

## ON GAP FUNCTIONS OF VARIATIONAL INEQUALITY IN A BANACH SPACE

SANGHO KUM AND GUE MYUNG LEE

**ABSTRACT.** In this paper we are concerned with theoretical properties of gap functions for the extended variational inequality problem (EVI) in a general Banach space. We will present a correction of a recent result of Chen et. al. without assuming the convexity of the given function  $\Omega$ . Using this correction, we will discuss the continuity and the differentiability of a gap function, and compute its gradient formula under two particular situations in a general Banach space. Our results can be regarded as infinite dimensional generalizations of the well-known results of Fukushima, and Zhu and Marcotte with some modifications.

### 1. Introduction

Given  $C$  closed and convex in  $\mathbf{R}^n$  and a function  $F : C \rightarrow \mathbf{R}^n$ , the variational inequality problem (in short, VI) is to find  $\bar{x} \in C$  such that

$$(1.1) \quad \langle F(\bar{x}), y - \bar{x} \rangle \geq 0 \text{ for all } y \in C,$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbf{R}^n$ . Due to its applications to such diverse areas as partial differential equations, mathematical economics, and operations research, VI has been investigated by many authors and various methods and algorithms to solve VI have been developed. Readers are referred to the comprehensive survey paper of Harker and Pang [5] and the references therein. In the past two decades, an interesting approach for solving VI has been made by introducing *gap functions*. A function  $\phi : C \rightarrow \mathbf{R} \cup \{+\infty\}$  is called a *gap function* for VI if

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- (i)  $\phi(x) \geq 0$  for all  $x \in C$ ; and
- (ii)  $\phi(\bar{x}) = 0$  if and only if  $\bar{x}$  solves VI.

With a gap function, one can reformulate VI as the following equivalent minimization problem:

$$\text{minimize } \phi(x) \quad \text{subject to } x \in C.$$

A gap function, e.g., Auslender's gap function [1], is in general nondifferentiable. However, Fukushima [4] proposed the projective gap function and provided an equivalent differentiable optimization problem formulation of VI, and treated a wide variety of applications. On the other hand, Zhu and Marcotte [8] studied a very general gap function which includes, as special cases, the projective function of Fukushima [4]. Recently, Chen et. al. [2] gave an interpretation of the meaning 'gap' in terms of Young's inequality. In addition, they dealt with a modification [2, Theorem 4.1] and an extension [2, Theorem 5.1] of Zhu and Marcotte [8, Theorems 3.1 and 5.1]. All the above results are presented in the Euclidean space  $\mathbf{R}^n$ .

In this paper we are concerned with theoretical properties of gap functions for the extended variational inequality problem (EVI) [2] in a general Banach space. To be more specific, first we are going to find a condition under which the function  $\phi$  (4.1) in Chen et. al. [2] can be a gap function for EVI. Then we will discuss the continuity and the differentiability of the gap function under two particular situations. As the beginning of our discussion, we provide a counterexample which shows that Chen et. al. [2, Theorem 4.1] is not correct.

## 2. Preliminaries

Consider the following EVI.

**EVI :** Let  $E$  be a real Banach space and  $E^*$  be its dual space. Given a nonempty closed convex subset  $C$  of  $E$ , a function  $F : C \rightarrow E^*$  and a proper convex lower semicontinuous function  $f : E \rightarrow \mathbf{R} \cup \{+\infty\}$ , find  $\bar{x} \in C$  such that

$$(2.1) \quad \langle F(\bar{x}), x - \bar{x} \rangle \geq f(\bar{x}) - f(x) \quad \text{for all } x \in C,$$

where  $\langle \cdot, \cdot \rangle$  denote the dual pairing on  $E \times E^*$ . If  $f$  is the indicator function on  $C$ , EVI (2.1) becomes VI (1.1).

For a locally Lipschitz function  $g : E \rightarrow \mathbf{R}$  on  $E$ , the Clarke generalized gradient [3]  $\partial g(x)$  is well defined for each  $x \in E$ . Some properties of  $\partial g$  necessary for our argument in the sequel are as follows.

