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Fuzzy semi-regular spaces and fuzzy δ-continuous functions

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Abstract

We introduce fuzzy semi-regular spaces. Furthermore, we investigate the relations among fuzzy super continuity, fuzzy δ-continuity and fuzzy almost continuity in fuzzy topological spaces in view of the definition of Sostak. We study some properties between them.

Key Words : Fuzzy semi-regular spaces, Fuzzy (super, δ-, almost) continuity

I. Introduction

Sostak [13] defined the fuzzy topology as an extension of Chang's fuzzy topology [5]. Ganguly and Saha [8] introduced the notions of fuzzy δ-cluster points and fuzzy θ-cluster points in fuzzy topological spaces in the sense of [5]. Kim and Park [9] introduced r- δ-cluster points and δ-closure operators in fuzzy topological spaces in view of the definition of Sostak. It is a good extension of the notions of [12].

In this paper, we introduce fuzzy semi-regular spaces. Furthermore, we investigate the relations among fuzzy super continuity, fuzzy δ-continuity and fuzzy almost continuity in fuzzy topological spaces in view of the definition of Sostak. In general, fuzzy super continuity implies fuzzy δ-continuity, fuzzy super continuity implies fuzzy continuity. But the converses need not be true. The notions of fuzzy δ-continuity and fuzzy continuity are independent

Throughout this paper, let X be a nonempty set, $I=[0,1]$ and $I_0=(0,1]$. For $\alpha\in I$, $\overline{\alpha}(x)=\alpha$ for all $x\in X$. Let $Pt(X)$ be the family of all fuzzy points in X.

Definition 1.1 [13] A function $\tau: I^X\to I$ is called a *fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tau(\overline{0})=\tau(\overline{1})=1$,

(O2) $\tau(\mu_1\wedge\mu_2)\ge\tau(\mu_1)\wedge\tau(\mu_2)$, for $\mu_1,\mu_2\in I^X$

(O3) $\tau(\bigvee_{i\in\Gamma}\mu_i)\ge\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for $\{\mu_i\}_{i\in\Gamma}\subset I^X$.

The pair (X,τ) is called a *fuzzy topological space* (for short, fts).

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Definition 1.2 [12] A map $C: I^X\times I_0\to I^X$ is called a *fuzzy closure operator* on X if for each $\lambda,\mu\in I^X$ and $r,s\in I_0$, it satisfies the following conditions:

(C1) $C(\overline{0},r)=\overline{0}$,

(C2) $\lambda\le C(\lambda,r)$,

(C3) $C(\lambda,r)\vee C(\mu,r)=C(\lambda\vee\mu,r)$,

(C4) $C(\lambda,r)\le C(\lambda,s)$, if $r\le s$.

The pair (X,C) is called a *fuzzy closure space.*

A fuzzy closure space (X,C) is *topological* if for $\lambda\in I^X$ and $r\in I_0$,

(C5) $C(C(\lambda,r),r)=C(\lambda,r)$.

Theorem 1.3 [12] Let (X,τ) be a fts. For each $r\in I_0,\lambda\in I^X$, we define an operator $C_\tau: I^X\times I_0\to I^X$ as follows:

$$C_\tau(\lambda,r)=\bigwedge\{\mu\mid\lambda\le\mu,\tau(\overline{1}-\mu)\ge r\}.$$

Then (X,C_τ) is a topological fuzzy closure space.

Theorem 1.4 [9] Let (X,τ) be a fts. For each $r\in I_0,\lambda\in I^X$, we define an operator

$$I_\tau(\lambda,r)=\bigvee\{\mu\mid\mu\le\lambda,\tau(\mu)\ge r\}.$$

For each $\lambda,\mu\in I^X$ and $r,s\in I_0$, it satisfies the following properties.

(1) $I_\tau(\overline{1}-\lambda,r)=\overline{1}-C_\tau(\lambda,r)$.

