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DESIGNING AN INTEGRATED STOCKING AND TIE-IN PROMOTION POLICY*

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ABSTRACT

Assume a company sells two products (A and B) in a retail market. The company adopts a specific promotion mechanism, Tie-in Promotion, in which product A's promotional discount coupon is distributed whenever a consumer purchases product B. Product A will later be sold at a markdown price when consumers eventually take the opportunity to redeem the coupon. In the integrated tie-in promotion and stocking policy, we assume managers of two products coordinate by sharing information on the demand forecast and deciding the order quantities and tie-in promotion program to maximize joint profits. The optimal integrated tie-in policy is analyzed. The integrated tie-in promotion model is then compared with two other base models: (1) a decentralized Newsboy model in which no promotion is considered, and (2) an individual promotion model in which managers design a promotion program to promote one of the two products directly. The factors that make an integrated tie-in promotion a better approach are studied.

1. INTRODUCTION

In this paper we consider an integrated tie-in promotion and stocking policy that links two products in a retail market. What are tie-in promotions? For example, in 1994, CoreStates Financial Corp. had a tie-in arrangement with Disney's Premier Cruise Lines. Customers who opened an account with the bank received free cruise tickets for two children when they booked a family cruise. The promotion motivated 5,000 customers to redeem their travel vouchers and yielded 42,000 new accounts (see Tellis [1998]). In another example, in 1983, Polaroid teamed up

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with Delta Airlines to increase sales of Polaroid Series 600 cameras. The "Buy Polaroid 600, Fly Delta Free" promotion generated the biggest Christmas sell-through for Polaroid in more than 10 years. About 33% (300,000) of the coupons were redeemed. Hundreds of thousands of these persons flew Delta (see Shimp and Delozier [1986]).

The major difference between a tie-in promotion and other consumer promotion programs (e.g., premium, price packs, rebates, etc.) is that the former is run by two cooperative partners, while the latter are mostly implemented by an individual party. Therefore, a tie-in promotion inherently requires two parties' integrated viewpoints and coordination to work well. The issue of coordination in the supply chain (or within decision-making facilities) has recently drawn much attention. Parlar and Weng (1997) describe coordination between a firm's manufacturing and supply departments; Kouvelis and Gutierrez (1997) study an internationally coordinated news-vendor problem; Weng (1995, 1997) analyzes channel coordination in pricing and stocking operations; and Lee (2001) considered a coordinated supply chain pricing, stocking, and return policy. Kohli and Park (1994), Dolan (1990), Rubin and Carter (1990), Joglekar (1988), Banerjee (1986), Lee and Rosenblatt (1986), and Monahans (1984) have considered synchronization of supply chain cycle time by providing a quantity discount. (See Thomas and Griffin [1996] for a review of supply chain coordination issues.) These models show that regardless of its simplicity, decentralized optimization can sometimes provide less than desirable consequences, show how a member's excessive strength can be transferred to the other party through a joint effort, and create a synergy effect to jointly optimize mutual benefit. The tie-in promotion program that we will study here fits perfectly into this kind of cooperative spirit.

In this paper, an integrated tie-in promotion and stocking policy is designed. We study the factors that make an integrated tie-in promotion a better approach by comparing the notion with two other models: (1) a decentralized Newsboy model in which no promotion is considered, and each product manager individually optimizes his own profit function; and (2) an individual promotion model in which one product manager designs a promotion program to promote his own product directly. Here again, each party individually optimizes his profit function. This paper is structured as follows. In § 2, a problem description, assumptions, and notations are presented and the objective functions of the tie-in promotion program are developed. In § 3, we develop and study the optimal integrated tie-in promotion model. In § 4, we provide a numerical analysis to show how and when integrated tie-in promotion helps to increase profits. Finally, a brief discussion is provided in § 5.

2. PROBLEM DESCRIPTION, ASSUMPTIONS, AND MODELS

In this section, we first discuss the setting for the problem and assumptions in the model. The model developed in this study represents an abstraction of sales promotion and inventory control as considered in its real-world application. Our purpose is to enhance understanding and to gain managerial insights into the effect of joint optimization in supply chain sales promotion and inventory control policies. Therefore, we emphasize developing and analyzing a relatively complete and general (but simple) set of premises that may hopefully capture the most essential aspects of this important issue. Future research should overcome any theoretical and empirical limitations of our model.

We consider two companies sells two products (A and B) in a retail market. The company adopts a specific promotion mechanism --Tie-in Promotion-- in which product A's promotional discount coupon is distributed whenever a consumer purchases product B. Product A will later be sold at a markdown price when consumers eventually take the opportunity to redeem the coupon. The customers consuming product B are divided into two exclusive sets: (a) Consumers who initially (before presentation of a tie-in promotion) intend to consume product B (type O (old)); and (b) consumers who initially do not intend to consume product B, but are later persuaded by the tie-in promotion to do so (type N (new)). Similarly, type O consumers are divided into two mutually exclusive consumer sets: (a.1) Those who are later persuaded by the tie-in promotion to consume both products (type OY), and (a.2) those who are later not persuaded by the tie-in promotion and consume only product B (type ON). Finally, type N consumers *are divided* into two mutually exclusive subsets: (b.1) those who consume both products (NY), and (b.2) those who consume only product B (NN) (see Figure 1)

During the tie-in promotion, the demand for the product B is assumed to be uncertain and dependent on the product A's markdown percentage. Standard approaches introduced a convenient way to model the price-dependent random demand. Specifically, the demand function is characterized by the two components. The first component, representing the mean or deterministic part of the random demand, is influenced by the price, whereas the second component, representing uncertainty or the shape of the random demand, is price independent (see, for example, Leland (1972), Young (1978), Lau and Lau (1988), Porteus (1990), and Emmons and Gilbert (1998)). Leland (1972), for example, has considered two price-dependent random demand functions--multiplicative and additive. The multiplicative model formulates random demand $d = h(P)\lambda$ as the product of a deterministic component $h(P)$ and a probabilistic component λ with