**PROPOSITION 2.1.** *If  $g$  is continuously (Gâteaux) differentiable at  $\bar{x}$ , then  $g$  is locally Lipschitz at  $\bar{x}$  and  $\partial g(\bar{x}) = \{\nabla g(\bar{x})\}$ , where  $\nabla g(\bar{x})$  is the Gâteaux derivative of  $g$  at  $\bar{x}$ .*

*Proof.* See Clarke [3, Propositions 2.2.2 and 2.2.4] □

**PROPOSITION 2.2.** *If  $g$  is convex and locally Lipschitz at  $\bar{x}$ ,  $\partial g(\bar{x})$  coincides with the subdifferential of  $g$  at  $\bar{x}$  in the sense of convex analysis.*

*Proof.* See Clarke [3, Proposition 2.2.7] □

**PROPOSITION 2.3.** *If  $g$  is locally Lipschitz at  $\bar{x}$ ,  $C$  is a convex subset of  $E$  and  $g$  attains a minimum over  $C$  at  $\bar{x}$ , then  $0 \in \partial g(\bar{x}) + N_C(\bar{x})$ , where  $N_C(\bar{x}) = \{x^* \in E^* \mid \langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in C\}$ , the normal cone of  $C$  at  $\bar{x}$ .*

*Proof.* See Clarke [3, Corollary and Proposition 2.4.4] □

**PROPOSITION 2.4.** *If  $g_i$  ( $i = 1, 2, \dots, n$ ) is a finite family of functions which is locally Lipschitz at  $\bar{x}$ , then their sum  $\Sigma g_i$  is also locally Lipschitz at  $\bar{x}$ , and  $\partial(\Sigma g_i)(\bar{x}) \subset \Sigma \partial g_i(\bar{x})$ .*

*Proof.* See Clarke [3, Proposition 2.3.3] □

Now we introduce an extra function  $\Omega : C \times C \rightarrow \mathbf{R}$ . Let  $\Omega(x, y)$  be nonnegative, and for each  $x \in C$ ,  $\Omega(x, \cdot)$  is continuously differentiable (not necessarily convex) on  $C$ . We further assume that  $\Omega(x, x) = 0$  and  $\nabla_y \Omega(x, x) = 0$  for all  $x \in C$ . Here  $\nabla_y \Omega(x, x)$  denotes the Gâteaux derivative of  $\Omega(x, y)$  with respect to the second variable, evaluated at  $y = x$ .

Our main goal is to show that the following function  $\phi : C \rightarrow \mathbf{R}$  is a gap function for EVI (2.1) and  $\phi$  is continuous and differentiable under suitable conditions;

$$(2.2) \quad \phi(x) := \sup_{y \in C} \{ \langle F(x), x - y \rangle + f(x) - f(y) - \Omega(x, y) \}.$$

### 3. An example and a correction

Chen et. al. [2, Theorem 4.1] asserted that the above  $\phi$  (2.2) is a gap function for EVI (2.1) under the following general assumptions on  $\Omega$  in the case that  $E = C = \mathbf{R}^n$ .

**Assertion.** Let  $\Omega : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$  be nonnegative and  $\Omega(x, x) = 0$  for all  $x \in \mathbf{R}^n$ . Let  $\Omega(x, \cdot)$  be convex for each  $x \in \mathbf{R}^n$ , and  $\mathbf{0} \in \partial_y \Omega(x, x)$ , where  $\partial_y \Omega(x, x)$  is the subdifferential of  $\Omega(x, y)$  with respect to the second variable, evaluated at  $y = x$ . Assume that  $f : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{+\infty\}$  is a proper convex lower semicontinuous function. Then  $\phi$  (2.2) is a gap function for EVI (2.1)

However, we can find a simple example which shows that Assertion is not correct.

**EXAMPLE 3.1.** Define two functions  $F$  and  $f : \mathbf{R} \rightarrow \mathbf{R}$  to be the identity function on  $\mathbf{R}$ . Define another function  $\Omega : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  to be

$$\Omega(x, y) = |x - y|.$$

We can easily check that all the conditions of Assertion are satisfied. Indeed,  $\partial_y \Omega(x, x) = [-1, 1]$  contains the origin  $0$ . By a direct computation, we get

$$\begin{aligned} \phi(x) &:= \sup_{y \in \mathbf{R}} \{ \langle F(x), x - y \rangle + f(x) - f(y) - \Omega(x, y) \} \\ &= \sup_{y \in \mathbf{R}} \{ x(x - y) + x - y - |x - y| \} \\ &= \delta_{[-2, 0]}(x), \end{aligned}$$

where  $\delta_{[-2, 0]}(x)$  is the indicator function of the closed interval  $[-2, 0]$  defined by

$$\delta_{[-2, 0]}(x) = \begin{cases} 0, & \text{if } x \in [-2, 0], \\ +\infty, & \text{if } x \notin [-2, 0]. \end{cases}$$

