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Fuzzy r-Pre-semineighborhoods and Fuzzy r-Pre-semicontinuous Maps

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Abstract

In this paper, we introduce the concepts of fuzzy r-pre-semiopen and r-pre-semiclosed sets. With them we define fuzzy r-pre-semiinterior and r-pre-semiclosure. We also introduce and investigate the properties of a fuzzy r-pre-semicontinuous map, a fuzzy r-pre-semiopen map and a fuzzy r-pre-semiclosed map. These concepts are generalizations of the Bai Shi-Zhong's fuzzy pre-semicontinuity.

Key Words : fuzzy r-pre-semiopen, fuzzy r-pre-semicontinuous

I. Introduction and preliminaries

As a generalization of a set, the concept of a fuzzy set was introduced by Zadeh[1]. Chang[2] and Lowen[3] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Some authors[4,5,6,7] introduced other definitions of a fuzzy topology as a generalization of Chang's fuzzy topology or Lowen's fuzzy topology.

Bai Shi-Zhong[8] introduced and studied fuzzy pre-semiopen sets and fuzzy pre-semicontinuous maps in Chang's fuzzy topology. Also Bai Shi-Zhong and Wang Wan Liang[9] established some other properties of a fuzzy pre-semicontinuous map by the concept of fuzzy pre-semiopen q-neighborhoods in Chang's fuzzy topology.

In this paper, we introduce the concepts of fuzzy r-pre-semiopen and r-pre-semiclosed sets. With them we define fuzzy r-pre-semiinterior and r-pre-semiclosure. We also introduce and investigate the properties of a fuzzy r-pre-semicontinuous map, a fuzzy r-pre-semiopen map and a fuzzy r-pre-semiclosed map. These concepts are generalizations of the Bai Shi-Zhong's fuzzy pre-semicontinuity.

We will denote the unit interval $[0,1]$ of the real line by I and $I_0=(0,1]$. A member μ of I^X is called a fuzzy set in X. For any $\mu\in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively.

Definition 1.1 ([9]) Let μ be a fuzzy set in a fuzzy topological space (X, T). Then μ is said to be

(1) *fuzzy pre-semiopen* if $\mu\leq \operatorname{sint}(\operatorname{cl}(\mu))$,

(2) *fuzzy pre-semiclosed* if $\operatorname{scl}(\operatorname{int}(\mu))\leq\mu$.

Definition 1.2([9]) Let x_α be a fuzzy point of a fuzzy topological space (X, T). Then a fuzzy set μ of X is called

(1) *a fuzzy pre-semineighborhood* of x_α if there is a fuzzy pre-semiopen set ρ in X such that $x_\alpha\in\rho\leq\mu$,

(2) a *fuzzy quasi-pre-semineighborhood* of x_α if there is a fuzzy pre-semiopen set ρ in X such that $x_\alpha q\rho\leq\mu$.

Definition 1.3([9]) Let $f: (X, T)\to(Y, U)$ be a map from a fuzzy topological space X to a fuzzy topological space Y. Then f is said to be

(1) fuzzy *pre-semicontinuous* if $f^{-1}(\mu)$ is a fuzzy pre-semiopen set in X for each fuzzy open set μ in Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy pre-semiclosed set in X for each fuzzy closed set μ in Y,

(2) fuzzy *pre-semiopen* if $f(\mu)$ is a fuzzy pre-semiopen set in Y for each fuzzy open set μ in X,

(3) fuzzy *pre-semiclosed* if $f(\mu)$ is a fuzzy pre-semiclosed set in Y for each fuzzy closed set μ in X.

II. Fuzzy r-pre-semiopen sets

In this section, we are going to define fuzzy r-pre-semiopen and fuzzy r-pre-semiclosed sets, and investigate some of their properties.

Definition 2.1 Let μ be a fuzzy set in a fuzzy topological space (X, T) and $r\in I_0$. Then μ is called

(1) a *fuzzy r-pre-semiopen* set if

$\mu\leq \operatorname{sint}(\operatorname{cl}(\mu, r), r)$,

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(2) a *fuzzy* r*-pre-semiclosed* set if
$\mu \geq \text{scl}(\text{int}(\mu, r), r)$.

Theorem 2.2 *Let* μ *be a fuzzy set in a fuzzy topological space* (X, T) *and* $r \in I_0$. *Then the following two statements are equivalent:*

(1) *A fuzzy set* μ *is fuzzy* r*-pre-semiopen.*
(1) *A fuzzy set* μ^c *is fuzzy* r*-pre-semiclosed.*

Proof. (1)⇒(2) Let μ be a fuzzy r-pre-semiopen set. Then $\mu \leq \text{sint}(\text{cl}(\mu, r), r)$. Since $\text{sint}(\mu, r)^c = \text{scl}(\mu^c, r)$ and $\text{cl}(\mu, r)^c = \text{int}(\mu^c, r)$, we have

$$\mu^c \geq \text{sint}(\text{cl}(\mu, r), r)^c = \text{scl}(\text{cl}(\mu, r)^c, r) = \text{scl}(\text{int}(\mu^c, r), r).$$

Therefore μ^c is fuzzy r-pre-semiclosed.