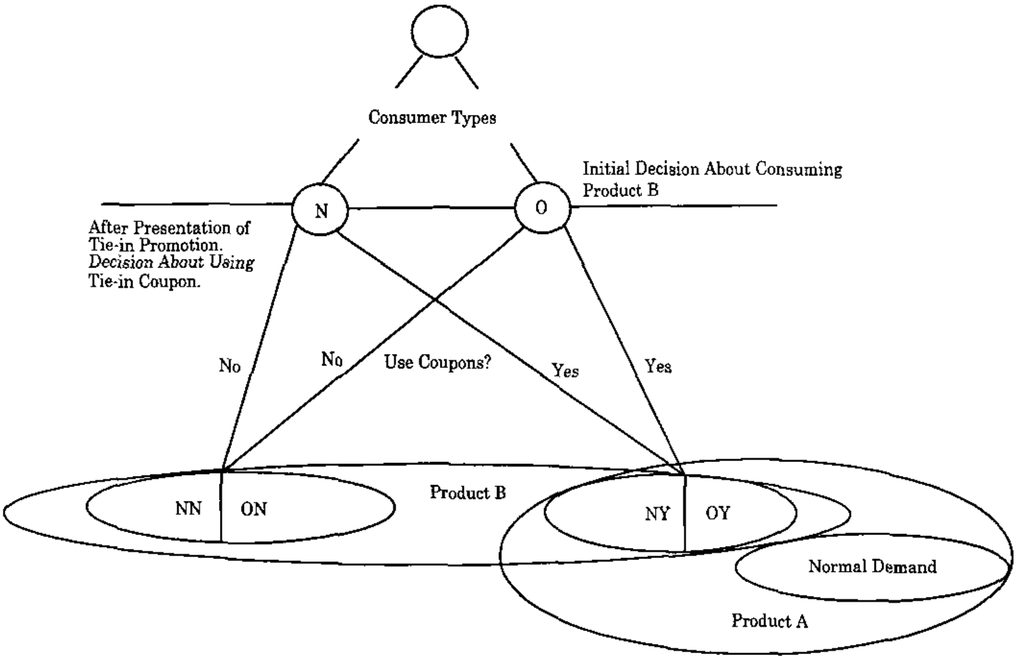

Figure 1. Consumer Types

$E(\lambda)$ =1. Here, $h(P)$, a decreasing function of price P, denotes the mean of the random demand, and λ denotes the uncertainty of the random demand. It implies that the variance of the random demand increases as demand increases (price decreases). This assumption is supported by Lau and Lau (1988), in which they *argue that for a low price (high demand) level beyond the normal operating range*, the random demand may have a large variance due to lack of past experience to draw on. The second demand function considered in Leland (1972) takes an additive form $d = h(P) + \lambda$ and $E(\lambda) = 0$. When demand is additively separable, the variance of random demand is constant across all possible prices. This assumption may be overly simplistic and not practical. In this work we use the first approach--a general multiplicative demand function--to formulate the random demand. Let α denote product A's promotional markdown price ratio (we use *markdown price ratio* to distinguish it from *markdown rate* $\bar{\alpha} = 1 - \alpha$). For example, if product *A* with a list price of \$10 is priced at 35% ($\bar{\alpha} = 0.35$) off the original face value, when consumers redeem the coupon, then α=0.65. Let $\eta(\alpha)$ denote product *B*'s expected demand as a function of α. Previous research suggests that, in general, promotion produces a short-term increase in sales, but at a diminishing rate (see, for example, Tellis [1998]); therefore, we assume $\eta(\alpha)$ to be concave, decreasing in α, i.e., $\partial\eta(\alpha)/\partial\alpha \le 0$ and $\partial^2\eta(\alpha)/\partial\alpha^2 \le 0$ for $\alpha \le 1$. Specifically, the realized random demand of product *B* in a tie-in promotion period is formulated as a product of mean of the random demand $\eta(\alpha)$ and a probabilistic component X with $E(X) = 1$. We let $g(x)$ denote the probability density of X.

Now let us formulate the product *A*'s random demand as follows. Product *A*'s total sales volume consists of two components: *Normal Selling* and *Coupon Redemption* (see Figure 1). The consumers who do not receive tie-in promotion discounts generate normal selling volume. That is, they consume only product *A*. We let $f(y)$ denote the probability density of product *A*'s *normal selling* demand, which is known but does not assume any specific form, and let μ denote expected *normal selling demand*. Let $\Phi(\alpha|\eta)$ denote the expected coupon redemption demand, which amounts to $\Phi(\alpha|\eta) = E_{OY}(\alpha) + E_{NY}(\alpha)$ (Table 1 summarizes expected demands of various customer types). We assume $\Phi(1|\eta) = 0$, $\partial\Phi(\alpha|\eta)/\partial\alpha \le 0$ and $\partial^2(\alpha\Phi(\alpha|\eta))/\partial\alpha^2 \le 0$. Now, under the given premises, we see that the benefits of tie-in are for (1) Product *B*--increased sales generated by type *NN* consumers, and for (2) Product *A*--increased sales generated by type *NY* and *OY* consumers.

Table 1. Expected Demands for Various Consumer T+ypes

$$E_O + E_N(\alpha) = \eta(\alpha) \qquad E_O = E_{OY}(\alpha) + E_{ON}(\alpha) = \eta(1)$$
$$E_N(\alpha) = E_{NY}(\alpha) + E_{NN}(\alpha) = \eta(\alpha) - \eta(1)$$

E_X Denotes expected demand of consumer type x, for example, E_o represent expected demand of consumer type o.

A Newsboy-type setting is used for formulating the problem (See Heyman and Sobel [1990, Ch. 12] for an excellent review of the Newsboy problem.). The parameters of the model are given as follows. Let N, n be the manufacturing costs per unit, Q,q the order quantities, D,d the selling prices, and E,e the emergency backorder manufacturing costs for products A and B, respectively. We assume that the unfilled demand is backordered and met from the emergency production run. For product A, this includes not only the unfilled demands from *normal sales*, but also the unfilled demands from the *coupon redemption* (consumer types NY and OY). Finally, we assume $D > E > N$ and $d > e > n$. Of course, the above managerial scenario is one of many possible scenarios in the real world. Again, we focus our attention on studying the tie-in alliance issue by analyzing a relatively complete and general but simple model.

Let us now consider the coordinated tie-in promotion model. The company initially produces q units of product B. If the random demand $(x\eta)$ during the selling season exceeds the availability (q), an emergency production session is initiated, and the profit equals $dx\eta - nq - e(x\eta - q)$. On the other hand, if $x\eta < q$, the resulting profit is computed by subtracting the cost of goods sold nq from the sales revenue $dx\eta$. Dropping arguments x and y for $g(x)$ and $f(y)$, the expected profit functions for the product B is given by the following expressions.

$$\Pi(q,\alpha)_B = d\eta(\alpha) \; - nq - \int_{\frac{q}{\eta}}^{\infty} e((x\eta(\alpha)) - q)dG \tag{1}$$

The product A's expected profit can be obtained in a similar fashion. The company initially produces Q units of product A. If the normal selling demand $(y + x\Phi)$ exceeds the availability (Q), an emergency production session is initiated, and the profit equals $D(y + \alpha x\Phi) - NQ - E(y + x\Phi - Q)$. On the other hand, if $y + x\Phi \le Q$, the resulting profit is computed by subtracting the cost of goods sold NQ from the sales revenue $D(y + \alpha x\Phi)$. The expected profit functions for the product B is given by the following expressions.

$$\begin{aligned}\Pi(Q,\alpha)_A = D(\mu + \alpha\Phi(\alpha|\eta)) \; -NQ - \int_0^Q \int_{\frac{Q-y}{\Phi}}^{\infty} E(x\Phi(\alpha|\eta) - Q + y)\,dGdF \\ - \int_Q^{\infty}\int_0^{\infty} E(x\Phi(\alpha|\eta) - Q + y)\,dGdF \end{aligned} \tag{2}$$

In the next section we will study the integrated tie-in promotion model first, in which we assume that the managers for the two products coordinate, that is, share information on the demand forecast and jointly decide on order quantities and a promotion program so as to maximize mutual benefits.

3. THE OPTIMAL INTEGRATED TIE-IN PROMOTION POLICY

We first analyze the optimal manufacturing policy for the expected profit function (1). It can be shown that the necessary condition for product *B's* manufacturing quantity is given by the following expression:

$$q^*(\alpha) = \eta(\alpha) G^{-1}\left(1 - \frac{n}{e}\right) \quad (3)$$

We see that $q^*(\alpha)$ increases as the markdown rate $\overline{\alpha}$ increases and the ratio n/e decreases. These results all make intuitive sense. Now, substituting the optimal order quantity $q^*(\alpha)$ into the expected profit function (1) reveals that

$$\Pi(q^*(\alpha))_B = \eta(\alpha)\left[d - e\int_{G^{-1}(\xi)}^{\infty} x dG\right],$$

where $\xi = 1 - n/e$. We see that $\Pi(q^*(\alpha))_B$ is concave in α. Adding expected profit functions (1) and (2) reveals that the joint optimization problem equals:

$$\max_{Q,\alpha} \Pi(Q, \alpha \,|\, q^*(\alpha))_J = \Pi(q^*(\alpha))_B + \Pi(Q, \alpha)_A \quad (4)$$

We now may analyze the optimality conditions for equation (4). Let $\omega(Q,\alpha) = (Q - y)/\Phi(\alpha \,|\, \eta)$. It can be shown that the necessary condition for the markdown rate that optimizes (4) satisfies equation (5) (see Appendix 1):

$D\Phi(\alpha \,|\, \eta) = \Delta$, where

$$\Delta(\alpha) = -\frac{\partial \Phi(\alpha \,|\, \eta)}{\partial \alpha}\left(D\alpha - E\left(1 - \int_0^Q \int_0^{\omega(Q,\alpha)} x dG dF\right)\right) - \frac{\partial \eta(\alpha)}{\partial \alpha}\left(d - e\int_{G^{-1}(\xi)}^{\infty} x dG\right) \quad (5)$$

Here, $-\Delta(\alpha)$ represents the cost penalty from increasing α by a unit. For example, the first term represents the marginal profit loss net of emergency production cost savings from decreasing the redemption demand (by decreasing $\overline{\alpha}$).

We now look at the manufacturing quantity that maximizes the expected joint profit function (4). It can be shown that the necessary condition for Q that maximizes (4) at a given α satisfies equation (6) (see Appendix 2):

$$1 - \frac{N}{E} = \int_0^Q G(\omega(Q,\alpha)) dF \quad (6)$$

As in the case for q^*, Q^* also increases as markdown rate $\bar{\alpha}$ increases and ratio N/E decreases. Proposition 1 provides us with the properties of the optimal solutions.