(2) If $I_\tau(C_\tau(\lambda,r),r)=\lambda$, then $C_\tau(I_\tau(\overline{1}-\lambda,r),r)=\overline{1}-\lambda$.

(3) $I_\tau(\overline{1},r)=\overline{1}$.

(4) $I_\tau(\lambda,r)\le\lambda$.

(5) $I_\tau(\lambda,r)\wedge I_\tau(\mu,r)=I_\tau(\lambda\wedge\mu,r)$.

(6) $I_\tau(\lambda,r)\ge I_\tau(\lambda,s)$, if $r\le s$.

(7) $I_\tau(I_\tau(\lambda,r),r)=I_\tau(\lambda,r)$.

Definition 1.5 [7,9] Let (X,τ) be a fts. For

$\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$,

(1) μ is called a r-open Q_τ-neighborhood of x_t if $x_t \, q \, \mu$ with $\tau(\mu) \geq r$.

(2) μ is called a r-open R_τ-neighborhood of x_t if $x_t \, q \, \mu$ with $\mu = I_\tau(C_\tau(\mu, r), r)$.

We denote

$$Q_\tau(x_t, r) = \{\mu \in I^X \mid x_t \, q \, \mu, \ \tau(\mu) \geq r\},$$
$$R_\tau(x_t, r) = \{\mu \in I^X \mid x_t \, q \, \mu = I_\tau(C_\tau(\mu, r), r)\}.$$

Definition 1.6 [9] Let (X, τ) be a fts.
For $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$,

(1) x_t is called a *r-cluster point* of λ if for every $\mu \in Q_\tau(x_t, r)$, we have $\mu \, q \, \lambda$.

(2) x_t is called a r-δ-cluster point of λ if for every $\mu \in R_\tau(x_t, r)$, we have $\mu \, q \, \lambda$.

(3) A δ-closure operator is a function $D_\tau : I^X \times I_0 \to I^X$ defined as follows:

$$D_\tau(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a } r-\delta-\text{cluster point of } \lambda\}.$$

Theorem 1.7 [9] Let (X, τ) be a fts.
For each $\lambda, \mu \in I^X$ and $r \in I_0$, we have the following properties.

(1) $D_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \ \mu = C_\tau(I_\tau(\mu, r), r)\}$.

(2) $C_\tau(\lambda, r) = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a } r-\text{cluster point of } \lambda\}$.

(3) x_t is a r-δ-cluster point of λ iff $x_t \in D_\tau(\lambda, r)$.

(4) x_t is a r-cluster point of λ iff $x_t \in C_\tau(\lambda, r)$.

(5) $R_\tau(x_t, r) \subset Q_\tau(x_t, r)$.

(6) $C_\tau(I_\tau(\lambda, r), r) = C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r)$.

(7) $I_\tau(C_\tau(\mu, r), r) = I_\tau(C_\tau(I_\tau(C_\tau(\mu, r), r), r), r)$.

Theorem 1.8 [9] Let (X, τ) be a fts. For each $\lambda, \rho, \mu \in I^X$ and $r \in I_0$, we have the following properties.

(1) If $\rho = C_\tau(I_\tau(\rho, r), r)$, then $D_\tau(\rho, r) = \rho$.

(2) $\lambda \leq C_\tau(\lambda, r) \leq D_\tau(\lambda, r)$.

(3) If $\tau(\lambda) \geq r$, then $C_\tau(\lambda, r) = D_\tau(\lambda, r)$.

(4) $D_\tau(\lambda, r) \leq D_\tau(\mu, r)$, if $\lambda \leq \mu$.

II. Fuzzy semi-regular spaces

Definition 2.1 A fts (X, τ) is called *fuzzy semi-regular* iff for each $\mu \in Q_\tau(x_t, r)$, there exists $\rho \in Q_\tau(x_t, r)$ such that $I_\tau(C_\tau(\rho, r), r) \leq \mu$.

