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The Ordering of Conditionally Multivariate Random Vectors[1]

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Abstract

In this paper, we will introduce multivariate versions of bivariate conditionally positive dependence and the partial ordering is developed among conditionally positive lower orthant dependent(*CPLOD*) random vectors. This permits us to measure the degree of *CPLOD*-ness and to compare pairs of *CPLOD* random vectors. Some proper ties and closure under certain statistical operations are derived.

Keywords : *CPQD*, *CPOD*, *CPLOD* ordering, convolution, ↑ *CSI*.

1. Introduction

Lehmann [13] introduced the concepts of positive(negative) dependence together with some other dependent concepts. Since then, much works has been done on the subject and its extensions and numerous multivariate inequalities have been obtained. In other words, a great many papers have been devoted of various generalizations of Lehmann's concepts to finite-dimensional distributions. For references of available results, see Karlin and Rinott[12], Ebrahimi and Ghosh[7] and Shaked[16] and Sampson[15] and Baek[2]. Recently, Brady and Singpurwalla[5] introduced some new conditionally independent and positive(negative) quadrant dependence concepts of random variables. These concepts are qualitative form of dependence which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate tests of hypothesis, sequential testing.

In this paper we will study multivariate versions of bivariate conditionally positive dependence, namely conditionally independent and positive(negative) quadrant dependence introduced by Brady and Singpurwalla[5] and the partial ordering of conditionally po sitive

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lower orthant dependent (*CPLOD*) is developed to compare pairs of conditionally positive orthant dependent (*CPLOD*) random vectors. As indicated above, since *CPLOD* is a qualitative form of dependence(i.e., it simply indicates whether the pair of random variables are mutually conditionally positive dependent or not), it would seem difficult, or impossible to compare different pairs of random variables as to their "degree of *CPLOD*-ness". Therefore, the main goal of this paper is to develop a partial ordering which permits us to compare pairs of dependence structures of a new *CPLOD* random vector of interest as to their degree of *CPLOD*-ness(the exact definition is given in Section 3).

In section 2, some definitions and preliminary results are given. Conditionally independent, positively orthant dependent (*CPOD*) and positive associated (*CPA*) random variables are introduced in Section 3 together with some basic properties used throughout this paper. Certain closure properties of *CPLOD* ordering are derived in Section 4. It is shown that *CPLOD* ordering is preserved under convolution, mixture of a certain type, transformation of the random variables by increasing functions, limit in distribution, and other operations of interest in statistics.

2. Preliminaries

An important principle of probability theory is that the notions of dependence and independence are conditional, the conditioning being done on some observable or unobservable quantity, say Θ . It is common to think of Θ as a parameter and this is the point of view that we adopt. Brady and Singpurwalla[5] introduced some concepts of conditional dependence between random variables. Let $\underline{X}$ and $\underline{Y}$ be two random vectors, of dimension p and q, respectively.

We start by stating the definitions of conditional independence and positive(negative) dependence provided by Brady and Singpurwalla[5].

Definition 2.1[5]. The random vector $\underline{X}=(X_1,\cdots,X_p)$ is $\theta\in I_1$ conditionally indep endent of $\underline{Y}=(Y_1,\cdots,Y_q)$ and $\theta\in I_2(\theta\in I_3)$ conditionally positive(negative) dependent on $\underline{Y}$, denoted by $\{(\underline{X} \amalg \underline{Y}) \mid \theta\in I_1,\ >\theta\in I_2,\ <\theta\in I_3\}$ if

(a) $P(\underline{X}\in A\mid \underline{Y}\in B,\ \theta\in I_1)=P(\underline{X}\in A\mid\theta\in I_1\}$,

(b) $P(\underline{X}\in A\mid \underline{Y}\in B,\ \theta\in I_2)\geq P(\underline{X}\in A\mid\theta\in I_2)$, and

(c) $P(\underline{X}\in A\mid \underline{Y}\in B,\ \theta\in I_3)\leq P(\underline{X}\in A\mid\theta\in I_3)$, $\forall A,\ B,\ \theta$,

where A, B are open upper sets (U is an upper set if $\underline{a}\in U$, and $\underline{a}<\underline{b}$ implies $\underline{b}\in U$ (Shaked, [16]).