Proposition 1. (Proof. See Appendix 3.)

(a) $\partial Q/\partial\alpha|_{\Pi(2)} \leq 0$, where $\Pi(2) := \partial\Pi(Q,\alpha \mid q^*)_J / \partial Q = 0$, and $\partial\alpha/\partial Q|_{\Pi(1)} \leq 0$, where $\Pi(1) := \partial\Pi(Q,\alpha \mid q^*)_J / \partial\alpha = 0$,

(b) The objective function $\Pi(Q,\alpha \mid q^*)_J$ is strictly concave; therefore, it has a unique optimal solution.

(c) Let α_C be the unit elastic point such that $(\Phi + \alpha\, \partial\Phi/\partial\alpha)|_{\alpha=\alpha_C} = 0$. The optimal markdown price ratio α^* decreasing (increasing) in D if $\alpha^* > \alpha_C$ ($\alpha^* < \alpha_C$). That is, for elastic markdown price ratios ($\alpha^* > \alpha_C$), markdown rate $\bar{\alpha}^*$ increases as D increases, and vice versa.

The optimal markdown rate $\bar{\alpha}^*$:

(c.1) Increasing in d (c.2) Decreasing in N

(c.3) Decreasing in n □

Proposition 1.a tells us that the optimal markdown rate $\bar{\alpha}$ increases as Q increases. A little reflection leads us to expect this result since the greater the order quantity, the more likely leftovers that will be generated and more discounts will be needed to stimulate higher redemption demand. We also can obtain this result by observing condition (5). The term $-\partial\Phi/\partial\alpha\, E(\bullet)$ in equation (5) represents product A's savings of emergency production costs as a result of increasing α. Equation (5) reveals that α increases as E increases or Q decreases. According to this result, when product A's emergency production cost is relatively high or manufacturing quantity is relatively small, there is a relatively high possibility of getting into a high-cost emergency production session. If so, it is more rational to decrease the promotional demand (by decreasing $\bar{\alpha}^*$) so as to avoid expensive emergency production.

To obtain the optimal (Q^*, α^*), we need to solve equations (5) and (6) simultaneously. Unfortunately, neither equation provides closed form solutions. However, it is possible to do so numerically. Proposition 1(a) reveals that the optimal (Q^*, α^*) are negatively related; therefore, the lower bound of the simultaneous solution of order quantity $Q^*(\alpha^*)$ must be given by the upper bound of $\alpha^*(Q^*)$.

The necessary condition (5) reveals that the α^* must be less than the α_{UB} that satisfies $\Phi+(\alpha-E/D)\partial\Phi/\partial\alpha=0$; therefore, a reasonable upper bound of $\alpha^*(Q^*)$ is α_{UB}. Now, the optimal solution can be obtained from an iterative procedure that utilizes a simple approach that begins by substituting α_{UB} into equation (6) to obtain Q_{LB} (lower bound of $Q^*(\alpha^*)$), followed by a straightforward iterative procedure. Since both equations give unique solutions and since the bounds for the solutions are well defined, the iterative procedure must give a unique simultaneous solution for each variable.

Let us now furnish a numerical analysis to study effects of two parameters' ($\partial\Phi/\partial\overline{\alpha}$ and $\partial\eta/\partial\overline{\alpha}$) variations on the optimal solutions (these parameters' variations were not discussed in Proposition 1). The following assumptions are used: Let $g(x)$ and $f(y)$ be uniformly distributed over [0,2] and $[0,2\mu]$. Although our choice of uniform distribution is somewhat arbitrary, it is easy to visualize the problem. The uniform distribution, because of its simplicity also has been used extensively by Emmons and Gilbert (1998) in their recent work of supply chain return policy. To keep things as simple as possible, we further assume that the expected redemption demand for product A and the expected demand for product B are linear, and are, respectively, of forms $\Phi(\alpha|\eta)=\Phi_0(1-\alpha)$, $\eta(\alpha)=\eta_0(1-\alpha)+\alpha\eta_1$ and $\eta_0>\eta_1$. We let $\eta]_{\overline{\alpha}}=\eta_\Delta=\eta_0-\eta_1$. The following base parameters are used in the numerical experiment.

Table 2. Base Parameters Used in the Numerical Experiment

N	n	E	e	D	d	Φ_0	η_0	η_1	$\mu=H_1$	H_0
14	14	24	24	40	40	400	800	700	400	800

$\partial\Phi/\partial\overline{\alpha}=\Phi_0$ and $\partial\eta/\partial\overline{\alpha}=\eta_\Delta$ Variation Figures 2 and Figures 3 summarize two experiments regarding η_Δ (rate of change of $E_N(\alpha)$) and Φ_0 (rate of change of $E_{OY}(\alpha)+E_{NY}(\alpha)$) variations. For Φ_0 variations, we keep other parameters constant and change Φ_0 from 100 to 700 in increments of 100 (that is, 100, 200, ...,700). Notic that here keeping η_Δ as a constant (keeping η_0 and η_1 as a constant) and increasing Φ_0 are equivalent to increasing the coupon redemption rate. Therefore, only product A benefits from the tie-in promotion. Figure 2 gives the ratio between the optimal value and the lowest value in the range of parameter variation. For example, $Q^* = 242$ when $\Phi_0 = 200$, and the lowest $Q^* = 199$ when $\Phi_0 = 100$; therefore, the corresponding value for $\Phi_0 = 200$ equals $242/199 = 1.2$. Figure 2 shows that Q increases while $\bar{\alpha}$ decreases as Φ_0 increases. It tells us that if coupon redemption rate is relatively high, the tie-in alliance could take advantage of this opportunity and then reduce tie-in promotion efforts ($\bar{\alpha}$). Analytically, equation (5) reveals that the optimal solution satisfies $D + \tilde{\Delta}(\alpha) = 0$, where $\tilde{\Delta} = D\alpha(\partial\Phi/\partial\alpha)/\Phi + d(\partial\eta/\partial\alpha)/\Phi$ (for the sake of simplicity, we let $E = e = 0$). Figure 4 illustrates the $\tilde{\Delta}$ shift from I to II as Φ increases (by increasing Φ_0); therefore, α increases from α_I to α_{II}. For η_Δ variation, we keep other parameters constant and change η_1 from 750 to 500 in increments of -50 (that is, 750, 700,...,500). Notice that here, keeping Φ_0 as a constant and increasing η_Δ (decreasing η_1) is equivalent to increasing demand sensitivity of $E_N(\alpha)$ ($\partial E_N(\alpha)/\partial\bar{\alpha} = \eta_\Delta$); therefore, $\bar{\alpha}$ increases (see Figure 4, where α decreases from α_I to α_{III}) to generate more new demand for product B, and coupon redemption demand for product A.

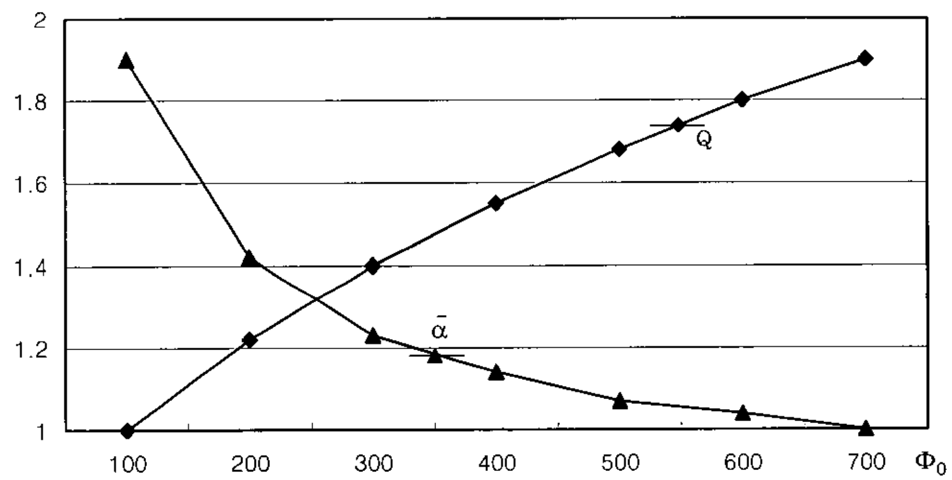

Figure 2. Optimal Policies for Φ_0 Variation

So far we have studied the jointly optimal tie-in policy. In the following sections, we provide a numerical experiment to compare expected profits from Coordinated Tie-in Promotion (hereafter, CP) with two other models -- (1) Individual Promotion Model (IP), and (2) Individual Newsboy Model (NP) -- so as to further enhance our intuition about the model.

