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Optimum Strategies in Discrete Red & Black

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Abstract

In discrete red and black, you can stake any amount s in your possession, but the value of s takes positive integer value. Suppose your goal is N and your current fortune is f, with $0 < f < N$. You win back your stake and as much more with probability p and lose your stake with probability, $q = 1 - p$. In this paper, we consider optimum strategies for this game with the value of p less than ½ where the house has the advantage over the player, and with the value of p greater than ½ where the player has the advantage over the house. The optimum strategy at any f when $p < ½$ is to play boldly, which is to bet as much as you can. The optimum strategy when $p > ½$ is to bet 1 all the time.

Keywords : stochastic process, bold play, timid play, gambler's ruin

I . Introduction

In a game called red and black, you can stake any amount s in your possession. Suppose your goal is 1 and your current fortune is f, $0 < f < 1$. You win back your stake and as much more with probability p and lose your stake with probability $q(=1-p)$. This problem was first considered by Coolidge (1909), and the optimum strategy when $p < ½$ was presented by Dubins and Savage (1965). They showed that the bold play is optimal when $p < ½$, and provided the basic idea of proving this theorem. Ahn (2000) considered the optimum strategy when $p > ½$.

In this paper we consider the discrete type of red and black. In discrete red and black, your

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fortune, stake, and goal are all integers. Now, your goal is N and your current fortune is f, $0<f<N$. You can stake any amount s in your possession. There is a lot of similarity between continuous and discrete cases. We can easily see that the optimum strategy in discrete case when $p<½$ is also to play boldly as in continuous case, which is to bet as much as you can. Some of the ideas of continuous case can be adopted to explain the phenomena of discrete case. We will consider an optimum strategy at any f when $p<½$ in section 2, and where $p>½$ in section 3. In section 4, we make a conclusion and present some of the stochastic problems related to this paper which needs to be studied in the future.

II. Optimum Strategy with $p<½$

This is the case where the house has the advantage over the player. The first strategy to consider is to bet a small amount each time. But, this is a bad strategy if we compute the gambler's ruin probability as it is shown in Ahn (2000). The second strategy is to bet as much as you can. We will call this a bold strategy since you bet you entire fortune f or enough to reach N whichever is least. A bet function $S(f)$ under bold strategy can be written as

$$\begin{aligned} S(f) &= f \qquad f \le [N/2], \\ &= N-f \qquad f > [N/2]. \end{aligned}$$

where $[N/2]$ denotes the maximum integers not exceeding $N/2$.

Theorem 1. The bold strategy at f is optimal for $p<½$

The proof of the theorem 1 is essentially the same as one in continuous case (See Dubins and Savage, 1965, and Ahn, 2000). To prove the Theorem 1 we will first define a function $Q(f)$ to denote a probability of reaching N at any f between 0 and N under the bold strategy. $Q(f)$ is continuous, and non-decreasing. And, $Q(0)=1$, and $Q(N)=1$.

If we derive $Q(f)$ more generally

i) $f \le [N/2]$: Bet f:

$$Q(f)=p \cdot Q(2f)+q \cdot Q(0)=p+Q(2f). \tag{1}$$

ii) $f>[N/2]$: Bet $N-f$:

$$Q(f)=p \cdot Q(N)+q \cdot Q(2f-N)=p+q \cdot Q(2f-N). \tag{2}$$

We now consider a strategy such that you bet s at first and then play boldly. Then, the probability of reaching N under this strategy will be $p \cdot Q(f+s)+q \cdot Q(f-s)$. To prove theorem 1 is equivalent to show that

$$p \cdot Q(f+s)+q \cdot Q(f-s) \leq Q(f), \tag{3}$$

for all integers f and s between 0 and N.

Dubins and Savage(1965) proved (3) in continuous case with the goal, 1 and f and s taking any values between 0 and 1. They showed that it suffices to establish (3) for binary rational values of f and s, that is, for numbers of the form $K \cdot 2^{-n}$ where K and n, non-negative integers and $K \cdot 2^{-n} \leqq 1$. We can now see that the proof of (3) in discrete case is merely a subset of that in continuous case. In continuous case the fortune f and the stake s can take any real numbers between 0 and 1. On the other hand the f and s in discrete case can take only integer values between 0 and N. Now, we will divide the values of f and s by N so that the goal becomes 1. Then the fortune space in discrete case will be $D_f = \{0, 1/N, 2/N, 3/N, \ldots, 1\}$. Since the fortune space C_f in continuous case is any real numbers between 0 and 1, D_f will be a subset of C_f. Therefore the proof in discrete case can be substituted with that in continuous case which is given in Dubins and Savage(1965) and Ahn (2000).