Assume that $p=q=1$. Then Definition 2.1 can be stated in terms of the joint and marginal distribution functions of X and Y. Let

$$F(x,\ y\mid\theta)=P(X\le x,\ Y\le y\mid\theta),$$
$$G(x\mid\theta)=P(X\le x\mid\theta),\ \text{and}$$
$$H(y\mid\theta)=P(Y\le y\mid\theta).$$

Then Definition 2.1 is equivalent to

Definition 2.2[5]. The pair (X, Y) is $\theta\in I_1$ conditionally independent and $\theta\in I_2$ ($\theta\in I_3$) conditionally positive(negative) quadrant dependent ($CPQD$($CNQD$)),denoted by $\{X \amalg Y \mid \theta\in I_1,\ > \theta\in I_2,\ < \theta\in I_3\}$ if

(a) $F(x,\ y\mid\theta\in I_1) = G(x\mid\theta\in I_1)H(y\mid\theta\in I_1)$,

(b) $F(x,\ y\mid\theta\in I_2)\ge G(x\mid\theta\in I_2)H(y\mid\theta\in I_2)$, and

(c) $F(x,\ y\mid\theta\in I_3)\le G(x\mid\theta\in I_3)H(y\mid\theta\in I_3)$.

We close this section by stating the following lemma as per Brady and Singpurwalla[5].

Lemma 2.3[5]. If conditions (a), (b) and (c) of Definition 2.2 hold and if the conditional expectations $E(XY\mid\theta)$, $E(X\mid\theta)$ and $E(Y\mid\theta)$ exist, then Definition 2.2 implies that

(a) $E(XY\mid\theta\in I_1)=E(X\mid\theta\in I_1)E(Y\mid\theta\in I_1)$,

(b) $E(XY\mid\theta\in I_2)\ge E(X\mid\theta\in I_2)E(Y\mid\theta\in I_2)$, and

(c) $E(XY\mid\theta\in I_3)\le E(X\mid\theta\in I_3)E(Y\mid\theta\in I_3)$.

A strengthening of Lemma 2.3 is

Lemma 2.4[5]. Let f, g be increasing functions of X and Y, respectively. Then Definition 2.2. implies that

(a) $Cov(f(X),\ g(Y)\mid\theta\in I_1) = 0$,

(b) $Cov(f(X),\ g(Y)\mid\theta\in I_2)\ge 0$, and

(c) $Cov(f(X),\ g(Y)\mid\theta\in I_3)\le 0$.

Proof. This follows by an extension of a proof by Lehmann[13].

3. Definitions and Properties

In this section, we present definitions, notations, and basic facts used the throughout this paper. We now extend the bivariate case of the $CPQD$ to the multivariate case for a sequence of random vectors.

Definition 3.1. A random vector $\underset{\sim}{X}=(X_1,\cdots,X_n)$ is $\theta\in I_1$ conditionally independent and

$\theta \in I_2$ conditionally positive lower orthant dependent (*CPLOD*) if

(a) $P(X_1 \le x_1, \cdots, X_n \le x_n \mid \theta \in I_1) = \prod_{i=1}^{n} P(X_i \le x_i \mid \theta \in I_1)$,

(b) $P(X_1 \le x_1, \cdots, X_n \le x_n \mid \theta \in I_2) \ge \prod_{i=1}^{n} P(X_i \le x_i \mid \theta \in I_2)$.

A random vector $\underset{\sim}{X} = (X_1, \cdots, X_n)$ is $\theta \in I_2$ conditionally positive upper orthant depend ent (*CPUOD*) if

(b′) $P(X_1 > x_1, \cdots, X_n > x_n \mid \theta \in I_2) \ge \prod_{i=1}^{n} P(X_i > x_i \mid \theta \in I_2)$.

A random vector $\underset{\sim}{X}$ is $\theta \in I_2$ conditionally positive orthant dependent (*CPOD*) if $\underset{\sim}{X}$ is *CPLOD* and *CPUOD*.

Theorem 3.2. Let (a) $\underset{\sim}{X} = (X_1, \cdots, X_n)$ and $\underset{\sim}{Z} = (Z_1, \cdots, Z_n)$ be θ conditionally independent and *CPOD*, (b) $\underset{\sim}{X}$ be conditionally independent of $\underset{\sim}{Z}$ given θ. Then $\underset{\sim}{X} + \underset{\sim}{Z}$ is θ conditionally independent and *CPOD* on I_1 and I_2.