Example 2.2 Define fuzzy topologies $\tau_i : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.7}, \\ 0, & \text{otherwise}. \end{cases}$$

(1) A fts (X, τ_1) is fuzzy semi-regular from: for $\overline{0.4} \in Q_{\tau_1}(x_t, r)$ with $t > 0.6$ and $0 < r \leq \frac{2}{3}$, there exists $\overline{0.4} \in Q_{\tau_1}(x_t, r)$ and $\overline{0.4} = I_{\tau_1}(C_{\tau_1}(\overline{0.4}, r), r)$.

(2) A fts (X, τ_2) is not fuzzy semi-regular from: for $\overline{0.7} \in Q_{\tau_2}(x_t, r)$ with $t > 0.3$ and $0 < r \leq \frac{2}{3}$, for each $\rho \in Q_{\tau_2}(x_t, r)$, $\overline{1} = I_{\tau_2}(C_{\tau_2}(\rho, r), r) \nleq \overline{0.7}$.

Theorem 2.3 A fts (X, τ) is fuzzy semi-regular iff $D_\tau(\lambda, r) = C_\tau(\lambda, r)$, for each $\lambda \in I^X$ and $r \in I_0$.

Proof. From Theorem 1.8(2), we only show that $D_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.

Suppose there exist $\lambda \in I^X$, $r \in I_0$ such that $D_\tau(\lambda, r) \nleq C_\tau(\lambda, r)$.

Then there exist $x \in X$, $t \in I_0$ such that

$$D_\tau(\lambda, r)(x) > t > C_\tau(\lambda, r)(x). \quad \text{(A)}$$

Since $C_\tau(\lambda, r)(x) < t$, x_t is not a r-cluster point of λ. Then there exists $\mu \in Q_\tau(x_t, r)$ such that $\lambda \leq \overline{1} - \mu$. Since (X, τ) is fuzzy semi-regular, for $\mu \in Q_\tau(x_t, r)$, there exists $\rho \in Q_\tau(x_t, r)$ such that $I_\tau(C_\tau(\rho, r), r) \leq \mu$. Thus

$$\begin{aligned} \lambda &\leq \overline{1} - \mu \\ &\leq \overline{1} - I_\tau(C_\tau(\rho, r), r) \\ &\leq C_\tau(I_\tau(\overline{1} - \rho, r), r). \end{aligned}$$

It follows

$$\begin{aligned} D_\tau(\lambda, r) &\leq D_\tau(C_\tau(I_\tau(\overline{1} - \rho, r), r)) \\ &\quad \text{(by Theorem 1.8(1))} \\ &= C_\tau(I_\tau(\overline{1} - \rho, r), r) \\ &\leq C_\tau(\overline{1} - \rho, r) \\ &= \overline{1} - \rho. \end{aligned}$$

Thus, $D_\tau(\lambda, r)(x) \leq (\overline{1} - \rho)(x) < t$. It is a contradiction for the equation (A).

Conversely, for each $\mu \in Q_\tau(x_t, r)$,

$$t > (\overline{1} - \mu)(x) = C_\tau(\overline{1} - \mu, r)(x).$$

Since $D_\tau(\overline{1} - \mu, r) = C_\tau(\overline{1} - \mu, r)$, then x_t is not a r-δ-cluster point of $\overline{1} - \mu$. Then there exists $\rho \in R_\tau(x_t, r)$ and $\rho = I_\tau(C_\tau(\rho, r), r) \leq \mu$.
Since $\rho \in R_\tau(x_t, r) \subset Q_\tau(x_t, r)$ from Theorem 1.7(5), (X, τ) is fuzzy semi-regular.

III. Fuzzy δ-continuous and fuzzy super continuous mappings

Definition 3.1 Let (X, τ) and (Y, η) be fts's. Let $f: X \to Y$ be a function. Then

(1) f is called *fuzzy continuous* iff $\eta(\mu) \le \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) f is called *fuzzy open* iff $\tau(\lambda) \le \eta(f(\lambda))$ for each $\lambda \in I^X$.