(2)⇒(1) Let μ^c be r-pre-semiclosed. Then $\mu^c \geq \text{scl}(\text{int}(\mu^c, r), r)$.
Since $\text{scl}(\mu, r)^c = \text{sint}(\mu^c, r)$ and $\text{int}(\mu^c, r) = \text{cl}(\mu, r)^c$, we have

$$\mu \leq \text{scl}(\text{int}(\mu^c, r), r)^c = \text{sint}(\text{int}(\mu^c, r)^c, r) = \text{sint}(\text{cl}(\mu, r), r).$$

Thus μ is fuzzy r-pre-semiopen.

Remark 2.3

(1) Every fuzzy r-preopen(r-preclosed) set μ is fuzzy r-pre-semiopen (r-pre-semiclosed).
(2) Every fuzzy r-semiopen(r-semiclosed) set μ is fuzzy r-pre-semiopen (r-pre-semiclosed).

Proof. (1) Let μ be a fuzzy r-preopen set. Then $\mu \leq \text{int}(\text{cl}(\mu, r), r) \leq \text{sint}(\text{cl}(\mu, r), r)$.
Therefore μ is fuzzy r-pre-semiopen.
(2) Let μ be a fuzzy r-semiopen set. Then $\mu = \text{sint}(\mu, r)$. Thus we have $\mu = \text{sint}(\mu, r) \leq \text{sint}(\text{cl}(\mu, r), r)$. Hence μ is fuzzy r-pre-semiopen.

The following examples show that the converses are not true.

Example 2.4 Let $X = \{x\}$ and μ_1, μ_2 and μ_3 be fuzzy sets of X defined as

$$\mu_1(x) = \frac{1}{4}, \quad \mu_2(x) = \frac{1}{3}, \quad \mu_3(x) = \frac{1}{5}.$$

Define $T: I^X \to I$ by

$$T(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is a fuzzy topology on X

(1) Since

$$\text{cl}(\text{int}(\mu_2, \tfrac{1}{2}), \tfrac{1}{2}) = \text{cl}(\mu_1, \tfrac{1}{2}) = \mu_1^c \geq \mu_2,$$

μ_2 is fuzzy $\frac{1}{2}$-semiopen. Thus μ_2 is fuzzy $\frac{1}{2}$-pre-semiopen. But μ_2 is not fuzzy $\frac{1}{2}$-preopen, because

$$\text{int}(\text{cl}(\mu_2, \tfrac{1}{2}), \tfrac{1}{2}) = \text{int}(\mu_1^c, \tfrac{1}{2}) = \mu_1 \ngeq \mu_2.$$

(2) Since

$$\text{int}(\text{cl}(\mu_3, \tfrac{1}{2}), \tfrac{1}{2}) = \text{int}(\mu_1^c, \tfrac{1}{2}) = \mu_1 \geq \mu_3,$$

μ_3 is fuzzy $\frac{1}{2}$-preopen. Thus μ_3 is fuzzy $\frac{1}{2}$-pre-semiopen. However μ_3 is not fuzzy $\frac{1}{2}$-semiopen, because

$$\text{cl}(\text{int}(\mu_3, \tfrac{1}{2}), \tfrac{1}{2}) = \text{cl}(\tilde{0}, \tfrac{1}{2}) = \tilde{0} \ngeq \mu_3.$$

Theorem 2.5
(1) *Any union of fuzzy* r*-pre-semiopen sets is fuzzy* r*-pre-semiopen.*
(2) *Any intersection of fuzzy* r*-pre-semiclosed sets is fuzzy* r*-pre-semiclosed.*

Proof. (1) Let $\{\mu_i\}$ be a collection of fuzzy r-pre-semiopen sets. Then for each i,
$\mu_i \leq \text{sint}(\text{cl}(\mu_i, r), r)$.
Thus we have

$$\bigvee \mu_i \leq \bigvee \text{sint}(\text{cl}(\mu_i, r), r) \leq \text{sint}(\bigvee \text{cl}(\mu_i, r), r) = \text{sint}(\text{cl}(\bigvee \mu_i, r), r).$$

Hence $\bigvee \mu_i$ is fuzzy r-pre-semiopen.

(2) Let $\{\mu_i\}$ be a collection of fuzzy r-pre-semiclosed sets. Then μ_i^c is fuzzy r-pre-semiopen. By (1), $\bigvee \mu_i^c = (\bigwedge \mu_i)^c$ is fuzzy r-pre-semiopen. Hence $\bigwedge \mu_i$ is fuzzy r-pre-semiclosed.