Proof. $P(X_1 + Z_1 \le x_1, \cdots, X_n + Z_n \le x_n \mid \theta \in I_i)$

$$= \int \cdots \int P(X_1 \le x_1 - z_1, \cdots, X_n \le x_n - z_n \mid \theta \in I_i)\, dH_{z_1, \cdots, z_n \mid \theta \in I_2}(z_1, \cdots, z_n \mid \theta \in I_i)$$

$$= \prod_{i=1}^{n} P(X_i + Z_i \le x_i \mid \theta \in I_1)$$

$$\ge \int \cdots \int \prod_{i=1}^{n} P(X_i \le x_i - z_i \mid \theta \in I_2)\, dH_{z_1, \cdots, z_n \mid \theta \in I_2}(z_1, \cdots, z_n \mid \theta \in I_2)$$

$$\ge \int \cdots \int \prod_{i=1}^{n} P(X_i \le x_i - z_i \mid \theta \in I_2)\, dH_{z_i \mid \theta \in I_2}(z_i \mid \theta \in I_2)$$

$$= \prod_{i=1}^{n} P(X_i + Z_i \le x_i \mid \theta \in I_2).$$ Similarly $\underset{\sim}{X} + \underset{\sim}{Z}$ is *CPUOD*.

Definition 3.3. Let f, g be increasing functions of random vector $\underset{\sim}{X} = (X_1, \cdots, X_n)$. Then $\underset{\sim}{X}$ is $\theta \in I_1$ conditionally independent and $\theta \in I_2$ conditionally positive associated (*CPA*) if

(a) $Cov(f(\underset{\sim}{X}), g(\underset{\sim}{X}) \mid \theta \in I_1) = 0$,

(b) $Cov(f(\underset{\sim}{X}), g(\underset{\sim}{X}) \mid \theta \in I_2) \ge 0$..

From Definition 3.3 of conditional independence and positively associated it is not difficult to show that:

Property 1. Increasing functions of a sequence of θ conditionally independent and *CPA* on I_1 and I_2 are θ conditionally independent and *CPA* on I_1 and I_2.

Property 2. The set of consisting of a single random variable is θ conditionally independent and *CPA* on I_1 and I_2.

Property 3. Any subset of θ conditionally independent and *CPA* on I_1 and I_2 are θ conditionally independent and *CPA* on I_1 and I_2.

Property 4. The union of θ conditionally independent and *CPA* on I_1 and I_2 are θ conditionally independent and *CPA* on I_1 and I_2.

Proof. The proof will be given for *CPA* on I_2. Let the components of $\underline{X}=(X_1,\cdots,X_n)'$ and $\underline{Y}=(Y_1,\cdots,Y_n)'$ be *CPA*, respectively and $\underline{X},\underline{Y}$ be independent given θ. Let us define the $V_1=g_1(\underline{X},\underline{Y})$, $V_2=g_2(\underline{X},\underline{Y})$ where g_1, g_2 are non decreasing in each argument. Then it is easy to check that

$$Cov(V_1, V_2|\theta\in I_2)=E[Cov(V_1, V_2|\underline{X}, \theta\in I_2)]+Cov[E(V_1|\underline{X}, \theta\in I_2), E(V_2|\underline{X}, \theta\in I_2)]$$

It is clear that for every fixed $\underline{X}=\underline{x}$ the conditional covariance of V_1, V_2 is nonnegative when $\theta\in I_2$ and that conditional expectations of V_1, V_2 are increasing in $\underline{x}$ (in each argument) when $\theta\in I_2$. Therefore the right-hand side of (3.1) is nonnegative.

Definition 3.4. A random vector $\underline{X}$ is θ conditionally stochastically increasing (*CSI*) in the random vector $\underline{Y}$ if $E(f(\underline{X})\,|\,\underline{Y}=\underline{y},\theta)$ is increasing in $\underline{y}$ for all real valued increasing function f given θ .

Theorem 3.5. Let (a) $\underline{X}=(X_1,\cdots,X_n)$ given $\underline{Y}=(Y_1,\cdots,Y_n)$, be *CPOD* on I_2, (b) X_i be *CSI* in $\underline{Y}$ for $i=1,2,\cdots,n$, and $\underline{Y}$ be *CPA* on I_2. Then (1) $(\underline{X},\underline{Y})$ are *CPOD* on I_2, and (2) in particular, $\underline{X}$ is *CPOD* on I_2.

Unfortunately, the elementary proof Theorem 3.5. for the bivariate case *CPQD* does not extend to higher dimensions *CPOD* on I_2. For this reason we present the following Lemma 3.6.