(3) f is called *fuzzy closed* iff $\tau(\overline{1}-\lambda) \le \eta(\overline{1}-f(\lambda))$ for each $\lambda \in I^X$.

(4) f s called *fuzzy super continuous* iff for $\mu \in Q_\eta(f(x)_t, r)$, there exists $\lambda \in Q_\tau(x_t, r)$ such that $f(I_\tau(C_\tau(\lambda, r), r)) \le \mu$.

(5) f is called *fuzzy almost continuous* iff for $\mu \in R_\eta(f(x)_t, r)$, there exists $\lambda \in Q_\tau(x_t, r)$ such that $f(\lambda) \le \mu$.

(6) f is called *fuzzy δ-continuous* iff for each $\mu \in R_\eta(f(x)_t, r)$, there exists $\lambda \in R_\tau(x_t, r)$ such that $f(\lambda) \le \mu$.

Theorem 3.2 Let (X, τ) and (Y, η) be fts's. Let $f: X \to Y$ be a function. For each $\lambda \in I^X, \mu \in I^Y$, $r \in I_0$, the following statements are equivalent:

(1) A map f is fuzzy super continuous.

(2) $f(D_\tau(\lambda, r)) \le C_\eta(f(\lambda), r)$.

(3) $D_\tau(f^{-1}(\mu), r) \le f^{-1}(C_\eta(\mu, r))$.

(4) $D_\tau(f^{-1}(\mu), r) = f^{-1}(\mu)$, for each set $\mu = C_\eta(\mu, r)$.

(5) $D_\tau(\overline{1}-f^{-1}(\mu), r) = \overline{1}-f^{-1}(\mu)$, for each set $\mu = I_\eta(\mu, r)$.

Proof. (1) $\Rightarrow$ (2) Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that $f(D_\tau(\lambda, r)) \not\le C_\eta(f(\lambda), r)$. Then there exist $y \in Y$, $t \in I_0$ such that

$$f(D_\tau(\lambda, r))(y) > t > C_\eta(f(\lambda), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, it contradicts to the fact that $f(D_\tau(\lambda, r))(y) = 0$. Hence $f^{-1}(\{y\}) \ne \emptyset$. There exists $x \in f^{-1}(\{y\})$ such that

$$\begin{aligned} f(D_\tau(\lambda, r))(y) &\ge D_\tau(\lambda, r)(x) \\ &> t > C_\eta(f(\lambda), r)(f(x)). \end{aligned} \qquad \text{(B)}$$

Since $C_\eta(f(\lambda), r)(f(x)) < t$, by Theorem 1.7(4), $f(x)_t$ is not a r-cluster point of $f(\lambda)$. Then there exists $\mu \in Q_\eta(f(x)_t, r)$ such that $f(\lambda) \le \overline{1}-\mu$. Since f is fuzzy super continuous, for $\mu \in Q_\eta(f(x)_t, r)$, there exists $\nu \in Q_\tau(x_t, r)$ such that $f(I_\tau(C_\tau(\nu, r), r)) \le \mu$.

Thus, $f(\lambda) \le \overline{1} - f(I_\tau(C_\tau(\nu, r), r))$ implies

$$\lambda \le \overline{1} - I_\tau(C_\tau(\nu, r), r).$$

For each $\nu \in Q_\tau(x_t, r)$, since $\nu \le I_\tau(C_\tau(\nu, r), r)$, by Theorem1.7(7), $I_\tau(C_\tau(\nu, r), r) \in R_\tau(x_t, r)$.

Hence x_t is not a r- δ-cluster point of λ, by Theorem 1.7(3), $D_\tau(\lambda, r)(x) < t$. It is a contradiction for the equation (B).

(2) $\Rightarrow$ (3) For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then

$$\begin{aligned} f(D_\tau(f^{-1}(\mu), r)) &\le C_\eta(f(f^{-1}(\mu)), r) \\ &\le C_\eta(\mu, r). \end{aligned}$$

It implies

$$\begin{aligned} D_\tau(f^{-1}(\mu), r)) &\le f^{-1}(f(D_\tau(f^{-1}(\mu), r))) \\ &\le f^{-1}(C_\eta(\mu, r)). \end{aligned}$$

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are easily proved from Theorem 1.8(2) and Theorem 1.4(1).