On the other hand,  $\bar{x}$  is a solution of EVI (2.1) if and only if

$$\begin{aligned} \langle F(\bar{x}), x - \bar{x} \rangle &\geq f(\bar{x}) - f(x) \quad \forall x \in \mathbf{R} \Leftrightarrow \bar{x}(x - \bar{x}) \geq \bar{x} - x \quad \forall x \in \mathbf{R} \\ &\Leftrightarrow (\bar{x} + 1)(x - \bar{x}) \geq 0 \quad \forall x \in \mathbf{R} \\ &\Leftrightarrow \bar{x} = -1. \end{aligned}$$

Thus the unique solution of EVI (2.1) is  $\bar{x} = -1$ . This implies that  $\phi(x) = \delta_{[-2,0]}(x)$  is not a gap function for EVI (2.1). Therefore Assertion does not hold even in the one dimensional case  $\mathbf{R}$ .

REMARKS 3.1. (i) The mistake in the proof of Assertion in Chen et. al. [2, Theorem 4.1] resides in the statement that “the solution  $\mathbf{x}$  of the convex optimization problem also solves the following variational inequality :

$$\text{Find } \mathbf{x} \in \mathbf{R}^n \text{ such that } \langle q, y - \mathbf{x} \rangle \geq 0 \quad \forall y \in \mathbf{R}^n,$$

where  $q \in \partial_y \Phi(x, x)$ ” (see Chen et. al. [2, p.667]). However this is not true in general if  $\Phi(x, \cdot)$  is not (continuously) differentiable.

(ii) Even though  $F$  is a  $C^\infty$  monotone operator and  $f$  is also a  $C^\infty$  convex real valued function,  $\phi$  is not a gap function for EVI.

We change the assumption on  $\Omega(x, \cdot)$  to get a correction of Chen et. al. [2, Theorem 4.1] in a Banach space. The point of change is to impose the differentiability condition on  $\Omega(x, \cdot)$  and to remove the convexity of  $\Omega(x, \cdot)$  instead.

THEOREM 3.1. *Let  $E$  be a real Banach space and  $E^*$  be its dual space. Let  $C$  be a nonempty closed convex subset of  $E$ ,  $F : C \rightarrow E^*$  be a function, and  $f : E \rightarrow \mathbf{R}$  be a convex continuous function. Let  $\Omega : C \times C \rightarrow \mathbf{R}$  be nonnegative and for each  $x \in C$ ,  $\Omega(x, \cdot)$  be continuously Gâteaux differentiable (not necessarily convex) on  $C$ . Assume further that for each  $x \in C$ ,  $\Omega(x, x) = 0$  and  $\nabla_y \Omega(x, x) = 0$ . Then  $\phi$  (2.2) is a gap function for EVI (2.1).*

*Proof.* Since  $\Omega(x, x) = 0$  for each  $x \in C$ , it is clear that  $\phi(x) \geq 0$  for each  $x \in C$ . Now assume that  $\bar{x}$  solves EVI (2.1). Then

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq f(\bar{x}) - f(x) \text{ for all } x \in C.$$

As  $\Omega(\bar{x}, x) \geq 0$  for all  $x \in C$ ,

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq f(\bar{x}) - f(x) - \Omega(\bar{x}, x) \text{ for all } x \in C,$$

and  $\phi(\bar{x}) \leq 0$ . Hence  $\phi(\bar{x}) = 0$ .

Conversely, assume that  $\phi(\bar{x}) = 0$ . This implies that

$$\langle F(\bar{x}), \bar{x} - y \rangle + f(\bar{x}) - f(y) - \Omega(\bar{x}, y) \leq 0 \text{ for all } y \in C.$$

Thus  $\bar{x}$  is a solution of the following nonconvex and nondifferentiable optimization problem :

$$\text{minimize } \langle F(\bar{x}), y \rangle + f(y) + \Omega(\bar{x}, y) \quad \text{subject to } y \in C.$$

By Propositions 2.1 to 2.4, we have

$$0 \in F(\bar{x}) + \partial f(\bar{x}) + \nabla_y \Omega(\bar{x}, \bar{x}) + N_C(\bar{x}).$$

Since  $\nabla_y \Omega(\bar{x}, \bar{x}) = 0$ , there exists an  $x^* \in \partial f(\bar{x})$  such that  $-F(\bar{x}) - x^* \in N_C(\bar{x})$ . Hence  $\langle -F(\bar{x}) - x^*, x - \bar{x} \rangle \leq 0$  for all  $x \in C$ , that is,

$$(3.1) \quad \langle F(\bar{x}), x - \bar{x} \rangle + \langle x^*, x - \bar{x} \rangle \geq 0 \text{ for all } x \in C.$$