Definition 2.6 Let (X, T) be a fuzzy topological space. For each $r \in I_0$ and $\mu \in I^X$, the *fuzzy* r*-pre-semiclosure* is defined by

$$\text{pscl}(\mu, r) = \bigwedge \{\rho \in I^X \mid \mu \leq \rho, \ \rho \text{ is fuzzy } r-\text{pre}-\text{semi closed}\}$$

and the *fuzzy* r*-pre-semiinterior* is defined by

$$\text{psint}(\mu, r) = \bigvee \{\rho \in I^X \mid \mu \geq \rho, \ \rho \text{ is fuzzy } r-\text{pre}-\text{semi open}\}.$$

Obviously $\text{pscl}(\mu, r)$ is the smallest fuzzy r-pre-semiclosed set which contains μ and $\text{psint}(\mu, r)$ is the greatest fuzzy r-pre-semiopen set which is contained in μ. Also, $\text{pscl}(\mu, r) = \mu$ for any fuzzy r-pre-semiclosed set μ and $\text{psint}(\mu, r) = \mu$ for any fuzzy r-pre-semiopen set μ.

Moreover, we have the following results.

Theorem 2.7 Let (X, T) be a fuzzy topological space and

$$\mathrm{pscl} : I^X \times I_0 \rightarrow I^X$$

the *fuzzy r-pre-semiclosure operator in* (X, T). Then for $\mu \in I^X, \rho \in I^X$ and $r \in I_0$

(1) $\mathrm{pscl}(\tilde{0}, r) = \tilde{0}$, $\mathrm{pscl}(\tilde{1}, r) = \tilde{1}$.
(2) $\mathrm{pscl}(\mu, r) \geq \mu$.
(3) $\mathrm{pscl}(\mu \vee \rho, r) \geq \mathrm{pscl}(\mu, r) \vee \mathrm{pscl}(\rho, r)$.
(4) $\mathrm{pscl}(\mathrm{pscl}(\mu, r), r) = \mathrm{pscl}(\mu, r)$.

Proof. (1), (2) and (4) are obvious.
(3) Since $\mu \leq \mu \vee \rho$ and $\rho \leq \mu \vee \rho$, $\mathrm{pscl}(\mu, r) \leq \mathrm{pscl}(\mu \vee \rho, r)$ and $\mathrm{pscl}(\rho, r) \leq \mathrm{pscl}(\mu \vee \rho, r)$.
Hence
$\mathrm{pscl}(\mu \vee \rho, r) \geq \mathrm{pscl}(\mu, r) \vee \mathrm{pscl}(\rho, r)$.

The fuzzy r-pre-semiinterior operator satisfies the condition of the interior operator.

Theorem 2.8 *Let μ be a fuzzy set in a fuzzy topological space X and $r \in I_0$. We have*

(1) $\mathrm{psint}(\mu, r)^c = \mathrm{pscl}(\mu^c, r)$,
(2) $\mathrm{pscl}(\mu, r)^c = \mathrm{psint}(\mu^c, r)$.

Proof. (1) Since $\mathrm{psint}(\mu, r) \leq \mu$ and $\mathrm{psint}(\mu, r)$ is fuzzy r-pre-semiopen, $\mu^c \leq \mathrm{psint}(\mu, r)^c$ and $\mathrm{psint}(\mu, r)^c$ is fuzzy r-pre-semiclosed. Thus

$$\mathrm{pscl}(\mu^c, r) = \mathrm{pscl}(\mathrm{psint}(\mu, r)^c, r) = \mathrm{psint}(\mu, r)^c.$$

Conversely, since $\mu^c \leq \mathrm{pscl}(\mu^c, r)$ and $\mathrm{pscl}(\mu^c, r)$ is fuzzy r-pre-semiclosed in X, $\mathrm{pscl}(\mu^c, r)^c \leq \mu$ and $\mathrm{pscl}(\mu^c, r)^c$ is fuzzy r-pre-semiopen. Thus

$$\mathrm{pscl}(\mu^c, r)^c = \mathrm{psint}(\mathrm{pscl}(\mu^c, r)^c, r) \leq \mathrm{psint}(\mu, r).$$

Hence $\mathrm{psint}(\mu, r)^c = \mathrm{pscl}(\mu^c, r)$.
(2) Similar to (1).