(5) $\Rightarrow$ (1) Let $\mu \in Q_\eta(f(x)_t, r)$. Since $\mu = I_\eta(\mu, r)$, by (5),

$$\overline{1} - f^{-1}(\mu) = D_\tau(\overline{1} - f^{-1}(\mu), r).$$

Since $f(x)_t \, q \, \mu$, we have $x_t \, q \, f^{-1}(\mu)$, i.e,

$$t > (\overline{1} - f^{-1}(\mu))(x) = D_\tau(\overline{1} - f^{-1}(\mu), r)(x).$$

So, x_t is not a r- δ-cluster point of $\overline{1} - f^{-1}(\mu)$. Then there exists $\nu \in R_\tau(x_t, r) \subset Q_\tau(x_t, r)$ such that $\overline{1} - f^{-1}(\mu) \le \overline{1} - \nu$.

Hence $\nu \le f^{-1}(\mu)$ implies

$$f(\nu) = f(I_\tau(C_\tau(\nu, r), r)) \le \mu.$$

The following theorem is similarly proved as Theorem 3.2.

Theorem 3.3 Let (X, τ) and (Y, η) be fts's. Let $f: X \to Y$ be a function. For each $\lambda \in I^X, \mu \in I^Y$ and $r \in I_0$, the following statements are equivalent:

(1) A map f is fuzzy continuous.

(2) $f(C_\tau(\lambda, r)) \le C_\eta(f(\lambda), r)$.

(3) $C_\tau(f^{-1}(\mu), r) \le f^{-1}(C_\eta(\mu, r))$.

Theorem 3.4 Let (X, τ) and (Y, η) be fts's. Let $f: X \to Y$ be a function. For each $\lambda \in I^X, \mu \in I^Y$ and $r \in I_0$, the following statements are equivalent:

(1) A map f is fuzzy δ-continuous.

(2) $f(D_\tau(\lambda, r)) \le D_\eta(f(\lambda), r)$.

(3) $D_\tau(f^{-1}(\mu), r) \le f^{-1}(D_\eta(\mu, r))$.

(4) $D_\tau(f^{-1}(\mu), r) = f^{-1}(\mu)$, for each set $\mu = D_\eta(\mu, r)$.

(5) $D_\tau(\overline{1} - f^{-1}(\mu), r) = \overline{1} - f^{-1}(\mu)$, for each set $\mu = I_\eta(C_\eta(\mu, r), r)$.

Proof. (1) $\Rightarrow$ (2) Supposethere exist $\lambda \in I^X$, $r \in I_0$ such that

$f(D_\tau(\lambda, r)) \nleq D_\eta(f(\lambda), r)$.

Then there exist $y \in Y$, $t \in I_0$ such that

$$f(D_\tau(\lambda, r))(y) > t > D_\eta(f(\lambda), r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, it contradicts to the fact that $f(D_\tau(\lambda, r))(y) = 0$. Hence $f^{-1}(\{y\}) \neq \emptyset$. There exists $x \in f^{-1}(\{y\})$ such that

$$\begin{aligned} f(D_\tau(\lambda, r))(y) &\geq D_\tau(\lambda, r)(x) \\ &> t > D_\eta(f(\lambda), r)(f(x)). \end{aligned} \quad \text{(C)}$$

Since $D_\eta(f(\lambda), r)(f(x)) < t$, by Theorem 1.7(3), $f(x)_t$ is not a r-δ-cluster point of $f(\lambda)$. Then there exists $\mu \in R_\eta(f(x)_t, r)$ such that $f(\lambda) \leq \overline{1} - \mu$.