Because  $f$  is convex and  $x^* \in \partial f(\bar{x})$ , we have

$$(3.2) \quad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \text{ for all } x \in C.$$

From (3.1) and (3.2),  $\langle F(\bar{x}), x - \bar{x} \rangle + f(x) - f(\bar{x}) \geq 0$  for all  $x \in C$ . Therefore  $\bar{x}$  solves EVI (2.1), as desired.  $\square$

REMARKS 3.2. We would like to point out that Theorem 3.1 generalizes Zhu and Marcotte [8, Theorem 3.1] in the following manners: (i) By adopting the Clarke generalized gradient [3], we could remove the strong convexity assumption on  $\Omega(x, \cdot)$ , which is essential in Zhu and Marcotte [8, Theorem 3.1]. Thus it is not necessary to suppose the strong convexity assumption on  $\Omega(x, \cdot)$  only for the result that  $\phi$  is a gap function for EVI (2.2); (ii) Theorem 3.1 is obtained in a Banach space whereas Zhu and Marcotte [8, Theorem 3.1] is done in  $\mathbf{R}^n$ ; (iii) Theorem 3.1 is an extension of Zhu and Marcotte [8, Theorem 3.1] from VI to EVI.

#### 4. The continuity and differentiability of $\phi$

In this section we are going to investigate the continuity and differentiability of  $\phi$  for two special cases. The first case (Theorem 4.1) deals with the projective gap function of Fukushima [4] in a Hilbert space  $H$  under the assumption that  $C = E = H$ . The second one (Theorem 4.3) is concerned with an infinite dimensional modification of Chen et. al. [2, Theorem 3.3] in a reflexive Banach space. We derive the first case.

THEOREM 4.1. *Let  $H$  be a real Hilbert space and  $H^*$  be its dual space. Let  $F : H \rightarrow H^*$  be a continuous operator, and  $A : H \rightarrow H^*$  be*

a continuous linear and symmetric operator in the sense that  $\langle Ax, y \rangle = \langle Ay, x \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the usual pairing on  $H^* \times H$ . Assume further that the bilinear form  $g : H \times H \rightarrow \mathbf{R}$  defined by  $g(x, y) = \langle Ax, y \rangle$  is coercive on  $H$ , that is, there exists  $\alpha > 0$  such that

$$g(x, x) = \langle Ax, x \rangle \geq \alpha \|x\|^2 \text{ for all } x \in H.$$

Then  $\phi(x) = \sup_{y \in H} \{ \langle F(x), x - y \rangle - \frac{1}{2} \langle A(x - y), x - y \rangle \}$  is a continuous gap function for VI (1.1). If  $F$  is continuously Gâteaux differentiable, then  $\phi$  is also continuously Gâteaux differentiable, and its gradient is given by

$$\nabla \phi(x) = J_*(Fx) \circ \nabla F(x).$$

Here  $J_*$  is the duality mapping from  $H^*$  to  $H^{**}$ .

*Proof.* First note that the bilinear form  $g(x, y) = \langle Ax, y \rangle$  defines an inner product on  $H$  and the induced norm  $\|x\|_A = \sqrt{g(x, x)} = \sqrt{\langle Ax, x \rangle}$  is equivalent to the original norm  $\|x\|$  on  $H$ . Thus we may assume that  $H$  is equipped with the norm  $\|x\|_A = \sqrt{g(x, x)} = \sqrt{\langle Ax, x \rangle}$  induced by the inner product  $g(x, y) = \langle Ax, y \rangle$ . Now we reformulate  $\phi$  (2.2) as follows;

$$\begin{aligned} \phi(x) &= \sup_{y \in H} \{ \langle F(x), x - y \rangle + f(x) - f(y) - \Omega(x, y) \} \\ &= \langle F(x), x \rangle + f(x) + \sup_{y \in H} \{ \langle -F(x), y \rangle - (f + \Omega_x)(y) \} \\ (4.1) \quad &= \langle F(x), x \rangle + f(x) + (f + \Omega_x)^*(-Fx), \end{aligned}$$

where  $\Omega_x(y) = \Omega(x, y)$  for each  $x, y \in E$ , and  $(f + \Omega_x)^*$  denotes the Fenchel conjugate of  $f + \Omega_x$  defined by  $(f + \Omega_x)^*(x^*) = \sup_{y \in E} \langle x^*, y \rangle - (f + \Omega_x)(y)$ . Taking  $f \equiv 0$  and  $\Omega(x, y) = \frac{1}{2} \langle A(x - y), x - y \rangle$  in (4.1), we obtain