Lemma 3.6. If $Y_1,\cdots,Y_m$ are *CPA* on I_2 and if $g_i=(y_1,\cdots,y_m)$ are nonnegative and increasing for $i=1,2,\cdots,k$, $k\geq 2$, then

$$E[\prod_{i=1}^{k} g_i(Y_1,\cdots,Y_m)\,|\,\theta\in I_2]\geq \prod_{i=1}^{k} E[g_i(Y_1,\cdots,Y_m)\,|\,\theta\in I_2] \tag{3.1}$$

Proof. We shall prove the lemma by mathematical induction.

Suppose $k=2$. By the definition of CPA (Def. 3.3), and the fact that $g_i=(y_1,\cdots,y_n)$ is increasing in all arguments, the inequality immediately follows from Property 1.

Now suppose (3.1) holds for $k-1$; i.e,

$$E[\prod_{i=1}^{k-1} g_i(Y_1,\cdots,Y_m)\,|\,\theta\in I_2]\ge\prod_{i=1}^{k-1}E[g_i(Y_1,\cdots,Y_m)\,|\,\theta\in I_2]. \tag{3.2}$$

Again by Property 1, $\prod_{i=1}^{k-1} g_i[(Y_1,\cdots,Y_m)|\theta\in I_2]$ and $g_k[(Y_1,\cdots,Y_m)|\theta\in I_2]$ are CPA.

It follows that

$$E[\prod_{i=1}^{k-1} g_i(Y_1,\cdots,Y_m)\,|\,\theta\in I_2]\ge E[g_k(Y_1,\cdots,Y_k)\,|\,\theta\in I_2]\,E[\prod_{i=1}^{k-1} g_i(Y_1,\cdots,Y_m)\,|\,\theta\in I_2]. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain the conclusion of the lemma.

Proof of Theorem 3.5.

(1) observe that

$$\begin{aligned}
&P(\bigcap_{i=1}^{n}(X_i>x_i),\bigcap_{j=1}^{m}(Y_j>y_j)\,|\,\theta\in I_2)\\
&=E_{\underline{Y}}\left[P\left(\bigcap_{i=1}^{n}(X_i>x_i)\,|\,\underline{Y},\theta\in I_2\right)\cdot I_{\left(\bigcap_{j=1}^{m}(Y_j>y_j)\,|\,\theta\in I_2\right)}\right]\\
&\ge E_{\underline{Y}}\left\{\prod_{i=1}^{n}P(X_i>x_i\,|\,\underline{Y},\theta\in I_2)\cdot I_{\left(\bigcap_{j=1}^{m}(Y_j>y_j)\,|\,\theta\in I_2\right)}\right\}\\
&\ge\prod_{i=1}^{n}\{E_{\underline{Y}}P(X_i>x_i\,|\,\underline{Y},\theta\in I_2)\cdot E_{\underline{Y}}I_{\left(\bigcap_{j=1}^{m}(Y_j>y_j)\,|\,\theta\in I_2\right)}\}\\
&=\prod_{i=1}^{n}P(X_i>x_i\,|\,\theta\in I_2)\cdot E_{\underline{Y}}I_{\left(\bigcap_{j=1}^{m}(Y_j>y_j)\,|\,\theta\in I_2\right)}\\
&\ge\prod_{i=1}^{n}P(X_i>x_i\,|\,\theta\in I_2)\prod_{j=1}^{m}P(Y_j>y_j\,|\,\theta\in I_2).
\end{aligned}$$

The first inequality follows assumption (a). The second and the third inequalities follow from assumptions (b), (c), together with Lemma 3.6. Thus $X_1,\cdots,X_n$, $Y_1,\cdots,Y_m$ are $CPUOD$ on I_2 and similarly $X_1,\cdots,X_n$, $Y_1,\cdots,Y_m$ are $CPLOD$ on I_2. (2) The result follows immediately from Property 3.

Corollary 3.7. Let (a) $\underline{X}$ given λ, be $CPOD$ on I_2 ,(b) X_i be CSI in λ for $i=1,\cdots,n$. Then $\underline{X}$ is $CPOD$ on I_2.

Proof. The proof is an immediate consequence of Theorem 3.5 and Property 2.

Let $\beta=\beta(F_{X_1}|\theta,\cdots,F_{X_n}|\theta)$ be the class of multivariate distribution functions H on R^n

having specified marginal distribution functions $F_{X_1},\cdots,F_{X_n}$ given θ.