Since f is fuzzy δ-continuous, for $\mu \in R_\eta(f(x)_t, r)$, there exists $\nu \in R_\tau(x_t, r)$ such that $f(\nu) \leq \mu$. Thus, $f(\lambda) \leq \overline{1} - f(\nu)$ implies $\lambda \leq \overline{1} - \nu$. Hence x_t is not a r-δ-cluster point of λ, by Theorem 1.7 (3), $D_\tau(\lambda, r)(x) < t$. It is a contradiction for (C).

(2) $\Rightarrow$(3) For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then

$$f(D_\tau(f^{-1}(\mu), r)) \leq D_\eta(f(f^{-1}(\mu)), r) \leq D_\eta(\mu, r).$$

It implies

$$\begin{aligned} D_\tau(f^{-1}(\mu), r)) &\leq f^{-1}(f(D_\tau(f^{-1}(\mu), r))) \\ &\leq f^{-1}(D_\eta(\mu, r)). \end{aligned}$$

(3) $\Rightarrow$(4) Since $f^{-1}(\mu) \leq D_\tau(f^{-1}(\mu), r))$ from Theorem 1.8(2), it is easily proved.

(4) $\Rightarrow$(5) For each $\mu = I_\eta(C_\eta(\mu, r), r) \in I^Y$, by Theorem 1.4(2), $C_\eta(I_\eta(\overline{1} - \mu, r), r) = \overline{1} - \mu$.

Thus, by Theorem 1.8(1),

$$\begin{aligned} D_\eta(\overline{1} - \mu, r) &= D_\eta(C_\eta(I_\eta(\overline{1} - \mu, r), r)) \\ &= C_\eta(I_\eta(\overline{1} - \mu, r), r) \\ &= \overline{1} - \mu. \end{aligned}$$

From (4), $D_\tau(\overline{1} - f^{-1}(\mu), r) = \overline{1} - f^{-1}(\mu)$.

(5) $\Rightarrow$(1) Let $\mu \in R_\eta(f(x)_t, r)$. From (5), since $\mu = I_\eta(C_\eta(\mu, r), r)$, we have

$$\overline{1} - f^{-1}(\mu) = D_\tau(\overline{1} - f^{-1}(\mu), r).$$

Since $f(x)_t \, q \, \mu$, we have $x_t \, q \, f^{-1}(\mu)$, i.e,
$t > (\overline{1} - f^{-1}(\mu))(x) = D_\tau(\overline{1} - f^{-1}(\mu), r)(x)$.

Thus, x_t is not a r-δ-cluster point of $\overline{1} - f^{-1}(\mu)$. Then there exists $\nu \in R_\tau(x_t, r)$ such that

$$\overline{1} - f^{-1}(\mu) \leq \overline{1} - \nu.$$

Hence $\nu \leq f^{-1}(\mu)$ implies $f(\nu) \leq \mu$.

The following theorem is similarly proved as Theorem 3.4.

Theorem 3.5 Let (X, τ) and (Y, η) be fts's.

Let $f: X \to Y$ be a function. For each $\lambda \in I^X$, $\mu \in I^Y$ and $r \in I_0$, the following statements are equivalent:

(1) A map f is fuzzy almost continuous.

(2) $f(C_\tau(\lambda, r)) \leq D_\eta(f(\lambda), r)$.

(3) $C_\tau(f^{-1}(\mu), r) \leq f^{-1}(D_\eta(\mu, r))$.

(4) $C_\tau(f^{-1}(\mu), r) = f^{-1}(\mu)$,
for each set $\mu = D_\eta(\mu, r)$.

(5) $C_\tau(\overline{1} - f^{-1}(\mu), r) = \overline{1} - f^{-1}(\mu)$,
for each set $\mu = I_\eta(C_\eta(\mu, r), r)$.

The following theorem is easily proved from Definition 3.1 and Theorem 1.8(2).

Theorem 3.6 Let (X, τ) and (Y, η) be fts's.