Theorem 2.9 *For a fuzzy set μ of a fuzzy topological space X and $r \in I_o$, we have*
(1)
$$\mathrm{psint}(\mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r), r) = \mathrm{psint}(\mathrm{pscl}(\mu, r), r),$$
(2)
$$\mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mathrm{psint}(\mu, r), r), r), r) = \mathrm{pscl}(\mathrm{psint}(\mu, r), r).$$

Proof. (1)Since $\mathrm{psint}(\mathrm{pscl}(\mu, r), r)$is fuzzy r-pre-semiopen and
$\mathrm{psint}(\mathrm{pscl}(\mu, r), r) \leq \mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r)$, it follows that

$$\mathrm{psint}(\mathrm{pscl}(\mu, r), r) = \mathrm{psint}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r) \leq \mathrm{psint}(\mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r), r).$$

Conversely, since $\mathrm{psint}(\mu, r)$ is fuzzy r-pre-semiclosed and
$\mathrm{psint}(\mathrm{pscl}(\mu, r), r) \leq \mathrm{pscl}(\mu, r)$,

$$\mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r) \leq \mathrm{pscl}(\mathrm{pscl}(\mu, r), r) = \mathrm{pscl}(\mu, r).$$

Thus

$$\mathrm{psint}(\mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r), r) \leq \mathrm{psint}(\mathrm{pscl}(\mu, r), r).$$

Hence

$$\mathrm{psint}(\mathrm{pscl}(\mathrm{psint}(\mathrm{pscl}(\mu, r), r), r), r) = \mathrm{psint}(\mathrm{pscl}(\mu, r), r).$$

(2) Similar to (1).

Let (X, T) be a fuzzy topological space. For an r-cut

$$T_r = \{\mu \in I^X \mid \tau(\mu) \geq r\},$$

it is obvious that (X, T_r) is a Chang's fuzzy topological space for all $r \in I_0$

Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Then a fuzzy topology $T^r : I^X \rightarrow I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise} \end{cases}$$

The next two theorems show that a fuzzy pre-semiopen set is a special case of a fuzzy r-pre-semiopen set.

Theorem 2.10 *Let μ be a fuzzy set in a fuzzy topological space (X, T) and $r \in I_0$. Then μ is fuzzy r-pre-semiopen (r-pre-semiclosed) in (X, T) if and only if μ is fuzzy pre-semiopen (pre-semiclosed) in (X, T_r).*

Proof. Straightforward.

Theorem 2.11 *Let μ be a fuzzy set in a Chang's fuzzy topological space (X, T) and $r \in I_0$. Then μ is fuzzy pre-semiopen (pre-semiclosed) in (X, T) if and only if μ is fuzzy r-pre-semiopen(r-pre-semiclosed) in (X, T^r).*

Proof. Straightforward.

III. Fuzzy r-pre-semineighborhoods

Now, we are going to introduce the concepts of fuzzy

r-pre-semineighborhoods, fuzzy r-quasi-pre-semineighborhoods and investigate their properties.

Definition 3.1 Let x_α be a fuzzy point of a fuzzy topological space (X, T) and $r\in I_0$. Then a fuzzy set μ in X is called

(1) *a fuzzy r-pre-semineighborhood of x_α* if there is a fuzzy r-pre-semiopen set ρ in X such that $x_\alpha\in\rho\leq\mu$,

(2) *a fuzzy r-quasi-pre-semineighborhood of x_α* if there is a fuzzy r-pre-semiopen set ρ in X such that $x_\alpha q\rho\leq\mu$.

Theorem 3.2 *Let (X, T) be a fuzzy topological space and $r\in I_0$. Then*

(1) a fuzzy set μ in X is fuzzy r-pre-semiopen if and only if μ is a fuzzy r-pre-semineighborhood of x_α for every fuzzy point x_α in μ,

(2) a fuzzy set μ in X is fuzzy r-pre-semiopen if and only if μ is a fuzzy r-quasi-pre-semineighborhood of x_α for every fuzzy point x_α such that $x_\alpha q\mu$..

Proof. (1) Let μ be a fuzzy r-pre-semiopen set in X and $x_\alpha\in\mu$. Put $\rho=\mu$. Then ρ is fuzzy r-pre-semiopen in X and $x_\alpha\in\rho\leq\mu$. Thus μ is a fuzzy r-pre-semineighborhood of x_α. Conversely, let $x_\alpha\in\mu$. Since μ is a fuzzy r-pre-semineighborhood of x_α, there is a fuzzy r-pre-semiopen set ρ_{x_α} in X such that $x_\alpha\in\rho_{x_\alpha}\leq\mu$. So we have

$$\mu=\bigvee\{x_\alpha \mid x_\alpha\in\mu\}\leq\bigvee\{\rho_{x_\alpha} \mid x_\alpha\in\mu\}\leq\mu.$$

Hence $\mu=\bigvee\{\rho_{x_\alpha} \mid x_\alpha\in\mu\}$. Since each ρ_{x_α} is fuzzy r-pre-semiopen, μ is fuzzy r-pre-semiopen.