We consider, β^+, a subclass of β, denoted by

$$\beta^+ = \{H(x_1,\cdots,x_n \mid \theta\in I_2) \mid \quad H \text{ is } CPLOD, H(x_1,\cdots,\infty\mid\theta\in I_2) = F_{X_1}(x_1\mid\theta\in I_2),\cdots, H(\infty,\cdots,x_n\mid\theta\in I_2) = F_{X_n}(x_n\mid\theta\in I_2).$$

We then define the *CPLOD* ordering of dependence within the class. Let H_1 and H_2 belong to β^+.

Definition 3.8. A random vector $\underset{\sim}{X}=(X_1,\cdots,X_n)$ or H_1 is more $\theta\in I_2$ conditionally positive lower orthant dependent (*CPLOD*) than is $\underset{\sim}{Y}=(Y_1,\cdots,Y_n)$or H_2 if for all $(x_1,\cdots,x_n)\in R^n$,

$$P(X_1\le x_1,\cdots,X_n\le x_n|\theta\in I_2)\ge P(Y_1\le x_1,\cdots,Y_n\le x_n|\theta\in I_2),$$

we write $\underset{\sim}{X}\rangle\,(CPLOD)\,\underset{\sim}{Y}$ or $H_1\rangle\,(CPLOD)\,H_2$.

4. Closure Properties of $(\beta^+, (\rangle(CPLOD))$

In this section we establish preservation of the *CPLOD* ordering under convolutions, mixtures, transformations of the random variables by increasing functions, limits in distribution, and other operations of interest in statistics.

Lemma 4.1. Let (a) $\underset{\sim}{X}=(X_1,\cdots,X_n)$ and $\underset{\sim}{Y}=(Y_1,\cdots,Y_n)$ have distributions H_1 and H_2 respectively, where H_1 and H_2 belong to β^+ such that $\underset{\sim}{X}\rangle\,(CPLOD)\,\underset{\sim}{Y}$ and (b) $\underset{\sim}{Z}=(Z_1,\cdots,Z_n)$ with an arbitrary *CPLOD* distribution function H conditionally independent of both $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ given θ. Then $\underset{\sim}{X}+\underset{\sim}{Z}\rangle\,(CPLOD)\,\underset{\sim}{Y}+\underset{\sim}{Z}$.

Proof. From Theorem 3.2 $\underset{\sim}{X}+\underset{\sim}{Z}$ and $\underset{\sim}{Y}+\underset{\sim}{Z}$ are *CPLOD.*

Next we need to show that for each $(x_1,\cdots,x_2)\in R^n$,

$$\underset{\sim}{X}+\underset{\sim}{Z}\rangle\,(CPLOD)\,\underset{\sim}{Y}+\underset{\sim}{Z} \tag{4.1}$$

Note that the left side of (4.1)

$$= \int\cdots\int P(X_1\le x_1-z_1,\cdots,X_n\le x_n-z_n\mid\theta\in I_2)\,dH_{z_1,\cdots,z_n|\theta\in I_2}(z_1,\cdots,z_n\mid\theta\in I_2)$$

$$\ge \int\cdots\int P(Y_1\le x_1-z_1,\cdots,Y_n\le x_n-z_n\mid\theta\in I_2)\,dH_{z_1,\cdots,z_n|\theta\in I_2}(z_1,\cdots,z_n\mid\theta\in I_2)$$

$$= P(Y_1+Z_1\le x_1,\cdots,Y_n+Z_n\le x_n\mid\theta\in I_2).$$

Theorem 4.2. Suppose that (a) the random vectors $\underset{\sim}{X}\rangle\,(CPLOD)\,\underset{\sim}{Y}$ and $\underset{\sim}{U}\rangle\,(CPLOD)\,\underset{\sim}{V}$, (b) $\underset{\sim}{U}$ is conditionally independent of both $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ given θ, and $\underset{\sim}{Y}$ is conditionally independent of $\underline{V}$ given θ. Then $\underline{X}+\underline{U} > (CPLOD)\, \underline{Y}+\underline{V}$.

Proof. By assumption $\underline{X} > (CPLOD)\, \underline{Y}$. Specifying $\underline{Z}$ to be $\underline{U}$, we apply Lemma 4.1 to obtain

$$\underline{X}+\underline{U} > (CPLOD)\, \underline{Y}+\underline{U} \tag{4.2}$$

Next, we use the assumption $\underline{U} > (CPLOD)\, \underline{V}$, specify $\underline{Z}$ to be $\underline{Y}$

$$\underline{Y}+\underline{U} > (CPLOD)\, \underline{Y}+\underline{V} \tag{4.3}$$

By combining (4.2) and (4.3), $\underline{X}+\underline{U} > (CPLOD)\, \underline{Y}+\underline{V}$.

