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A Central Limit Theorem for a Stationary Linear Process Generated by Linearly Positive Quadrant Dependent Process[1]

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Abstract

A central limit theorem is obtained for stationary linear process of the form $X_t=\sum_{j=0}^{\infty}a_j\varepsilon_{t-j}$, where $\{\varepsilon_t\}$ is a strictly stationary sequence of linearly positive quadrant dependent random variables with $E\varepsilon_t=0$, $E\varepsilon_t^2<\infty$ and $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty}|a_j|<\infty$ we also derive a functional central limit theorem for this linear process.

Keywords : Central limit theorem, linear process, linearly positive quadrant dependent sequence, stationary.

1. Introduction and Main Results

In the last years there has been growing interest in concepts of positive dependence for families of random variables. Such concepts are of considerable use in deriving inequalities in probability and statistics. Lehmann(1966) introduced a simple and natural definition of positive dependence : A sequence $\{\varepsilon_t,\ t\in Z^+\}$, $Z^+=\{0,1,2,3,\cdots\}$, of random variables is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real a_i, a_j and $i\neq j$ $P\{\varepsilon_i>a_i,\ \varepsilon_j>a_j\}\geq P\{\varepsilon_i>a_i\}P\{\varepsilon_j>a_j\}$. A stronger concept than PQD was introduced by Newman(1984) : A sequence $\{\varepsilon_t,\ t\in Z^+\}$ of random variables is said to be linearly positive quadrant dependent(LPQD) if for any disjoint subsets A, B $\subset Z^+$ and positive $r_j's$

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$$\sum_{i\in A} r_i \varepsilon_i \quad \text{and} \quad \sum_{j\in B} r_j \varepsilon_j \quad \text{are} \quad \text{PQD}. \tag{1}$$

Newman(1984) established a central limit theorem for the strictly stationary LPQD process $\{\varepsilon_t\}$ with $E(\varepsilon_t)=0$, $E(\varepsilon_t^2)<\infty$; more precisely, if $0<\sigma^2=E\varepsilon_1^2+2\sum_{t=2}^{\infty}E(\varepsilon_1\varepsilon_t)<\infty$ then $n^{-\frac{1}{2}}(\varepsilon_1+\cdots+\varepsilon_n)\to N(0,\sigma^2)$ as $n\to\infty$.

Let $\{X_t, t\in Z^+\}$ be a stationary linear process defined on a probability space $(\Omega,\mathcal{F},P)$ of the form

$$X_t=\sum_{j=0}^{\infty}a_j\varepsilon_{t-j}, \tag{2}$$

where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty}|a_j|<\infty$ and $\{\varepsilon_t\}$ is a strictly stationary process such that $E\varepsilon_t=0$ and $0<E\varepsilon_t^2<\infty$.

The linear processes are special importance in time series analysis and they arise from a wide variety of contexts (see, e.g., Hannan(1970) Ch.6). Applications to economics, engineering and physical sciences are extremely broad and a vast amount of literature is devoted to the study of the limit theorems for linear processes under various conditions on ε_t. For the linear processes, Fakhre-Zakeri and Lee(1992) and Fakhre-Zakeri and Farshidi(1993) established central limit theorems for the linear processes under the iid assumption on ε_t and Fakhre-Zakeri and Lee(1997) proved a functional central limit theorem under the strong mixing condition on ε_t.

In this paper, we establish a central limit theorem and a functional central limit theorem for a stationary linear process of the form (2), generated by a stationary LPQD process $\{\varepsilon_t\}$. More precisely, we will prove the following theorems :

Theorem 1. Let $\{X_t\}$ be a stationary linear process of the form (2), where $\{a_j\}$ is a sequence of constants with $\sum_{j=0}^{\infty}|a_j|<\infty$ and $\{\varepsilon_t\}$ is a strictly stationary LPQD process with $E\varepsilon_t=0$, $0<E\varepsilon_t^2<\infty$.
Assume

$$0<\sigma^2=E\,\varepsilon_1^2+2\sum_{t=2}^{\infty}E(\varepsilon_1\varepsilon_t)<\infty. \tag{3}$$

Then the linear process $\{X_t\}$ fulfills the central limit theorem, that is

$$n^{-\frac{1}{2}}\sum_{t=1}^{n}X_t\xrightarrow{D}N\Big(0,\sigma^2\Big(\sum_{j=0}^{\infty}a_j\Big)^2\Big)\text{ as } n\to\infty,$$

where $\xrightarrow{D}$ indicates convergence in distribution.