(2) Let μ be a fuzzy r-pre-semiopen set in X and $x_\alpha q\mu$. Put $\rho=\mu$. Then ρ is a fuzzy r-pre-semiopen in X and $x_\alpha q\rho\leq\mu$. Thus μ is a fuzzy r-quasi-pre-semineighborhood of x_α.

Conversely, let x_α be any fuzzy point in μ such that $\alpha<\mu(x)$. Then $x_{1-\alpha}q\mu$. By the hypothesis, μ is a fuzzy r-quasi-pre-semineighborhood of $x_{1-\alpha}$. Thus there is a fuzzy r-pre-semiopen set ρ_{x_α} in X such that $x_{1-\alpha}q\rho_{x_\alpha}\leq\mu$. Hence $\alpha<\rho_{x_\alpha}(x)$ and $\rho_{x_\alpha}\leq\mu$. So we have

$$\begin{aligned}\mu&=\bigvee\{x_\alpha \mid x_\alpha \text{ is a fuzzy point in } \mu \text{ such that } \alpha<\mu(x)\}\\ &\leq\bigvee\{\rho_{x_\alpha} \mid x_\alpha \text{ is a fuzzy point in } \mu \text{ such that } \alpha<\mu(x)\}\\ &\leq \mu.\end{aligned}$$

Hence

$$\mu=\bigvee\{\rho_{x_\alpha} \mid x_\alpha \text{ is a fuzzy point in } \mu \text{ such that } \alpha<\mu(x)\}.$$

Since each ρ_{x_α} is fuzzy r-pre-semiopen, μ is fuzzy r-pre-semiopen.

Theorem 3.3 *Let x_α be a fuzzy point in a fuzzy topological space (X, T) and $r\in I_0$. Then $x_\alpha\in \text{pscl}(\mu, r)$ if and only if $\rho q\mu$ for any fuzzy r-quasi-pre-semineighborhood ρ of x_α.*

Proof. Suppose that there is a fuzzy r-quasi-pre-semineighborhood ρ of x_α such that $\rho\not q\mu$. Then there is a fuzzy r-pre-semiopen set λ such that $x_\alpha q\lambda\leq\rho$. So $\lambda\not q\mu$ and hence $\mu\leq\lambda^c$. Since λ^c is fuzzy r-pre-semiclosed, $\text{pscl}(\mu, r)\leq \text{pscl}(\lambda^c, r)=\lambda^c$. On the other hand, since $x_\alpha q\lambda$, $x_\alpha\notin\lambda^c$. Hence $x_\alpha\notin \text{pscl}(\mu, r)$. It is a contradiction. Conversely, suppose $x_\alpha\notin \text{pscl}(\mu, r)$. Then there is a fuzzy r-pre-semiclosed set η such that $\mu\leq\eta$ and $x_\alpha\notin\eta$. Thus η^c is fuzzy r-pre-semiopen and $x_\alpha q\eta^c$, and hence η^c is a fuzzy r-quasi-pre-semineighborhood of x_α. By the hypothesis, $\eta^c q\mu$ and hence $\mu\nleq(\eta^c)^c=\eta$. It is a contradiction.

Remark 3.4

(1) Every fuzzy r-preneighborhood (r-quasi-preneighborhood) of x_α is a fuzzy r-pre-semineighborhood (r-quasi-pre-semineighborhood) of x_α.

(2) Every fuzzy r-semineighborhood(r-quasi-semineighborhood) of x_α is a fuzzy r-pre-semineighborhood(r-quasi-pre-semineighborhood) of x_α.

The next two theorems show the relation between a fuzzy pre-semineighborhood and a fuzzy r-pre- semineighborhood.

Theorem 3.5 *Let x_α be a fuzzy point in a fuzzy topological space (X, T) and $r\in I_0$. Then a fuzzy set μ is a fuzzy r-pre-semineighborhood(r-quasi-pre-semineighborhood) of x_α in (X, T) if and only if μ is a fuzzy pre-semineighborhood (quasi-pre-semineighborhood) of x_α in (X, T_r).*

Proof. Straightforward.

Theorem 3.6 *Let x_α be a fuzzy point of a Chang's fuzzy topological space (X, T) and $r\in I_0$. Then a fuzzy set μ is a fuzzy pre-semineighborhood (quasi-pre-semineighborhood) of x_α in (X, T) if and only if μ is a fuzzy r-pre-semineighborhood(r*

-quasi-pre-semineighborhood) of x_α *in* (X, T^r).

Proof. Straightforward.

Ⅳ. Fuzzy r-pre-semicontinuous maps

In this section, we investigate the properties of r-pre-semicontinuity, r-pre-semiopen and r-pre-semiclosed maps in fuzzy topological spaces and obtain the equivalent conditions of them.