The following is an application of Theorem 4.2 which is very important *CPLOD* in compound distributions which arise naturally in stochastic processes.

Application 4.3. Let (a) $(N_1, \cdots, N_n)$ be a k-variate variable with components assuming values in the set $\{1,2,\cdots\}$ and let (b) $\{X_{i1}, \cdots, X_{ik}) : i \geq 1\}$ and $\{Y_{i1}, \cdots, Y_{ik}) : i \geq 1\}$ be sequence of nonnegative independent $k-$variate given θ with have distribution functions H_1 and H_2, respectively, where H_1 and H_2 belong to β^+ such that $H_1 > (CPLOD)\, H_2$. Then using the Theorem 4.2, we obtain that

$$\left(\sum_{i=1}^{N_1} X_{i1}, \cdots, \sum_{i=1}^{N_k} X_{ik}\right) > (CPLOD) \left(\sum_{i=1}^{N_1} Y_{i1}, \cdots, \sum_{i=1}^{N_k} Y_{ik}\right)$$

Example 1. Let $\{N_1(t) \cdots N_k(t) \mid t \geq 0\}$ be a k-variate poisson processes, i.e $N_1(t) = Z_1(t) + W(t), \cdots, N_k(t) = Z_k(t) + W(t)$ where $Z_1(t), \cdots, Z_k(t)$,and $W(t)$ are indepe ndent poisson processes given θ. Let $\{(X_{n1}, \cdots, X_{nk}) \mid n = 0,1,2,\cdots\}$ and $\{(Y_{n1}, \cdots, Y_{nk}) \mid n = 1,2,\cdots\}$ be the sequence of independent and identically distributed variables. Define the k-variate compound poisson processes $\{(X_1(t), \cdots, X_n(t)) \mid t \geq 0\}$ and $\{(Y_1(t), \cdots, Y_n(t)) \mid t \geq 0\}$ by

$$X_1(t) = \sum_{n=0}^{N_1(t)} X_{n1}, \cdots, X_n(t) = \sum_{n=0}^{N_k(t)} X_{nk} \text{ and } Y_1(t) = \sum_{n=0}^{N_1(t)} Y_{n1}, \cdots, Y_n(t) = \sum_{n=0}^{N_k(t)} Y_{nk}.$$

Then, consequently an application 4.3 implies $(X_1(t), \cdots, X_n(t)) > (CPLOD)\, (Y_1(t), \cdots, Y_n(t))$ for every $t \geq 0$ whenever $(X_{n1}, \cdots, X_{nk}) > (CPLOD)\, (Y_{1n}, \cdots, Y_{nk})$.

Our next result deals with the preservation of the *CPLOD* ordering under mixture. We may now define the subclass β_λ^+ of β^+ by $\beta_\lambda^+ =$

$$\{H_\lambda \mid H(x_1, \infty, \cdots, \infty \mid \theta \in I_2, \lambda) = F_{X_1}(x_1 \mid \theta \in I_2, \lambda), \cdots, H(\infty, \cdots, \infty, x_n \mid \theta \in I_2, \lambda) = F_{X_n}(x_n \mid \theta \in I_2, \lambda), H_\lambda \mid \lambda \text{ is } CPLOD, \text{ and } F_{X_1}, \cdots, F_{X_n} \text{ are } CSI \text{ in } \lambda\}$$

Now consider $(\beta_\lambda^+, >(CPLOD))$. The following proposition shows that if two elements of β_λ^+ are ordered according to $>(CPLOD)$, then after mixing on λ when $\theta\in I_2$, the resulting elements in β^+ preserve the same order.

Proposition 4.4. Let the random vector $\underset{\sim}{X}|\lambda$ and $\underset{\sim}{Y}|\lambda$ belong to β_λ^+ and let $\underset{\sim}{X}|\lambda >(CPLOD)\underset{\sim}{Y}|\lambda$ for all λ. Then, unconditionally $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ belong to β^+ and $\underset{\sim}{X} > (CPLOD)\underset{\sim}{Y}$

Proof. From Corollary 3.7, $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are *CPLOD*.