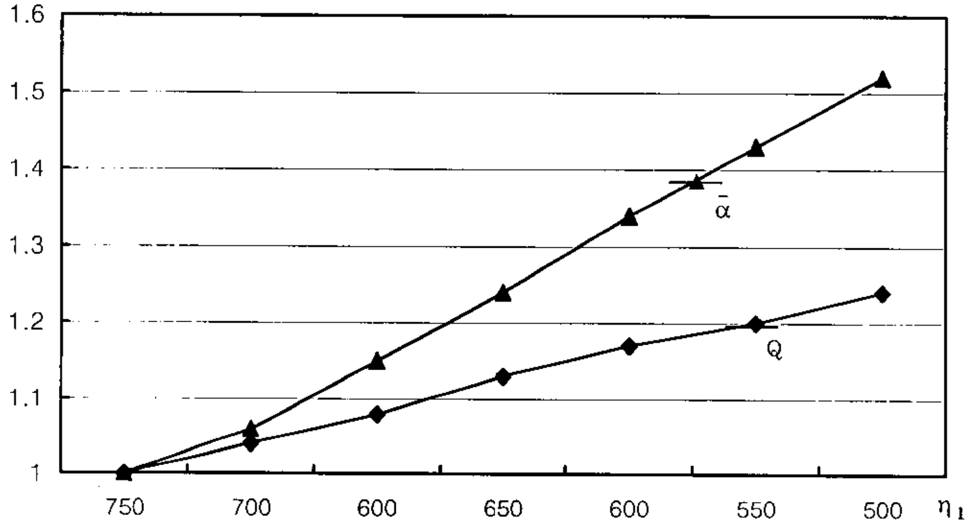

Figure 3. Optimal Policies for η_1 Variation

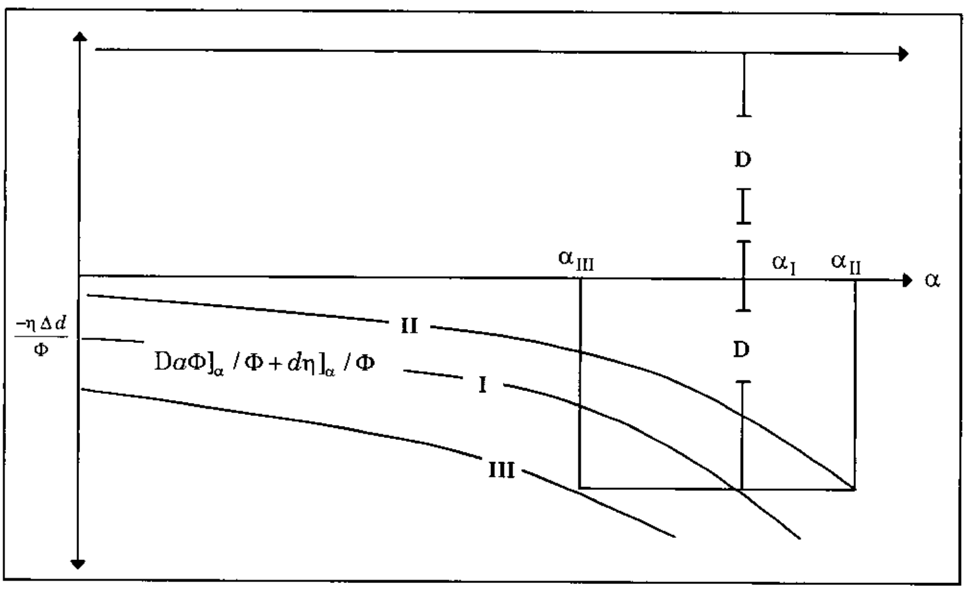

Figure 4. Parameter Variation Effects on α

4. COMPARISONS WITH OTHER MODELS

In this section, we first compare CP with IP in which managers design an individual promotion program to promote product A directly. We assume that the demand density function of product A in IP is a multiplicative type. The realized random demand is $y\mathrm{H}(\alpha)$ with $E(y)=1$. We let $f(y)$ denote the probability density of y. Here, α is product A's IP markdown price ratio, and $\mathrm{H}(\alpha)$ is the expected demand. We assume $\partial\mathrm{H}(\alpha)/\partial\alpha \le 0$ and $\partial^2(\alpha\mathrm{H}(\alpha))/\partial\alpha^2 \le 0$. We next compare CP with a simple Newsboy model (NP) in which no promotion of any kind is considered and the two parties solve simple Newsboy problems to individually design their order quantities. The expected profit functions and optimal policies for NP and IP are provided in Table 3.

Table 3. Optimal Policies

Objective Functions for NP		Optimal Policies
Product B	$\Pi(q)\big\|_{B;IP} = d\eta(1) - nq - \int_q^{\infty} e(x-q)dG$	$q^* = G^{-1}(1-n/e)$
Product A	$\Pi(Q)\big\|_{A;IP} = D\mathrm{H}(1) - NQ - \int_Q^{\infty} E(y-Q)dF$	$Q^* = F^{-1}(1-N/E)$
Objective Functions for IP		**Optimal Policies**
Product B	$\Pi(q)\big\|_{B;IP} = d\eta(1) - nq - \int_q^{\infty} e(x-q)dG$	$q^* = G^{-1}(1-n/e)$
Product A	$\Pi(Q,\alpha)\big\|_{A;IP} = \alpha D\mathrm{H}(\alpha) - NQ - \int_{\frac{Q}{\mathrm{H}(\alpha)}}^{\infty} E(y\mathrm{H}(\alpha)-Q)dF$	$Q^* = \mathrm{H}(\alpha)F^{-1}(1-N/E)$, and $D\{\mathrm{H}(\alpha)+\alpha\mathrm{H}(\alpha)]_\alpha\} = \int_{\frac{Q}{\mathrm{H}(\alpha)}}^{\infty} Ey\mathrm{H}(\alpha)]_\alpha dF$

See Appendix 4 for the derivation of the necessary and Sufficient conditions for IP.

Numerical experiments are now shown in which the expected profits of CP are compared with those of IP and NP. The following assumptions are made for NP, and IP, respectively:

Problem Set IP: The notations used in the following paragraph are based on those in Table 3. Let $g(x)$ and $f(y)$ be uniformly distributed over $[0, 2\eta(1)]$ and $[0, 2]$. We assume that the expected demand for product A is linear, and is of the form $\mathrm{H}(\alpha) = \mathrm{H}_0(1-\alpha)+\alpha\mathrm{H}_1$, $\mathrm{H}_0 > \mathrm{H}_1$. We let $\partial\mathrm{H}/\partial\overline{\alpha} = \mathrm{H}_\Delta = \mathrm{H}_0 - \mathrm{H}_1$, and $\mu = \mathrm{H}(1)$ (we use μ to denote product A's expected normal selling demand in CP). The average demand for product B can be obtained by $\eta(1)$ (defined in CP, equals E_O).

Problem Set NP: Let $g(x)$ and $f(y)$ be uniformly distributed over $[0, 2\eta(1)]$ and $[0, 2\mathrm{H}(1)]$.

We now show the effects of the parameters' variation on the optimal values. We first solve three problems and find the optimum values $(Q_{CP}, \alpha_{CP}, q_{CP})$, $(Q_{IP}, \alpha_{IP}, q_{IP})$, and (Q_{NP}, q_{NP}) for CP, IP and NP, respectively. We then substitute these values into a joint expected profit function and compute the joint profits $\Pi_{J;CP}$, $\Pi_{J;IP}$ and $\Pi_{J;NP}$. Finally, we compute the Percentage Increase Ratios (PIR) in the expected profits: $(\Pi_{J;CP}/\Pi_{J;NP} - 1) \times 100$ (PIR/NP) and $(\Pi_{J;CP} / \Pi_{J;IP} - 1) \times 100$ (PIR/IP). The base parameters summarized in Table 2 are used again here. However, we will change the value of base parameters when it is necessary.