Let $f: X \to Y$ be a function.Then we have the following implications.

$$\begin{array}{ccc} f \text{ is S-map} & \Rightarrow & f \text{ is C-map} \\ \Downarrow & & \Downarrow \\ f \text{ is } \delta\text{-map} & \Rightarrow & f \text{ is A-map} \end{array}$$

where fuzzy super continuous(for short S-map), fuzzy continuous(C-map), fuzzy δ-continuous (δ-map), fuzzy almost continuous (A-map).

Example 3.7 Define fuzzy topologies $\tau_i: I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.4}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.3}, \\ 0, & \text{otherwise}. \end{cases}$$

$$\tau_3(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.4}, \\ 0, & \text{otherwise}. \end{cases}$$

From Theorem 1.3 and Theorem 1.7(1), we obtain $C_{\tau_1}, D_{\tau_1}: I^X \times I_0 \to I^X$ as follows:

$$C_{\tau_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0}, r \in I_0, \\ \overline{0.6}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.6}, \\ & \quad 0 < r \leq \frac{2}{3}, \\ \overline{0.7}, & \text{if } \overline{0.6} \ngeq \lambda \leq \overline{0.7}, \\ & \quad 0 < r \leq \frac{2}{3}, \\ \overline{1}, & \text{otherwise}, \end{cases}$$

$$D_{\tau_1}(\lambda, r) = \begin{cases} \overline{0}, & \text{if } \lambda = \overline{0}, r \in I_0, \\ \overline{0.6}, & \text{if } \overline{0} \neq \lambda \leq \overline{0.6}, \\ & \quad 0 < r \leq \frac{2}{3}, \\ \overline{1}, & \text{otherwise}, \end{cases}$$

$D_{\tau_1}=D_{\tau_3}=C_{\tau_3}$.

Furthermore,

$$D_{\tau_2}(\lambda, r)=\begin{cases}\overline{0}, & \text{if } \lambda=\overline{0}, r\in I_0,\\ \overline{0.7}, & \text{if } \overline{0}\neq\lambda\leq\overline{0.7},\ 0<r\leq\frac{2}{3},\\ \overline{1}, & \text{otherwise},\end{cases}$$

(1) Since $D_{\tau_1}=D_{\tau_3}=C_{\tau_3}$, by Theorem 3.4(2) and Theorem 3.5(2), the identity function

$$id_X:(X,\tau_3)\to(X,\tau_1)$$

is fuzzy almost continuous and fuzzy δ-continuous.
It is neither fuzzy continuous nor fuzzy super continuous because $D_{\tau_3}\not\leq C_{\tau_1}$, $C_{\tau_3}\not\leq C_{\tau_1}$ from Theorem 3.2(2) and Theorem 3.3(2).

(2) Since $C_{\tau_1}\leq D_{\tau_2}=C_{\tau_2}$, by Theorem 3.5(2) and Theorem 3.3(2), the identity function

$$id_X:(X,\tau_1)\to(X,\tau_2)$$

is fuzzy almost continuous and fuzzy continuous.
It is neither fuzzy δ-continuous nor fuzzy super continuous because $D_{\tau_1}\not\leq D_{\tau_2}=C_{\tau_2}$, from Theorem 3.2(2) and Theorem 3.3(2).

(3) From(1) and (2), the notions of fuzzy δ-continuity and fuzzy continuity are independent.

Theorem 3.8 Let (X,τ) and (Y,η) be fts's.
Let $f:X\to Y$ be a function.
(1) If f is fuzzy super continuous and fuzzy-open, then, for each $\mu\in I^Y$, $r\in I_0$,

$$D_\tau(f^{-1}(\mu),r)=f^{-1}(C_\eta(\mu,r)).$$

(2) If f is fuzzy continuous and fuzzy-open, then, for each $\mu\in I^Y$, $r\in I_0$,

$$C_\tau(f^{-1}(\mu),r)=f^{-1}(C_\eta(\mu,r)).$$

Proof. (1) From Theorem 3.2(3), we only show that, for each $\mu\in I^Y$, $r\in I_0$,