Definition 4.1 Let $f: (X, T) \to (Y, U)$ be a map from a fuzzy topological space X to a fuzzy topological space Y and $r \in I_0$. Then f is said to be

(1) fuzzy r-pre-semicontinuous if $f^{-1}(\mu)$ is a fuzzy r-pre-semiopen set in X for each fuzzy r-open set μ in Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-pre-semiclosed set in X for each fuzzy r-closed set μ in Y,

(2) fuzzy r-pre-semiopen if $f(\mu)$ is a fuzzy r-pre-semiopen set in Y for each fuzzy r-open set μ in X,

(3) fuzzy r-pre-semiclosed if $f(\mu)$ is a fuzzy r-pre-semiclosed set in Y for each fuzzy r-closed set μ in X.

Remark 4.2 It is obvious that every fuzzy r-precontinuous(r-preopen, r-preclosed) map is also fuzzy r-pre-semicontinuous(r-pre-semiopen, r-pre-semiclosed). Also, every fuzzy r-semicontinuous(r-semiopen, r-semiclosed) map is also fuzzy r-pre-semicontinuous(r-pre-semiopen, r-pre-semiclosed). The converses are false by the following examples.

Example 4.3 Let $X = \{x\}$ and μ_1 and μ_2 be fuzzy sets of X defined as

$$\mu_1(x) = \frac{1}{4}, \quad \mu_2(x) = \frac{1}{3}.$$

Define $T_1 : I^X \to I$ and $T_2 : I^X \to I$ by

$$T_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly T_1 and T_2 are fuzzy topologies on X.

Consider the map $f: (X, T_1) \to (X, T_2)$ defined by $f(x) = x$.

(1) Then f is fuzzy $\frac{1}{2}$-pre-semicontinuous and not fuzzy $\frac{1}{2}$-precontinuous.

(2) f is fuzzy $\frac{1}{2}$-pre-semiopen and not fuzzy $\frac{1}{2}$-semiopen.

(3) f is fuzzy $\frac{1}{2}$-pre-semiclosed and not fuzzy $\frac{1}{2}$-semiclosed.

Consider the map $g: (X, T_2) \to (X, T_1)$ defined by $g(x) = x$.

(4) Then g is fuzzy $\frac{1}{2}$-pre-semicontinuous and not fuzzy $\frac{1}{2}$-semicontinuous.

(5) g is fuzzy $\frac{1}{2}$-pre-semiopen and not fuzzy $\frac{1}{2}$-preopen.

(6) g is fuzzy $\frac{1}{2}$-pre-semiclosed and not fuzzy $\frac{1}{2}$-preclosed.

The definition of fuzzy r-pre-semicontinuity can be restated in terms of fuzzy r-pre-semiclosure and fuzzy r-pre-semiinterior.

Theorem 4.4 *Let* $f: (X, T) \to (Y, U)$ *be a map and* $r \in I_0$. *Then the following statements are equivalent:*

(1) f *is fuzzy* r*-pre-semicontinuous.*

(2) $f(\operatorname{pscl}(\rho, r)) \le \operatorname{cl}(f(\rho), r)$ *for each fuzzy set* ρ *in* X.

(3) $\operatorname{pscl}(f^{-1}(\mu), r) \le f^{-1}(\operatorname{cl}(\mu, r))$ *for each fuzzy set* μ *in* Y.

(4) $f^{-1}(\operatorname{int}(\mu, r)) \le \operatorname{psint}(f^{-1}(\mu), r)$ *for each fuzzy set* μ *in* Y.

Proof. (1)$\Rightarrow$(2) Let ρ be a fuzzy set in X. Since $\operatorname{cl}(f(\rho), r)$ is a fuzzy r-closed set in Y, $f^{-1}(\operatorname{cl}(f(\rho), r))$ is a fuzzy r-pre-semiclosed set in X.

Since $\rho \le f^{-1}f(\rho)$,

$$\begin{aligned} \operatorname{pscl}(\rho, r) &\le \operatorname{pscl}(f^{-1}f(\rho), r) \\ &\le \operatorname{pscl}(f^{-1}(\operatorname{cl}(f(\rho), r)), r) \\ &= f^{-1}(\operatorname{cl}(f(\rho), r)). \end{aligned}$$

Hence we have

$$\begin{aligned} f(\operatorname{pscl}(\rho, r)) &\le ff^{-1}(\operatorname{cl}(f(\rho), r)) \\ &\le \operatorname{cl}(f(\rho), r). \end{aligned}$$

(2)$\Rightarrow$(3) Let μ be a fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. By (2),

$$f(\operatorname{pscl}(f^{-1}(\mu), r)) \le \operatorname{cl}(ff^{-1}(\mu), r) \le \operatorname{cl}(\mu, r)$$

Thus we have

$$\begin{aligned} \operatorname{pscl}(f^{-1}(\mu), r) &\le f^{-1}f(\operatorname{pscl}(f^{-1}(\mu), r)) \\ &\le f^{-1}(\operatorname{cl}(\mu, r)). \end{aligned}$$