$\partial\eta/\partial\overline{\alpha} = \eta_\Delta$, $\partial\Phi/\partial\overline{\alpha} = \Phi_0$, and $\partial H/\partial\overline{\alpha} = \mathrm{H}_\Delta$ Variation. Figure 5 summarizes three experiments regarding η_Δ (rate of change of $E_N(\alpha)$) and Φ_0 (rate of change of $\Phi(\alpha) = E_{OY} + E_{NY}$) variations. In the first experiment we let $\eta_\Delta = \Phi_0$, and provide $\eta_\Delta = \Phi_0$ variation; the second experiment analyzes η_Δ variation, keeping Φ_0 as a constant; and the third experiment demonstrates Φ_0 variation, keeping η_Δ as a constant. For $\eta_\Delta = \Phi_0$ variation, we keep other parameters constant and change $\eta_\Delta = \Phi_0$ from 100 to 400 in increments of 50 (that is, 100, 150,...,400). Figure 5 reveals that CP payoffs (PIR/NP) increase as $\eta_\Delta = \Phi_0$ increases. Figure 5 also reveals (here, we let $\eta_1 = 700$ and $\eta_0 = 800$) that reducing Φ_0 while keeping η_Δ as a constant will decrease PIR/NP. We see that reducing Φ_0 while keeping η_Δ as a constant is equivalent to reducing the ability of tie-in promotion in attracting redemption demand. We also see from Figure 5 that reducing η_Δ (by increasing η_1) while keeping Φ_0 as a constant is equivalent to reducing the effectiveness of tie-in promotion to allure the type N customers, and consequently reduces PIR/NP (here, we let $\Phi_0 = 400$ and $\eta_0 = 800$).

The numerical experiment tells us that an integrated tie-in promotion should be adopted when $\partial\eta/\partial\overline{\alpha} = \eta_\Delta$ and $\partial\Phi/\partial\overline{\alpha} = \Phi_0$ are both relatively high in their values--that is, when product A's price markdown can successfully attract large numbers of (1) consumer type N (high $\partial\eta/\partial\overline{\alpha} = \eta_\Delta$) and (2) redemption demand (high $\partial\Phi/\partial\overline{\alpha} = \Phi_0$). One such example is two highly related, complementary products such as a Thanksgiving turkey and white wine.

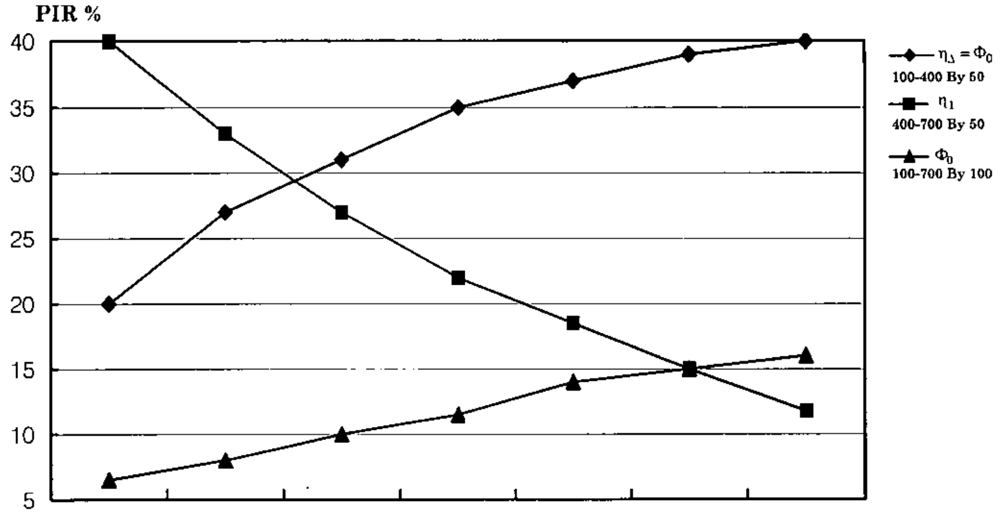

Figure 5. PIR for CP VS NP

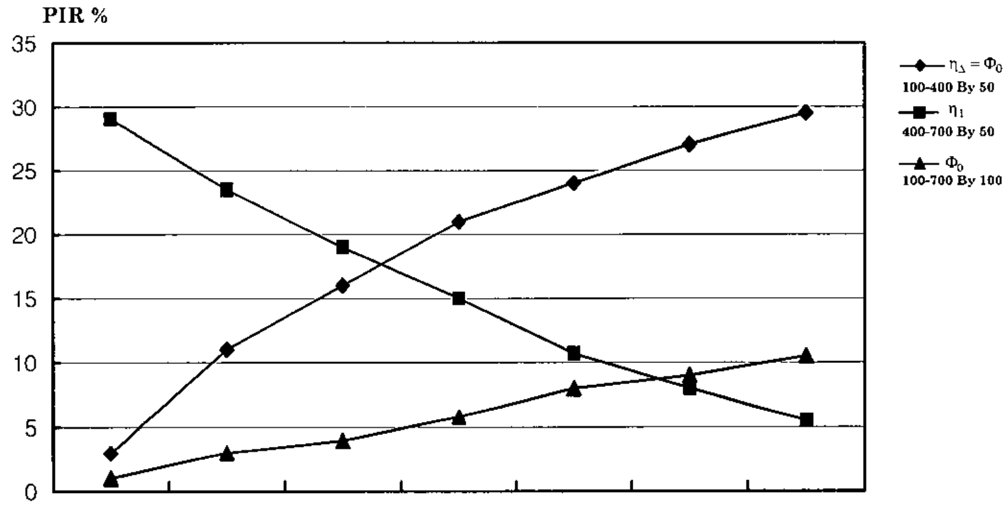

Figure 6. PIR for CP VS IP

***N, n* variation.** For N, n variation, we keep other parameters constant and change N from \$10 to \$22 in increments of \$2 (that is, \$10, \$12, ..., \$22), and n from \$10 to \$22 in increments of \$2. Here, we let $\Phi_0 = 400$, $\eta_1 = 700$, $\eta_0 = 800$, $H_0 = 1000$ and $\mu(= H(1)) = 200$. Figure 7 gives PIR/NP. It reveals that tie-in alliance payoffs are most significant when the business environment is relatively favorable for product A (low N). This result makes intuitive sense. When N is relatively low, promoting product A's sales will no doubt increase the joint profit (Notice that CP promotes both products). However, Figure 8 also tells us that if $\partial \mathrm{H}/\partial \overline{\alpha}$ is relatively high (here, $\mathrm{H}_\Delta = \mathrm{H}_0 - \mathrm{H}_1 = 800$, $\Phi_0 = 400$ and $\eta_\Delta = 100$) and N is extremely low, then we might as well promote product A directly, instead of circumventing as in the tie-in promotion. Let us verify this argument by modifying the parameter values to $\mathrm{H}_0 = 600$ ($\mathrm{H}_\Delta = 400$). Our numerical experiment then shows that PIR/IP increases as N decreases. This result is obvious. The sales increase to product A from CP is more than that of IP in the modified case; therefore, PIR/IP increases as N decreases. This result confirms again the earlier findings that an integrated tie-in promotion should be adopted when $\partial\eta/\partial\overline{\alpha} = \eta_\Delta$ and $\partial\Phi/\partial\overline{\alpha} = \Phi_0$ are both relatively high in their values.

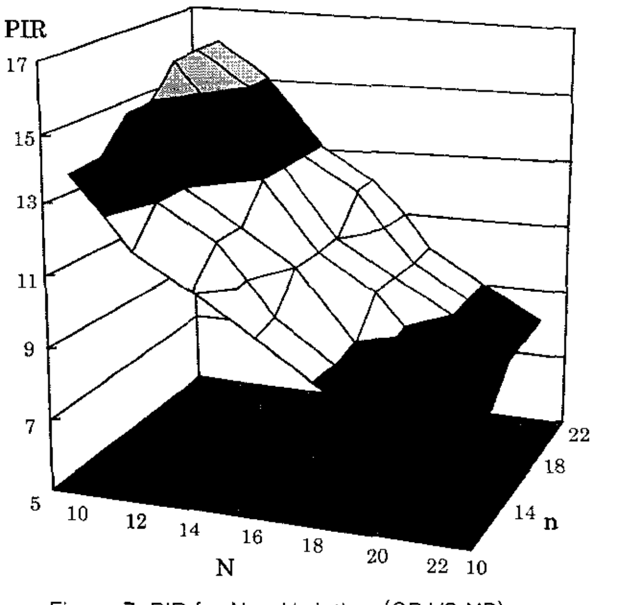

Figure 7. PIR for N, n Variation (CP VS NP)

Figure 8 reveals that PIR/IP shows an increasing trend when n decreases. When n is relatively low, promoting product B will increase the joint profit. We see that CP promotes not only product A, but also product B. Therefore, PIR/IP shows an increasing trend as n decreases. Figure 7 reveals that PIR/NP increases as n increases. The reason for this result can be obtained from Figure 9. Figure 9 illustrates $q_{CP} - q_{NP}$ for a specific case in which $N = 10$. We vary n from 10 to 22. We observe that q_{CP} (through the process of joint optimization) shows a steeper decreasing trend than q_{NP} as n increases. Equation (3) and Table 3 reveal that both q_{CP} and q_{NP} decrease as n increases. However, CP adjusts more dynamically to the parameter variation. This timely adjusting strategy causes PIR/NP to increase as n increases.