$$D_\tau(f^{-1}(\mu),r)\geq f^{-1}(C_\eta(\mu,r)).$$

Suppose there exist $\mu\in I^Y$, $r\in I_0$ such that

$$D_\tau(f^{-1}(\mu),r)\not\geq f^{-1}(C_\eta(\mu,r)).$$

Then there exist $x\in X$, $t\in I_0$ such that

$$D_\tau(f^{-1}(\mu),r)(x)<t<C_\eta(\mu,r)(f(x)). \qquad \text{(D)}$$

Since $D_\tau(f^{-1}(\mu),r)(x)<t$, by Theorem 1.7(3), x_t is not a r- δ-cluster point of $f^{-1}(\mu)$.
Hence there exists $\rho\in R_\tau(x_t,r)$ such that

$$f^{-1}(\mu)\leq\overline{1}-\rho.$$

Furthermore, $x_t\, q\, \rho$ implies $f(x)_t\, q\, f(\rho)$.
Since f is fuzzy-open, $\eta(f(\rho))\geq\tau(\rho)\geq r$.
Thus, $f(\rho)\in Q_\eta(f(x)_t,r)$.
Since $f^{-1}(\mu)\leq\overline{1}-\rho$ iff $\mu\leq\overline{1}-f(\rho)$,
$f(x)_t$ is not a r-cluster point of μ,
that is, $C_\eta(\mu,r)(f(x))<t$.
It is a contradiction for the equation (D).
(2) It is similar to (1).

Theorem 3.9 Let (X,τ), (Y,η) and (Z,γ) be fts's.Let $f:X\to Y$ and $g:Y\to Z$ be functions.
If g and f are fuzzy super continuous (resp. fuzzy continuous, fuzzy δ-continuous),
then $g\circ f$ is fuzzy super continuous(resp. fuzzy continuous, fuzzy δ-continuous).

Proof. Let g and f be fuzzy super continuous.
For each $\lambda\in I^X$, $r\in I_0$,

$g(f(D_\tau(\lambda,r)))\leq g(C_\eta(f(\lambda),r))$

(f is fuzzy super continuous) $\leq g(D_\eta(f(\lambda),r))$

(by Theorem 1.8(2)) $\leq C_\gamma(g(f(\lambda)),r)$.

(g is fuzzy super continuous)

Thus, $g\circ f$ is fuzzy super continuous.
Others are similarly proved.

In general, the composition of two fuzzy almost continuous functions need not be fuzzy almost continuous from the following example.

Example 3.10 Define fuzzy topologies $\tau_i:I^X\to I$ as in Example 3.7. The identity functions $id_X:(X,\tau_3)\to(X,\tau_1)$ and $id_X:(X,\tau_1)\to(X,\tau_2)$ are fuzzy almost continuous, but $id_X:(X,\tau_3)\to(X,\tau_2)$ is not fuzzy almost continuous because,

$$\text{for } \overline{0.3}\in R_{\tau_2}(x_{0.8},\tfrac{2}{3})$$

$$\text{and for all } \lambda\in\{\overline{0.4},\overline{1}\}\subset Q_{\tau_3}(x_{0.8},\tfrac{2}{3}),$$

$$\text{we have } \lambda\not\leq\overline{0.3}.$$

From Theorem 2.3, we can obtain the following theorems.

Theorem 3.11 Let (X,τ) and (Y,η) be fts's. Let $f:X\to Y$ be a function and (X,τ) be fuzzy semi-regular. Then
(1) f is fuzzy super continuous iff f is fuzzy continuous.
(2) f is fuzzy δ-continuous iff f is fuzzy almost continuous.

Theorem 3.12 Let (X,τ) and (Y,η) be fts's. Let $f:X\to Y$ be a function and (Y,η) be fuzzy semi-regular. Then

(1) f is fuzzy super continuous iff f is fuzzy δ-continuous.
(2) f is fuzzy continuous iff f is fuzzy almost continuous.

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