Now,

$$
\begin{aligned}
&P(X_1\le x_1,\cdots,X_n\le x_n\,|\,\theta\in I_2)\\
&=E_\lambda P(X_1\le x_1,\cdots,X_n\le x_n\,|\,\theta\in I_2,\ \lambda)\\
&\ge E_\lambda P(Y_1\le x_1,\cdots,Y_n\le x_n\,|\,\theta\in I_2,\ \lambda)\\
&=P(Y_1\le x_1,\cdots,Y_n\le x_n\,|\,\theta\in I_2).
\end{aligned}
$$

Thus $\underset{\sim}{X} > (CPLOD)\underset{\sim}{Y}$.

Definition 4.5. Functions $f: R^m\to R^n$ are increasing if they increase in each of their arguments when all other arguments are fixed.

Then we show that the *CPLOD* ordering is invariant under transformations of increasing real valued functions.

Theorem 4.6. Let (a) $(X_{i1},X_{i2},\cdots,X_{ik})$ and $(Y_{i1},Y_{i2},\cdots,Y_{ik})$, $i=1,2,\cdots,n$ be θ independent k-variate random variables given θ with have distribution functions H_1 and H_2 respectively, where H_1 and H_2 belong to β^+ such that $H_1>(CPLOD)H_2$, and (b) $g_j: R^n\to R$, $j=1,2,\cdots,k$ are increasing functions. Then $X_j=g_j(X_{i1},\cdots,X_{nj})>(CPLOD)Y_j=g_j(Y_{i1},\cdots,Y_{nj})$, for $j=1,2,\cdots,k$.

Proof. The proof will be given for the case $k=n=2$. For the general k and n, proof is similar. First we will show that

$(g_1(X_{11},X_{21}),\ g_2(X_{12},X_{22}))$ and $(g_1(Y_{11},Y_{21}),\ g_2(Y_{12},Y_{22}))$ are *CPLOD*.

Now,

$$
\begin{aligned}
&P(g_1(X_{11},X_{21})\le x_1,\ g_2(X_{12},X_{22})\le x_2\,|\,\theta\in I_2)\\
&=EP(g_1(X_{11},x_{21})\le x_1,\ g_2(X_{12},x_{22})\le x_2\,|\,\theta\in I_2,\ X_{21},X_{22})\\
&\ge E[P(g_1(X_{11},x_{21})\le x_1\,|\,\theta\in I_2,\ X_{21}),\ P(g_2(X_{12},x_{22})\le x_2\,|\,\theta\in I_2,\ X_{22})]\\
&\ge \prod_{i=1}^{2} EP(g_i(X_{1i},x_{2i})\le x_i\,|\,\theta\in I_i,\ X_{2i})\\
&=P(g_1(X_{11},X_{21})\le x_1\,|\,\theta\in I_2)\,P(g_2(X_{12},X_{22})\le x_2\,|\,\theta\in I_2),
\end{aligned}
$$

so that $g_1(X_{11}, X_{21})$, $g_2(X_{12}, X_{22})$are *CPLOD,* and similarly $g_1(Y_{11}, Y_{21})$, $g_2(Y_{12}, Y_{22})$ are also *CPLOD.*

Next we need to show that for each $(x_1, x_2) \in R^2$,

$$\begin{aligned} &P(g_1(X_{11}, X_{21}) \le x_1,\ g_2(X_{12}, X_{22}) \le x_2 \mid \theta \in I_2) \\ &= EP(g_1(X_{11}, x_{21}) \le x_1,\ g_2(X_{12}, x_{22}) \le x_2 \mid \theta \in I_2,\ X_{21}, X_{22}) \\ &\ge EP(g_1(Y_{11}, y_{21}) \le x_1,\ g_2(Y_{12}, y_{22}) \le x_2 \mid \theta \in I_2,\ Y_{21}, Y_{22}) \\ &= P(g_1(Y_{11}, Y_{21}) \le x_1,\ g_2(Y_{12}, Y_{22}) \le x_2 \mid \theta \in I_2). \end{aligned}$$

In Theorem 4.7 we show that the *CPLOD* ordering is preserved under limits in distributions.