(3)$\Rightarrow$(4) Let μ be a fuzzy set in Y. then μ^c is a fuzzy set in Y. By (3),

$$\mathrm{pscl}(f^{-1}(\mu)^c, r) = \mathrm{pscl}(f^{-1}(\mu^c), r) \leq f^{-1}(\mathrm{cl}(\mu^c, r)).$$

Thus we have

$$f^{-1}(\mathrm{int}(\mu, r)) = f^{-1}(\mathrm{cl}(\mu^c, r)^c) = f^{-1}(\mathrm{cl}(\mu^c, r))^c \leq \mathrm{pscl}(f^{-1}(\mu)^c, r)^c = \mathrm{psint}(f^{-1}(\mu), r).$$

(4) ⇒(1) Let μ be a fuzzy r-open set in Y. Then $\mathrm{int}(\mu, r) = \mu$. By(4),

$$f^{-1}(\mu) = f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{psint}(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu) = \mathrm{psint}(f^{-1}(\mu), r)$. Hence $f^{-1}(\mu)$ is a fuzzy r-pre-semiopen set in X. Therefore f is fuzzy r-pre-semicontinuous.

Theorem 4.5 *Let $f: (X, T) \to (Y, U)$ be a bijection and $r \in I_0$. Then the following statements are equivalent:*

(1) f is fuzzy r-pre-semicontinuous.

(2) $f(\mathrm{pscl}(\rho, r)) \leq \mathrm{cl}(f(\rho), r)$ for each fuzzy set ρ of X.

(3) $\mathrm{pscl}(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{cl}(\mu, r))$ for each fuzzy set μ of Y.

(4) $f^{-1}(\mathrm{int}(\mu, r)) \leq \mathrm{psint}(f^{-1}(\mu), r)$ for each fuzzy set μ of Y.

(5) $\mathrm{int}(f(\rho), r) \leq f(\mathrm{psint}(\rho, r))$ for each fuzzy set ρ of X.

Proof. By the above theorem, it suffices to show that(4) is equivalent to (5). Let ρ be any fuzzy set in X. Then $f(\rho)$ is a fuzzy set in Y. Since f is one-to-one,

$$f^{-1}(\mathrm{int}(f(\rho), r)) \leq \mathrm{psint}(f^{-1}f(\rho), r) = \mathrm{psint}(\rho, r).$$

Since f is onto,

$$\mathrm{int}(f(\rho), r) = ff^{-1}(\mathrm{int}(f(\rho), r)) \leq f(\mathrm{psint}(\rho, r)).$$

Conversely, let μ be any fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. Since f is onto,

$$\mathrm{int}(\mu, r) = \mathrm{int}(ff^{-1}(\mu), r) \leq f(\mathrm{psint}(f^{-1}(\mu), r)).$$

Since f is one-to-one,

$$f^{-1}(\mathrm{int}(\mu, r)) \leq f^{-1}f(\mathrm{psint}(f^{-1}(\mu), r)) = \mathrm{psint}(f^{-1}(\mu), r).$$

Hence the theorem follows.

Theorem 4.6 *Let $f: (X, T) \to (Y, U)$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) f is a fuzzy r-pre-semiopen map.

(2) $f(\mathrm{int}(\rho, r)) \leq \mathrm{psint}(f(\rho), r)$ for each fuzzy set ρ in X.

(3) $\mathrm{int}(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{psint}(\mu, r))$ for each fuzzy set μ in Y.

Proof. (1) ⇒(2) Let ρ be a fuzzy set in X. Then $\mathrm{int}(\rho, r)$ is a fuzzy r-open set in X. Since f is r-pre-semiopen, $f(\mathrm{int}(\rho, r))$ is a fuzzy r-pre-semiopen set in Y. Also since $f(\mathrm{int}(\rho, r)) \leq f(\rho)$,

$$f(\mathrm{int}(\rho, r)) = \mathrm{psint}(f(\mathrm{int}(\rho, r)), r) \leq \mathrm{psint}(f(\rho), r).$$

(2) ⇒(3) Let μ be a fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. By (2),

$$f(\mathrm{int}(f^{-1}(\mu), r)) \leq \mathrm{psint}(ff^{-1}(\mu, r)) \leq \mathrm{psint}(\mu, r).$$

Thus we have

$$\mathrm{int}(f^{-1}(\mu), r) \leq f^{-1}f(\mathrm{int}(f^{-1}(\mu), r)) \leq f^{-1}(\mathrm{psint}(\mu, r)).$$

(3) ⇒(1) Let ρ be a fuzzy r-open set in X. Then $\mathrm{int}(\rho, r) = \rho$ and $f(\rho)$ is a fuzzy set in Y. By (3),

$$\rho = \mathrm{int}(\rho, r) \leq \mathrm{int}(f^{-1}f(\rho), r) \leq f^{-1}(\mathrm{psint}(f(\rho), r)).$$

So we have

$$f(\rho) \leq ff^{-1}(\mathrm{psint}(f(\rho), r)) \leq \mathrm{psint}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \mathrm{psint}(f(\rho), r)$ and $f(\rho)$ is a fuzzy r-pre-semiopen set in Y. Therefore f is fuzzy r-pre-semiopen.