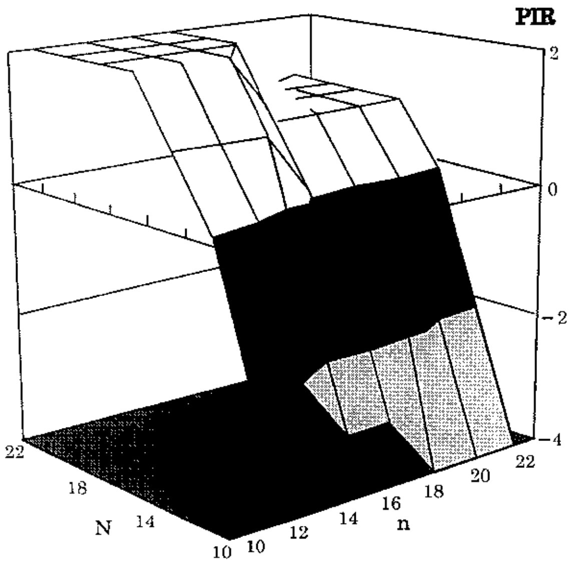

Figure 8. PIR for N, n Variation (CP VS IP)

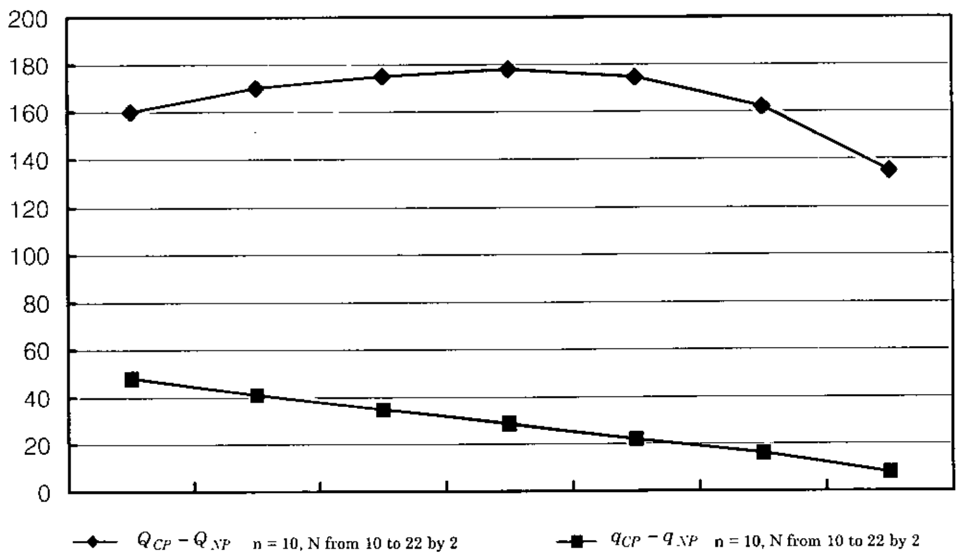

Figure 9. Difference in Decision Variables

So far we have assumed that $\partial\eta/\partial\overline{\alpha} = \eta_\Delta \le \partial\Phi/\partial\overline{\alpha} = \Phi_0$. However, in the real-world situation this relation might not necessarily hold. For example, in Section 1 we have provided an example of a tie-in alliance between CoreStates Financial Corp. and Disney's Premier Cruise Lines. The example reveals that the promotion motivated 5,000 customers to redeem their travel vouchers and yielded 42,000 new accounts. The tie-in generated far more consumers (consumer type N) of the product B (CoreStates Financial Corp) than did product A (redemption demand, Disney's Premier Cruise Lines). Therefore, in what follows we provide a numerical experiment in which we assume $\eta_\Delta \ge \Phi_0$, and keep all the previous assumptions intact. We let $\Phi_0 = 400$, $\eta_1 = 300$, $\eta_0 = 800$, $H_0 = 1000$ and $H_I = 300$. The numerical experiment that compares CP with NP is summarized in Figure 10. Now, comparing that to Figure 7, the numerical experiment shows completely different results toward n and N variations. First, PIR/NP shows an increasing trend when n decreases. Note that in the previous example the difference between q_{CP} and q_{NP} is not very large due to a low η_Δ value. In this example, however, q_{CP} is much higher than q_{NP} due to a high $\eta_\Delta = 500$ value; therefore, the benefit of the decrease in n is very significant. Second, PIR/NP increases as N increases. Figure 9 illustrates a specific case in which n =10. We vary N from 10 to 22. We observe that $Q_{CP} - Q_{NP}$ increases initially and then follows a decreasing trend as N increases. That is, initially Q_{NP} reveals an overly sensitive decreasing trend toward N increases (Equation (6) and Table 3 reveal that both Q_{CP} and Q_{NP} decrease as N increases), later showing overly insensitive behavior toward N increases. This contributes to the sub-optimality of model NP.

Although not provided in this paper, we also studied the effects of price (d, D) variations on the integrated tie-in promotion model. In conclusion, the numerical experiments reveal the following two findings:

(F1) First, PIR/NP is most significant when either (a) $\eta_\Delta \le \Phi_0$ and $N(d)$ decreases and $n(D)$ increases, or (b) $\eta_\Delta \ge \Phi_0$ and $N(d)$ increases, and $n(D)$ decreases.

(F2) Second findings, PIR/IP is most significant when either (a) $\partial\eta/\partial\overline{\alpha} = \eta_\Delta$ and $\partial\Phi/\partial\overline{\alpha} = \Phi_0$ are relatively high; $\partial H/\partial\overline{\alpha} = H_\Delta$ is relatively low; $N(n)$ decreases; $d(D)$ increases, or (b) $\partial\eta/\partial\overline{\alpha} = \eta_\Delta$ and $\partial\Phi/\partial\overline{\alpha} = \Phi_0$ are relatively low; $\partial H/\partial\overline{\alpha} = H_\Delta$ is relatively high; $N(d)$ increase; $n(D)$ decrease. In particular, if the first two conditions of (b) apply, and N decreases and D

increases, then IP can outperform CP.

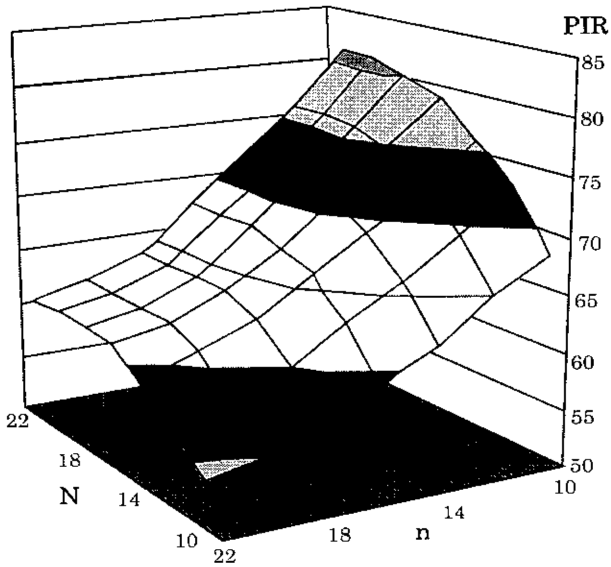

Figure 10. PIR for N, n Variation (CP VS NP)