Theorem 2. Let $\{X_t\}$ be a stationary linear process of the form (2) defined in Theorem 1. Let $S_n = \sum_{t=1}^{n} X_t$, $\tau^2 = \sigma^2 (\sum_{j=0}^{\infty} a_j)^2$ where σ^2 is defined as in (3) and $\sum_{j=0}^{\infty} |a_j| < \infty$. Define for $n \geq 1$ the stochastic process

$$\xi_n(u) = n^{-\frac{1}{2}} \tau^{-1} S_{[nu]}, \quad u \in [0, 1]. \tag{4}$$

Assume

$$\sum_{t=n+1}^{\infty} E\varepsilon_1 \varepsilon_t = O(n^{-\rho}) \qquad \text{for some} \quad \rho > 0, \tag{5}$$

and

$$E|\varepsilon_t|^s < \infty \qquad \text{for some} \quad s > 2. \tag{6}$$

Then the process $\{\xi_n\}$ satisfies the functional central limit theorem, that is, the process $\{\xi_n\}$ converges weakly to Wiener measure W on D[0,1] the space of all functions on $[0, 1]$, which have left hand limits and are continuous from the right.

2. Proofs

Proof of Theorem 1. Letting

$$\tilde{a}_j = \sum_{i=j+1}^{\infty} a_i$$

and

$$\tilde{\varepsilon}_t = \sum_{j=0}^{\infty} \tilde{a}_j \varepsilon_{t-j},$$

which is well-defined since $\sum_{j=0}^{\infty} |\tilde{a}_j| < \infty$, we have

$$\begin{aligned} X_t &= \sum_{j=0}^{\infty} a_j \varepsilon_{t-j} \\ &= a_0 \varepsilon_t + \sum_{j=1}^{\infty} a_j \varepsilon_{t-j} \\ &= (\sum_{j=0}^{\infty} a_j) \varepsilon_t - \tilde{a}_0 \varepsilon_t + \sum_{j=1}^{\infty} (\tilde{a}_{j-1} - \tilde{a}_j) \varepsilon_{t-j} \\ &= (\sum_{j=0}^{\infty} a_j) \varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t, \end{aligned}$$

which implies that

$$\sum_{t=1}^{n} X_t = (\sum_{j=0}^{\infty} a_j)(\sum_{t=1}^{n} \varepsilon_t) + \tilde{\varepsilon}_0 - \tilde{\varepsilon}_n. \tag{7}$$

Since $(\sum_{j=0}^{\infty} a_j)\varepsilon_t's$ are LPQD by Newman's central limit theorem for LPQD processes (see Theorem 12 of [9]) we have

$$\left(\sum_{j=0}^{\infty} a_j\right)\left(n^{-\frac{1}{2}} \sum_{t=1}^{n} \varepsilon_t\right) \xrightarrow{D} N\left(0,\ \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2\right), \tag{8}$$

where $\xrightarrow{D}$ indicates convergence in distribution.

Note that $\left(\sum_{j=0}^{\infty} a_j\right)^2$ is finite by the assumption $\sum_{j=0}^{\infty} |a_j| < \infty$.

Thus if

$$\widetilde{\varepsilon}_0 / \sqrt{n} \xrightarrow{P} 0 \text{ as } n \to \infty \tag{9}$$

and

$$\widetilde{\varepsilon}_n / \sqrt{n} \xrightarrow{P} 0 \text{ as } n \to \infty \tag{10}$$

hold then

$$\sum_{t=1}^{n} X_t \xrightarrow{D} N\left(0,\ \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2\right) \tag{11}$$

will follow.

To prove (9) and (10), it suffices to prove that

$$\widetilde{\varepsilon}_0 / \sqrt{n} \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty \tag{12}$$

and

$$\widetilde{\varepsilon}_n / \sqrt{n} \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty. \tag{13}$$

But (12) and (13) follow from $E(\widetilde{\varepsilon}_n)^2 < \infty$ (see Remark) and the fact that for any $\delta > 0$

$$\sum_{n=1}^{\infty} P(|\widetilde{\varepsilon}_n| / \sqrt{n} > \delta) = \sum_{n=1}^{\infty} P(|\widetilde{\varepsilon}_0| > \sqrt{n}\,\delta) < \infty,$$

because for an arbitrary random variable Z

$$EZ^2 < \infty \Leftrightarrow \sum_{n=1}^{\infty} P(|Z| > \sqrt{n}\,\delta) < \infty.$$

Remark.