Ⅲ. Optimum Strategy with $p > ½$

Theorem 2. The play such that you always bet 1 is optimal for the discrete case with $p > ½$.

Proof: The play such that you always bet 1 will be called timid play. Let a function $Q(f)$ denote the probability of reaching N using a timid play $Q(0)=0$, and $Q(N)=1$. In general, $Q(f)$ under timid play can be written as follows;

$$\begin{aligned} Q(f) &= p \cdot Q(2) && \text{if } f = 1, \\ &= q \cdot Q(f-1) + p \cdot Q(f+1) && \text{if } 2 \leq f \leq N-2, \\ &= q \cdot Q(N-2) + p && \text{if } f = N-1. \end{aligned}$$

According to Dubins and Savage (1965), it suffices to show that

$$p \cdot Q(f+s)+q \cdot Q(f-s) \leq Q(f) \text{ for all } f \text{ and } s, \tag{4}$$

where $1 \leq s \leq \min(f, N-s)$, s is a positive integer.

By the gambler's ruin probability (See Parzen, 1962, p233), the probability that f will go to 0 can be written as

$$\frac{(\frac{q}{p})^f-(\frac{q}{p})^N}{1-(\frac{q}{p})^N}, \quad p \neq q.$$

And, the probability of reaching N is,

$$Q(f) = 1 - \frac{(\frac{q}{p})^f-(\frac{q}{p})^N}{1-(\frac{q}{p})^N}$$

$$= \frac{1-(\frac{q}{p})^f}{1-(\frac{q}{p})^N}, \quad 1-(\frac{q}{p})^N > 0 \ . \tag{5}$$

Now, we consider a convex function $f(x) = x^s$. Let's choose three points A, B, and C on the convex function $f(x) = x^s$. The x value of point B is 1 and those of points A and C are $\frac{q}{p}$ and $\frac{p}{q}$, respectively. Since $p > 1/2$, $\frac{q}{p} < 1 < \frac{p}{q}$. Thus the point C is between A and B. We can now see that since $f(x)$ is a convex function

$$q\,(\frac{p}{q})^s + p(\frac{q}{p})^s \geq 1. \tag{6}$$

Note that the left hand side of (4) is a linear combination of two points A and B. Let's rewrite (5) by switching p and q of the first term of the left hand side.

$$q\,(\frac{q}{p})^{-s} + p(\frac{q}{p})^s \geq 1 \tag{7}$$

Multiplying the both sides of (6) by $-(\frac{q}{p})^f$ we get the following.

$$-q\,(\frac{q}{p})^{f-s} - p(\frac{q}{p})^{f+s} \leq -(\frac{q}{p})^f. \tag{8}$$

Adding $p + q$ to the left hand side and 1 to the right hand side we get,

$$q\,[\,1 - (\frac{q}{p})^{f-s}\,] + p\,[\,1 - (\frac{q}{p})^{f+s}\,] \leq 1 - (\frac{q}{p})^f\,. \tag{9}$$

Finally we divide the both sides of (9) by $1-(\frac{q}{p})^N$ which is positive.

$$q\,\frac{1-\left(\frac{q}{p}\right)^{f-s}}{1-\left(\frac{q}{p}\right)^{N}}+p\,\frac{1-\left(\frac{q}{p}\right)^{f+s}}{1-\left(\frac{q}{p}\right)^{N}}\le\frac{1-\left(\frac{q}{p}\right)^{f}}{1-\left(\frac{q}{p}\right)^{N}}.$$

Thus, we get the final inequality

$$p\cdot Q(f+s)+q\cdot Q(f-s)\le Q(f).$$

Q.E.D.

Ⅳ. Conclusion

The optimum strategy in discrete red and black is also to play boldly, thus to bet as much as you can when $p<½$, which is the same result as in continuous case. The optimum strategy when $p>½$ is to bet one all the time. The method of proving the Theorem 2 was to use the idea of convex function. But, there may be another efficient method. One of the next research problem may be to obtain the number of wins and loses or the number of the total bets before we reach the goal. This problem can also be related to many other problems in stochastic processes or the Markov random field.

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