Theorem 4.7 *Let $f: (X, T) \to (Y, U)$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) f is fuzzy r-pre-semiclosed.

(2) $\mathrm{pscl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r))$ for each fuzzy set ρ of X.

Proof. (1) ⇒(2) Let ρ be a fuzzy set in X. Then $\mathrm{cl}(\rho, r)$ is a fuzzy r-closed set in X. Since f is r-pre-semiclosed, $f(\mathrm{cl}(\rho, r))$ is a fuzzy r-pre-semiclosed set in Y. Since $f(\rho) \leq f(\mathrm{cl}(\rho, r))$,

$$\mathrm{pscl}(f(\rho), r) \leq \mathrm{pscl}(f(\mathrm{cl}(\rho, r)), r) = f(\mathrm{cl}(\rho, r)).$$

(2) ⇒(1) Let ρ be a fuzzy r-closed set in X. Then $\mathrm{cl}(\rho, r) = \rho$ and $f(\rho)$ is a fuzzy set in Y. By (2),

$$\mathrm{pscl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r)) = f(\rho).$$

So we have $f(\rho) \leq \mathrm{pscl}(f(\rho), r) \leq f(\rho)$. Thus $f(\rho) = \mathrm{pscl}(f(\rho), r)$ and $f(\rho)$ is a fuzzy r-pre-semiclosed set in Y. Therefore f is fuzzy r-pre-semiclosed.

Theorem 4.8 *Let $f: (X, T) \to (Y, U)$ be a bijection and $r \in I_0$. Then the following statements are equivalent:*

(1) f is fuzzy r-pre-semiclosed.

(2) $\mathrm{pscl}(f(\rho), r) \leq f(\mathrm{cl}(\rho, r))$ for each fuzzy set ρ in X.

(3) $f^{-1}(\mathrm{pscl}(\mu, r)) \leq \mathrm{cl}(f^{-1}(\rho), r)$ for each fuzzy set μ in Y.

Proof. By the above theorem, it suffices to show that (2) is equivalent to (3). Let μ be any fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. Since f is onto,

$$\mathrm{pscl}(\mu, r) = \mathrm{pscl}(ff^{-1}(\mu), r) \leq f(\mathrm{cl}(f^{-1}(\mu), r)).$$

Since f is one-to-one,

$$f^{-1}(\text{ pscl}(\mu, r)) \leq f^{-1}f(\text{ cl}(f^{-1}(\mu), r) = \text{cl}(f^{-1}(\mu), r).$$

Conversely, let ρ be any fuzzy set in X. Then $f(\rho)$ is a fuzzy set in Y. Since f is one-to-one,

$$f^{-1}(\text{ pscl}(f(\rho), r)) \leq \text{ cl}(f^{-1}f(\rho), r) = \text{cl}(\rho, r).$$

Since f is onto,

$$\text{pscl}(f(\rho), r)) = ff^{-1}(\text{ pscl}(f(\rho), r) \leq f(\text{ cl}(\rho, r)).$$

Hence the theorem follows.

The next two theorems show that a fuzzy pre-semi-continuous map is a special case of a fuzzy r-pre-semicontinuous map.

Theorem 4.9 *Let* $f : (X, T) \rightarrow (Y, U)$ *be a map from a fuzzy topological space* X *to a fuzzy topological space* Y *and* $r \in I_0$. *Then* f *is fuzzy* r*-pre-semicontinuous (* r *-pre-semiopen and* r*-pre-semiclosed, respectively) if and only if* $f : (X, T_r) \rightarrow (Y, U_r)$ *is fuzzy pre-semi-continuous(pre-semiopen and pre-semiclosed, respectively).*

Proof. Straightforward.

Theorem 4.10 *Let* $f : (X, T) \rightarrow (Y, U)$ *be a map from a Chang's fuzzy topological space* X *to a Chang's fuzzy topological space* Y *and* $r \in I_0$. *Then* f *is fuzzy pre-semicontinuous (pre-semiopen and pre-semiclosed, respectively) if and only if* $f : (X, T^r) \rightarrow (Y, U^r)$ *is fuzzy* r*-pre-semicontinuous(* r*-pre-semiopen and* r*-pre-semiclosed, respectively).*

Proof. Straightforward.

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