$$\begin{aligned}
E(\widetilde{\varepsilon}_t)^2 &= E\left(\sum_{j=0}^{\infty} \widetilde{a}_j \varepsilon_{t-j}\right)^2 \\
&= \sum_{j=0}^{\infty} \widetilde{a}_j^2\, E\varepsilon_{t-j}^2 + \sum_{k=0}^{\infty} \sum_{l=0,\ k\neq l}^{\infty} \widetilde{a}_k \widetilde{a}_l E(\varepsilon_{t-k}\varepsilon_{t-l}) \\
&\leq \left(\sum_{j=0}^{\infty} \widetilde{a}_j^2\right) E\varepsilon_1^2 + \sum_{k=0}^{\infty} \sum_{l=0\ (k\neq l)}^{\infty} |\widetilde{a}_k||\widetilde{a}_l| E(\varepsilon_{t-k}\varepsilon_{t-l}) \\
&\leq \left(\sum_{j=0}^{\infty} \widetilde{a}_j^2\right) E\varepsilon_1^2 + 2\sum\sum_{k<l} |\widetilde{a}_k||\widetilde{a}_l| \left(\sum_{t=2}^{\infty} E(\varepsilon_1 \varepsilon_t)\right)
\end{aligned}$$

$$\le \left(\sum_{j=0}^{\infty} \tilde{a}_j^2 \right) (E\varepsilon_1^2 + 2\sum_{t=2}^{\infty} E\varepsilon_1\varepsilon_t) < \infty \quad \text{by (3).}$$

Proof of Theorem 2. From (4) and (7)

$$\xi_n(u) = n^{-\frac{1}{2}} \tau^{-1} \sum_{t=1}^{[nu]} X_t$$

$$= n^{-\frac{1}{2}} \tau^{-1} \left(\sum_{j=0}^{\infty} a_j \right) \sum_{t=1}^{[nu]} \varepsilon_t + n^{-\frac{1}{2}} \tau^{-1} \tilde{\varepsilon}_0 - n^{-\frac{1}{2}} \tau^{-1} \tilde{\varepsilon}_{[nu]}$$

(14)
follows.

First note that $(\sum_{j=0}^{\infty} a_j)\,\varepsilon_t$'s are LPQD since ε_t's are LPQD.

According to (12) and (13) the second term and the third term in the right-hand side of (14) converge in probability to zero, i.e,

$$n^{-\frac{1}{2}} \tau^{-1} \tilde{\varepsilon}_{[nu]} \xrightarrow{P} 0 \tag{15}$$

and

$$n^{-\frac{1}{2}} \tau^{-1} \tilde{\varepsilon}_0 \xrightarrow{P} 0. \tag{16}$$

Hence by Theorem 4.2 of Billingsley (1968) it remains to prove

$$n^{-\frac{1}{2}} \tau^{-1} \left(\sum_{j=0}^{\infty} a_j \right) \sum_{t=1}^{[nu]} \varepsilon_t \xrightarrow{D} W, \tag{17}$$

where W is a Wiener measure on $D[0,1]$ the space of all functions on $[0,1]$ which have left limits and are continuous from the right.
Let

$$\tilde{X}_t = \sum_{j=0}^{\infty} a_j \varepsilon_t \text{ and } \quad \tilde{S}_n = \sum_{t=1}^{n} \tilde{X}_t.$$

Then

$\{\tilde{X}_t\}$ is a stationary LPQD process and from (5) and (6)

$$\sum_{t=n+1}^{\infty} E(\tilde{X}_1 \tilde{X}_t) = \left(\sum_{j=0}^{\infty} a_j \right)^2 \sum_{t=n+1}^{\infty} E(\varepsilon_1 \varepsilon_t)$$

$$= O(n^{-\rho}) \text{ for some } \rho > 0 \tag{18}$$

and for some $s > 2$,

$$E|\tilde{X}_t|^s = E\left| \left(\sum_{j=0}^{\infty} a_j \right) \varepsilon_t \right|^s = \left| \sum_{j=0}^{\infty} a_j \right|^s E|\varepsilon_t|^s$$

$$\le \left(\sum_{j=0}^{\infty} |a_j| \right)^s E|\varepsilon_t|^s \tag{19}$$

follow by assumption $\sum |a_j| < \infty$.

Thus it follows from (18) and (19) that $\{\widetilde{X}_t\}$ satisfies conditions of Corollary 2 of Birkel (1993). This implies that Theorem 2 holds for the sequence $\{\widetilde{\xi}_n\}$ according to Corollary 2 of Birkel(1993), where we define $\widetilde{\xi}_n$ as in (4), but $\hat{S}_{[nu]}$ replacing by $S_{[nu]}$, that is, (17) holds.
The following remark is Corollary 2 of Birkel(1993) :

Remark. Let $\{\varepsilon_t,\ t\geq 1\}$ be a stationary LPQD sequence with $E\varepsilon_t=0,\ 0<\varepsilon_t^2<\infty$. Assume that (5) and (6) hold. Then $\{\varepsilon_t,\ t\geq 1\}$ fulfills the functional central limit theorem.

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