Theorem 4.7. Suppose $H_n > (CPLOD)\, H_n'$ for every n, and H_n and H_n' converge weakly to H_1, H_1' respectively. Then

$$H_1 > (CPLOD)\, H_1'$$

Proof. For any $x_1, \cdots, x_n$ writing $\underline{X}_n = (X_{1n}, \cdots, X_{pn})$and $\underline{Y}_n = (Y_{1n}, \cdots, Y_{pn})$, $n \ge 1$,

$$\begin{aligned} &P(X_1 \le x_1, \cdots, X_p \le x_p \mid \theta \in I_2) \\ &= \lim_{n \to \infty} P(X_{1n} \le x_1, \cdots, X_{pn} \le x_p \mid \theta \in I_2) \\ &\ge \lim_{n \to \infty} P(Y_{1n} \le x_1, \cdots, Y_{pn} \le x_p \mid \theta \in I_2) \\ &= P(Y_1 \le x_1, \cdots, Y_p \le x_p \mid \theta \in I_2). \end{aligned}$$

Thus $H_1 > (CPLOD)\, H_1'$.

We now turn our attention to a simple but important property of the class β^+.

Theorem 4.8. Let H_1 and H_2 be both having the same one dimensional marginals, where H_1 and H_2 belong to β^+. Then if $H_\alpha = \alpha H_1 + (1-\alpha)H_2$, $\alpha \in (0,1)$, H_α is *CPLOD.*

Proof. We prove this result for *CPLOD.* By definition, the one-dimensional margina ls of H_α are the same as those of H_1 or H_2

$$\begin{aligned} &P_{H_\alpha}(X_1 \le x_1, \cdots, X_n \le x_n \mid \theta \in I_2) \\ &= \alpha P_{H_1}(X_{1n} \le x_1, \cdots, X_n \le x_n \mid \theta \in I_2) + (1-\alpha) P_{H_2}(X_{1n} \le x_1, \cdots, X_n \le x_n \mid \theta \in I_2) \\ &\ge \alpha \prod_{i=1}^{n} P_{H_1}(X_i \le x_i \mid \theta \in I_2) + (1-\alpha) \prod_{i=1}^{n} P_{H_2} P(X_i \le x_i \mid \theta \in I_2) \\ &= \prod_{i=1}^{n} P_{H_2}(X_i \le x_i \mid \theta \in I_2) \end{aligned}$$

Hence H_α is *CPLOD.*

References

[1] Ahmed, A., Langberg, N., Leon, R. and Proschan, F. (1978). Two concepts of positive dependence with applications in multivariate analysis. Technical Report AFOSR 78-6, Department of Statistics, Florida State University.

[2] Baek, J. I. (1997). A weakly dependence structure of multivariate processes. Statistics & Probability Letters, 34, 355-363

[3] Barlow, R. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, Inc., New York.

[4] Blomqvist, N.(1950). On a measure of dependence between two random variables. Annals of Mathematical Statistics, 21, 593-600.

[5] Brady, B. M. and Singpurwalla, N. D. (1990). Stochastically Monotone Dependence Topics in Statistical Dependence,(H. W Block, A. R. Sampson, T. H. Savits Ed.) Institute of Mathematical Statistics, 16, 93-102.

[6] Ebrahimi, N.(1982). The ordering of negative quadrant dependence. Communications in Statistics, 11(21), 2389-2399

[7] Ebrahimi, N. and Ghosh, M. (1981). Multivariate Negative Dependence. Communicat ions in Statistics, 10, 307-339.

[8] Esary, J. D., Proschan. F. and Walkup. D. W.(1967). Association of random variables applications. Annals of Mathematical Statistics, 38, 1466-1474.

[9] Friday, D. S.(1981). Dependence concepts for stochastic processes. Proceedings of Nato Advanced Study Institute Series. 5, 349-361.

[10] Glaz, J. and Johnson, B. M.(1982). Probability inequalities for multivariate distribut ions dependence structures. Technical Report, University of Connecticut.

[11] Holland, P. W. and Rosenbaum, P. R. (1986). Conditional Association and Unidimen sionality in Monotone Latent Variable Models. Annals of Statistics, 14, 1523-1543.

[12] Karlin, S. and Rinott, Y.(1980). Classes of orderings of measures and related correlation inequalities. Journal of Multivariate Analysis, 10, 467-498.

[13] Lehmann, E. (1966). Some Concepts of Dependence. Annals of Mathematical Stati stics, 37, 1137 -1153.

[14] Ross, S. M.(1983). Stochastic Processes. Wiley, New York.

[15] Sampson, A. R.(1983). Positive dependence properties of elliptically symmetric distributions. Journal of Multivariate Analysis. 13, 375-381.

[16] Shaked, M. (1982). A General Theory of Some Positive Dependence Notions. Journal of Multivariate Analysis, 12, 199-218.