4. Discussion and Conclusion

In this paper, an integrated tie-in promotion model is studied. Several findings were reported. First, our study reveals that an integrated tie-in promotion should be adopted when both $\partial\eta/\partial\overline{\alpha}=\eta_{\Delta}$ and $\partial\Phi/\partial\overline{\alpha}=\Phi_0$ have relatively high values. We see that reducing $\partial\Phi/\partial\overline{\alpha}=\Phi_0$ is equivalent to reducing the effectiveness of tie-in promotion to attract redemption demand. On the other hand, reducing $\partial\eta/\partial\overline{\alpha}=\eta_{\Delta}$ is equivalent to reducing the ability of tie-in promotion to allure type N customers. Hence, the benefits of a tie-in promotion are most significant when product A's price markdown can successfully attract large number of type N (high $\partial\eta/\partial\overline{\alpha}=\eta_{\Delta}$) customer and redemption demand (high $\partial\Phi/\partial\overline{\alpha}=\Phi_0$). Second, we have shown that the integrated tie-in promotion does not always promise a greater benefit. It tells us that IP can be a better option when $\partial H/\partial\overline{\alpha}=H_{\Delta}$ (measures the effectiveness of an individual promotion) is extremely high. Third, we have shown that PIR/NP is most significant when either (a)

$\eta_\Delta \le \Phi_0$ and $N(d)$ decreases and $n(D)$ increases, or (b) $\eta_\Delta \ge \Phi_0$ and $N(d)$ increases and $n(D)$ decreases. This result is particularly insightful when we note that according to our finding, two companies (or two divisions of a company) facing different business environments can form a tie-in alliance to diversify their risk. If case (b) applies, then the tie-in alliance benefit is most significant when the business environment is relatively favorable for product B (low n, high d) and relatively hostile for product A (high N, low D). For example, assume a company produces both Thanksgiving turkey and white wine. Now, assume that the Thanksgiving turkey (product A) faces a very competitive environment (its demand and profit margins are lowered due to overcompetition). This product could form a tie-in alliance with white wine (product B) so as to diversify its risk (given that the white wine business environment is very favorable). The 1984 promotion campaign between Polaroid and TWA is a good example of an alliance between two products facing different business environments. The tie-in was designed to move Polaroid's products (product B) during the Christmas season and to sell TWA Airline (product A) tickets during a low demand, post-holiday period (see Shimp and Delozier [1986]).

Finally, we have demonstrated that PIR/IP is most significant when either (a) η_Δ and Φ_0 are relatively high, and $\partial H/\partial \overline{\alpha} = H_\Delta$ is relatively low; N and n decrease; d and D increases, or (b) η_Δ and Φ_0 are relatively low, and H_Δ is relatively high; $N(d)$ increase; $n(D)$ decrease. In particular, if first two conditions of (b) apply, and N decreases and D increases, then IP can outperform CP.

Our focus so far does not allow us to study the possibility of a situation involving competitive environment so that the consumers can have the alternatives of tie-in promoted products offered by other companies. Another limitation is that the model only considers a single period situation. Generally, in a real-world application, a promotion operation may consist of more than one period. Future work on a progressive multiperiods tie-in promotion model could certainly shed further light on the topic.

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Appendix 1. Necessary and Sufficient Conditions of α Differentiating $\partial\Pi_J/\partial\alpha$ with respect to α, we then have

$$\frac{\partial \Pi_J}{\partial \alpha} = \left(d - e\int_{G^{-1}(\xi)}^{\infty} xdG\right)\frac{\partial \eta(\alpha)}{\partial \alpha} + D\left\{\Phi(\alpha|\eta) + (\alpha - E/D)\frac{\partial \Phi(\alpha|\eta)}{\partial \alpha}\right\}$$

$$+ \int_0^Q \int_0^{\omega} E\frac{\partial \Phi(\alpha|\eta)}{\partial \alpha} xdGdF .$$

The necessary condition (5) can be obtained from the first derivatives. Next, differentiating $\partial\Pi_J/\partial\alpha$ with respect to α again gives

$$\frac{\partial^2 \Pi_J}{\partial \alpha^2} = \left(d - e\int_{G^{-1}(\xi)}^{\infty} xdG\right)\frac{\partial^2 \eta(\alpha)}{\partial \alpha^2} + 2D\frac{\partial \Phi(\alpha|\eta)}{\partial \alpha} + (D\alpha - E)\frac{\partial^2 \Phi(\alpha|\eta)}{\partial \alpha^2}$$

$$+ \int_0^Q \int_0^{\omega} E\frac{\partial^2 \Phi(\alpha|\eta)}{\partial \alpha^2} xdGdF - \int_0^Q E\left(\omega\frac{\partial \Phi(\alpha|\eta)}{\partial \alpha}\right)^2 g(\omega) \Big/ \Phi(\alpha|\eta)dF .$$

The first term is negative since $\partial^2\eta/\partial\alpha^2 \le 0$ and $\Pi(q^*,\alpha)_B \ge 0$. We now consider two situations: $\partial^2\Phi/\partial\alpha^2 \le 0$ and $\partial^2\Phi/\partial\alpha^2 \ge 0$.

(1) Assume first $\partial^2\Phi/\partial\alpha^2 \ge 0$.

It is seen that $-E\,\partial^2\Phi/\partial\alpha^2 + \int_0^Q \int_0^{\omega} E\,\partial^2\Phi/\partial\alpha^2\, xdGdF =$

$$- \int_0^{\infty} \int_0^{\infty} E\,\partial^2\Phi/\partial\alpha^2\, xdGdF + \int_0^Q \int_0^{\omega} E\,\partial^2\Phi/\partial\alpha^2\, xdGdF \le 0 .$$

We also see that $2\partial\Phi/\partial\alpha + \alpha\,\partial^2\Phi/\partial\alpha^2 = \partial^2(\alpha\Phi)/\partial\alpha^2 + \partial\Phi/\partial\alpha \le 0$, since $\partial\Phi/\partial\alpha \le 0$ and $\partial^2(\alpha\Phi)/\partial\alpha^2 = \partial\Phi/\partial\alpha + \alpha\,\partial^2\Phi/\partial\alpha^2 \le 0$,

(2) Assuming now $\partial^2\Phi/\partial\alpha^2 \le 0$, then $\partial^2\Phi/\partial\alpha^2\,(D\alpha - E) \le 0$, since it is apparent that $\alpha D \ge E$.

Therefore, by (1) and (2), the second derivatives also is negative, and the profit function is concave.

Appendix 2. Necessary and Sufficient Conditions of Q

The partial derivative of Π_J with respect to Q gives:

$$\frac{\partial \Pi_J}{\partial Q} = E - N - E\int_0^Q G(\omega)\,dF\,.$$

Rearranging the term gives the necessary condition (6). We now consider the second order derivatives:

$$\frac{\partial^2 \Pi_J}{\partial Q^2} = -E\int_0^Q \frac{g(\omega)}{\Phi(\alpha|\eta)}\,dF < 0\,.$$

Appendix 3. Proof of Proposition 1

a) Derivation of $\partial Q/\partial\alpha|_{\Pi(2)} \le 0$

We see that $\partial^2\Pi_J/\partial Q\partial\alpha = E\int_0^Q g(\omega)\omega(Q,\alpha)(\partial\Phi/\partial\alpha)/\Phi dF$. Therefore, it follows that

$$\left.\frac{\partial Q}{\partial\alpha}\right|_{\Pi(2)} = -\frac{\partial^2\Pi_J}{\partial Q\partial\alpha}\bigg/\frac{\partial^2\Pi_J}{\partial Q\partial Q} = \frac{\int_0^Q g(\omega)(Q-y)\,(\partial\Phi(\alpha|\eta)/\partial\alpha)\,dF}{\Phi(\alpha|\eta)\int_0^Q g(\omega)\,dF} < 0. \qquad \text{(A3.1)}$$

Equation (A3.1) reveals that α and Q are negatively related, and the decreasing rate of Q increases as the elasticity of redemption demand Φ'/Φ increases. This makes sense intuitively. With high demand elasticity, managers can comfortably increase ordering quantity, since high demand elasticity implies that a little discount can satisfactorily digest all of the order quantity. The derivative of $\partial\alpha/\partial Q|_{\Pi(1)} \le 0$ can be obtained similarly. The implicit derivative reveals that

$$\left.\frac{\partial\alpha}{\partial Q}\right|_{\Pi(1)} = -\frac{\partial^2\Pi_J}{\partial\alpha\partial Q}\bigg/\frac{\partial^2\Pi_J}{\partial\alpha\partial\alpha} = -\int_0^Q g(\omega)\omega(Q,\alpha)\frac{\partial\Phi(\alpha|\eta)}{\partial\alpha}dF\bigg/\Phi(\alpha|\eta)\frac{\partial^2\Pi_J}{\partial\alpha\partial\alpha} < 0. \text{ (A3.2)}$$

