k=4, l=2, $\alpha = 0.05$)

Distribution	n	Statistic	θ					
			0.0	0.2	0.4	0.6	0.8	1.0
uniform	3	U_{RSS}	.037	.065	.108	.172	.222	.331
		U_{SRS}	.033	.051	.068	.099	.126	.164
	4	U_{RSS}	.041	.067	.142	.189	.312	.447
		U_{SRS}	.042	.049	.081	.106	.147	.205
	5	U_{RSS}	.033	.066	.151	.237	.323	.507
		U_{SRS}	.050	.059	.094	.118	.209	.238
normal	3	U_{RSS}	.029	.057	.103	.187	.250	.352
		U_{SRS}	.038	.054	.066	.092	.110	.193
	4	U_{RSS}	.041	.064	.120	.222	.310	.419
		U_{SRS}	.034	.064	.069	.095	.155	.204
	5	U_{RSS}	.035	.084	.137	.251	.361	.518
		U_{SRS}	.040	.059	.094	.118	.209	.238
double exponential	3	U_{RSS}	.035	.064	.146	.241	.329	.456
		U_{SRS}	.031	.056	.078	.135	.175	.243
	4	U_{RSS}	.041	.079	.162	.290	.437	.564
		U_{SRS}	.038	.062	.103	.141	.219	.306
	5	U_{RSS}	.031	.082	.187	.325	.507	.637
		U_{SRS}	.031	.074	.113	.158	.246	.367
logistic	3	U_{RSS}	.032	.062	.132	.170	.273	.382
		U_{SRS}	.033	.049	.077	.095	.121	.187
	4	U_{RSS}	.040	.066	.139	.237	.346	.459
		U_{SRS}	.032	.045	.084	.116	.182	.249
	5	U_{RSS}	.037	.077	.068	.260	.420	.583
		U_{SRS}	.036	.072	.095	.142	.206	.277
Cauchy	3	U_{RSS}	.041	.062	.103	.187	.257	.341
		U_{SRS}	.038	.049	.063	.119	.143	.184
	4	U_{RSS}	.030	.072	.133	.223	.319	.441
		U_{SRS}	.033	.064	.077	.126	.170	.210
	5	U_{RSS}	.036	.078	.141	.265	.384	.510
		U_{SRS}	.033	.061	.101	.125	.177	.246

Table 4.2 Empirical powers of Tests (c=3, k=5, l=2, $\alpha = 0.05$)

Distribution	n	Statistic	θ					
			0.0	0.2	0.4	0.6	0.8	1.0
uniform	3	U_{RSS}	.035	.084	.140	.216	.331	.477
		U_{SRS}	.034	.058	.079	.103	.156	.203
	4	U_{RSS}	.031	.069	.153	.275	.413	.541
		U_{SRS}	.037	.064	.080	.139	.186	.242
	5	U_{RSS}	.036	.082	.185	.333	.495	.643
		U_{SRS}	.035	.060	.104	.157	.192	.273
normal	3	U_{RSS}	.029	.077	.139	.220	.345	.446
		U_{SRS}	.041	.060	.082	.110	.157	.204
	4	U_{RSS}	.038	.075	.159	.264	.402	.545
		U_{SRS}	.039	.052	.102	.116	.193	.249
	5	U_{RSS}	.048	.078	.191	.337	.489	.672
		U_{SRS}	.035	.075	.091	.174	.216	.300
double exponential	3	U_{RSS}	.049	.101	.166	.316	.457	.634
		U_{SRS}	.039	.056	.096	.160	.205	.280
	4	U_{RSS}	.032	.104	.173	.376	.551	.759
		U_{SRS}	.043	.064	.174	.160	.259	.342
	5	U_{RSS}	.040	.119	.230	.458	.664	.812
		U_{SRS}	.039	.062	.118	.230	.302	.389
logistic	3	U_{RSS}	.038	.068	.154	.238	.353	.525
		U_{SRS}	.033	.051	.086	.116	.191	.241
	4	U_{RSS}	.036	.079	.162	.326	.424	.632
		U_{SRS}	.038	.055	.099	.147	.218	.286
	5	U_{RSS}	.032	.096	.184	.382	.547	.708
		U_{SRS}	.037	.045	.101	.138	.240	.342
Cauchy	3	U_{RSS}	.030	.065	.139	.224	.361	.467
		U_{SRS}	.038	.043	.084	.113	.166	.209
	4	U_{RSS}	.033	.071	.172	.288	.435	.609
		U_{SRS}	.033	.074	.080	.132	.185	.266
	5	U_{RSS}	.039	.094	.208	.362	.512	.692
		U_{SRS}	.038	.045	.091	.155	.242	.301

References

[1] Bohn, L. L. and Wolfe, D. A.(1992). Nonparametric two-sample procedures for ranked-set samples data, *Journal of the American Statistical Association*, **87(418)**, 552-561.

[2] Bohn, L. L. and Wolfe, D. A.(1994). The effect of imperfect judgement rankings on properties of procedures based on the ranked-set samples analog of Mann-Whitney-Wilcoxon statistic, *Journal of the American Statistical Association*, **89(425)**, 168-176.

[3] Dell, T. R. and Clutter, J. L.(1972). Ranked-set sampling theory with order statistics background, *Biometrics*, **28**, 545-553.

[4] Hettmansperger, T. P.(1995). The ranked-set sample sign test, *Journal of the Nonparametric Statistics*, **4**, 271-282.

[5] Kim, D. H., Kim, H. G. and Park, H. K.(2000). Nonparametric test for ordered alternatives on multiple ranked-set samples, *The Korean Communication in Statistics*, **7(2)**, 563-573.

[6] Kim, D. H., Kim, Y. C. and Kim, H. G.(1999). Page type test for ordered alternatives on multiple ranked-set samples, *The Korean Communication in Statistics*, **6(2)**, 479-486.

[7] Kim, D. H., Kim, Y. C. and Lee, G. B.(1999). Nonparametric c-sample test for ordered alternatives on ranked-set samples, *Journal of the Korean Data Analysis society*, **1**, 89-109.

[8] McIntyre, G. A.(1952). A method of unbiased selective sampling using ranked sets, *Australian Journal of Agricultural Research*, **3**, 385-390.

[9] Randles, R. H. and Wolfe, D. A.(1979). *Introduction to the Theory of Nonparametric Statistics*, New York : John Wiley.

[10] Stokes, S. L.(1980). Estimation of variance using judgement ordered ranked-set samples, *Biometrics*, **36**, 35-42.

[11] Stokes, S. L. and Sager, T. W.(1988). Characterization of a ranked-set samples with application to estimating distribution functions, *Journal of the American Statistical Association*, **83**, 374-381.

[12] Takahasi, K. and Wakimoto, K.(1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering, *Annals of the institute of statistical mathematics*, **20**, 1-31.