As noted in Appendix 1, the denominator in (A3.2) is negative, and by definition $\partial\Phi/\partial\alpha \le 0$; hence, (A3.2) is negative.

b) The Sufficient Condition

The determinant of the Hessian:

$$|\mathrm{H}| = \frac{\partial^2\Pi}{\partial Q^2}\frac{\partial^2\Pi}{\partial\alpha^2} - \left(\frac{\partial^2\Pi}{\partial\alpha\partial Q}\right)^2 > |\tilde{\mathrm{H}}| = \frac{\partial^2\Pi}{\partial Q^2}\frac{\partial^2\tilde{\Pi}}{\partial\alpha^2} - \left(\frac{\partial^2\Pi}{\partial\alpha\partial Q}\right)^2,$$

where

$$\frac{\partial^2\tilde{\Pi}}{\partial\alpha^2} = -\int_0^Q\left(\omega(Q,\alpha)\frac{\partial\Phi(\alpha|\eta)}{\partial\alpha}\right)^2\frac{Eg(\omega)}{\Phi(\alpha|\eta)}dF > \frac{\partial^2\Pi}{\partial\alpha^2}.$$

The result follows since

$$\frac{\partial^2\Pi}{\partial Q^2}\frac{\partial^2\tilde{\Pi}}{\partial\alpha^2}=\left(\int_0^Q \frac{E\omega(Q,\alpha)g(\omega)}{\Phi(\alpha|\eta)}\frac{\partial\Phi(\alpha|\eta)}{\partial\alpha}dF\right)^2=\left(\frac{\partial^2\Pi}{\partial\alpha\partial Q}\right)^2.$$

c) The Effect of Parameters' Variation on Markdown Rate

c.1) We first evaluate

$$\left.\frac{\partial\alpha^*}{\partial D}\right|_{\Pi(1)}=-\left(\Phi(\alpha|\eta)+\alpha\frac{\partial\Phi(\alpha|\eta)}{\partial\alpha}\right)\Big/\frac{\partial^2\Pi_J}{\partial\alpha^2}.$$

The expression shows us that $\partial\overline{\alpha}^*/\partial D$ is positive if $\overline{\alpha}^*<\overline{\alpha}_C$, and negative if $\overline{\alpha}^*>\overline{\alpha}_C$, where $(\Phi+\alpha\,\partial\Phi/\partial\alpha)=0$ at $\alpha=\alpha_c$.

c.2) $\left.\frac{\partial\alpha^*}{\partial d}\right|_{\Pi(1)}=\frac{-\partial\eta(\alpha)}{\partial\alpha}\Big/\frac{\partial^2\Pi_J}{\partial\alpha^2}\le 0$. Therefore, the markdown rate is increasing in product B's price.

c.3) $\left.\frac{\partial\alpha^*}{\partial N}\right|_{\Pi(1)}=-\frac{\partial^2\Pi_J}{\partial\alpha\partial Q}\frac{\partial Q}{\partial N}\Big/\frac{\partial^2\Pi_J}{\partial\alpha^2}\ge 0$.

Since $\left.\frac{\partial Q}{\partial N}\right|_{\Pi(2)}=\left[\frac{\partial^2\Pi_J}{\partial Q^2}\right]^{-1}\le 0$, and

$$\frac{\partial^2\Pi_J}{\partial\alpha\partial Q}=\int_0^Q E\omega(Q,\alpha)g(\omega)\frac{\partial\Phi(\alpha|\eta)}{\partial\alpha}\frac{\partial\omega(Q,\alpha)}{\partial Q}dF\le 0.$$

Therefore, the markdown rate is decreasing in product A's manufacturing cost.

c.4) $\left.\frac{\partial\alpha^*}{\partial n}\right|_{\Pi(1)}=\frac{\partial e\int_{G^{-1}(\xi)}^{\infty}x\,dG}{\partial n}\frac{\partial\eta(\alpha)}{\partial\alpha}\Big/\frac{\partial^2\Pi_J}{\partial\alpha^2}\ge 0$ since $\frac{\partial e\int_{G^{-1}(\xi)}^{\infty}x\,dG}{\partial n}\ge 0$

Appendix 4. Table 3: Necessary and Sufficient Conditions for IP

Let $\theta(\alpha) = Q/\mathrm{H}(\alpha)$. Differentiating $\Pi_{A;IP}$ with respect to α, and equating to zero, we then have

$$\frac{\partial \Pi_{A;IP}}{\partial \alpha} = D\left\{\mathrm{H}(\alpha) + \alpha \frac{\partial \mathrm{H}(\alpha)}{\partial \alpha}\right\} - \int_{\theta}^{\infty} Ey \frac{\partial \mathrm{H}(\alpha)}{\partial \alpha} dF = 0 .$$

Rearranging the term gives the necessary condition. Next, differentiating $\partial \Pi_{A;IP}/\partial \alpha$ with respect to α again gives

$$\frac{\partial^2 \Pi_{A;IP}}{\partial \alpha^2} = 2D\frac{\partial \mathrm{H}(\alpha)}{\partial \alpha} + \alpha D \frac{\partial^2 \mathrm{H}(\alpha)}{\partial \alpha^2} + E\theta(\alpha)\frac{\partial \mathrm{H}(\alpha)}{\partial \alpha} f(\theta) \frac{\partial \theta(\alpha)}{\partial \alpha} - \int_{\theta}^{\infty} Ey \frac{\partial^2 \mathrm{H}(\alpha)}{\partial \alpha^2} dF .$$

The sum of the first two terms is negative since $\partial^2(\alpha \mathrm{H})/\partial \alpha^2 \le 0$ and $\partial \mathrm{H}/\partial \alpha \le 0$. The third term is negative. To address the sign of the fourth term we need to consider two situations $\partial^2 \mathrm{H}/\partial \alpha^2 \le 0$ and $\partial^2 \mathrm{H}/\partial \alpha^2 \ge 0$. Assuming first $\partial^2 \mathrm{H}/\partial \alpha^2 \ge 0$, the fourth term is negative. Assuming $\partial^2 \mathrm{H}/\partial \alpha^2 \le 0$, the sum of the second and fourth terms is negative since $\alpha D \partial^2 \mathrm{H}/\partial \alpha^2 = \int_0^{\infty} D\alpha \partial^2 \mathrm{H}/\partial \alpha^2 \, y dF \ge \int_{\theta}^{\infty} Ey \partial^2 \mathrm{H}/\partial \alpha^2 \, dF$ since it is apparent that $\alpha D \ge E$.

We will now show that the objective function $\Pi(Q,\alpha)_{A;IP}$ is concave. Let

$$0 \ge \frac{\partial^2 \tilde{\Pi}_{A;IP}}{\partial \alpha^2} = \frac{-E\theta^2(\alpha) f(\theta)(\partial \mathrm{H}(\alpha)/\partial \alpha)^2}{\mathrm{H}(\alpha)} > \frac{\partial^2 \Pi_{A;IP}}{\partial \alpha^2} .$$

We see that $\dfrac{\partial^2 \Pi_{A;IP}}{\partial Q^2} = \dfrac{-Ef(\theta)}{\mathrm{H}(\alpha)}$. The determinant of the Hessian

$$|\mathrm{H}| = \frac{\partial^2 \Pi}{\partial Q^2} \frac{\partial^2 \Pi}{\partial \alpha^2} - \left(\frac{\partial^2 \Pi}{\partial \alpha \partial Q}\right)^2 > |\tilde{\mathrm{H}}| = \frac{\partial^2 \Pi}{\partial Q^2} \frac{\partial^2 \tilde{\Pi}}{\partial \alpha^2} - \left(\frac{\partial^2 \Pi}{\partial \alpha \partial Q}\right)^2 .$$

The result follows since

$$\frac{\partial^2 \Pi}{\partial Q^2} \frac{\partial^2 \tilde{\Pi}}{\partial \alpha^2} = \left[\frac{Ef(\theta)\theta(\alpha)(\partial H(\alpha)/\partial \alpha)}{H(\alpha)}\right]^2 = \left(\frac{\partial^2 \Pi}{\partial \alpha \partial Q}\right)^2 .$$