$$\begin{aligned} \phi(x) &= \sup_{y \in H} \{ \langle F(x), x - y \rangle - \frac{1}{2} \langle A(x - y), x - y \rangle \} \\ (4.2) \quad &= \langle F(x), x \rangle + \Omega_x^*(-Fx) = \frac{1}{2} \|Fx\|_*^2, \end{aligned}$$

where  $\|\cdot\|_*$  denotes the equivalent dual norm on  $H^*$  induced by  $\|x\|_A$

on  $H$ . Indeed, for each  $y^* \in H^*$ , we have

$$\begin{aligned}
 \Omega_x^*(y^*) &= \sup_{y \in H} [\langle y^*, y \rangle - \frac{1}{2} \langle A(y-x), y-x \rangle] \\
 &= \sup_{z \in H} [\langle y^*, z+x \rangle - \frac{1}{2} \langle Az, z \rangle] \\
 &= \langle y^*, x \rangle + \sup_{z \in H} [\langle y^*, z \rangle - \frac{1}{2} \|z\|_A^2] \\
 &= \langle y^*, x \rangle + \sup_{\lambda \geq 0} \sup_{\|z\|_A = \lambda} [\langle y^*, z \rangle - \frac{1}{2} \lambda^2] \\
 &= \langle y^*, x \rangle + \sup_{\lambda \geq 0} [\lambda \|y^*\|_* - \frac{1}{2} \lambda^2] \\
 &= \langle y^*, x \rangle + \frac{1}{2} \|y^*\|_*^2.
 \end{aligned}$$

Hence

$$\Omega_x^*(-Fx) = \langle -Fx, x \rangle + \frac{1}{2} \|Fx\|_*^2.$$

Therefore (4.2) does hold. Since  $\Omega(x, y) = \frac{1}{2} \|y-x\|_A^2$ ,  $\Omega(x, y)$  is non-negative and clearly  $\Omega(x, x) = 0$ . Moreover, for each  $x \in H$ , we can directly compute

$$\partial_y(\Omega_x)(y) = J(y-x) \circ Id_H$$

where  $J$  is the duality mapping defined by

$$J(x) = \{x^* \in H^* \mid \langle x^*, x \rangle = \|x^*\|_* \cdot \|x\|_A \text{ and } \|x^*\|_* = \|x\|_A\}$$

(see Phelps [7, 2.26 Example, p.27]), and  $Id_H$  is the identity operator on  $H$ . In particular, the duality mapping  $J$  is the canonical isomorphism between  $H$  and  $H^*$  (Actually  $J$  can be identified with the identity operator on  $H = H^*$  in this sense). Thus  $J$  is a single-valued continuous operator, so  $\partial_y(\Omega_x)(y) = J(y-x) = \nabla_y(\Omega_x)(y)$ , and  $\Omega_x$  is continuously Gâteaux differentiable. In addition,  $\nabla_y(\Omega_x)(x) = \nabla_y\Omega(x, x) = J(x-x) = J(0) = 0$ , which implies that  $\phi$  is a gap function for VI (1.1) by Theorem 3.1. Then the continuity and differentiability of  $\phi$  immediately follows from (4.2). The gradient of  $\phi$  is given by

$$\nabla\phi(x) = J_*(Fx) \circ \nabla F(x). \quad \square$$

REMARKS 4.1. (i) In the case  $H = \mathbf{R}^n$ , the operator  $A$  is represented by an  $n \times n$  positive definite symmetric matrix  $A$  (we denote it by  $A$  for



notational simplicity). In this case we can easily show that the bilinear form  $g(x, y) = y^t Ax$  is coercive. Hence the coercivity assumption on  $A$  in Theorem 4.1 is natural. Therefore Theorem 4.1 is a Hilbert space version of Fukushima [4, Theorem 3.2] with the modification  $C = H$ .

(ii) As pointed out in Chen et. al. [2], in Fukushima [4], the gradient formula requires the solution of another optimization problem. However, in the gradient formula of Theorem 4.1, we don't need to solve any optimization problem.

We need some more definitions and propositions to reach our second case. Let  $E$  be a Banach space. A function  $f : E \rightarrow \mathbf{R} \cup \{+\infty\}$  is said to be *coercive* when

$$\lim_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

**PROPOSITION 4.1.** *For a proper convex lower semicontinuous, and coercive function  $f : E \rightarrow \mathbf{R} \cup \{+\infty\}$ ,  $f^*(x^*) < +\infty$  for all  $x^* \in E^*$ , that is,  $\text{dom} f^* = E^*$ . In this case,  $f^*$  is a continuous convex real-valued function on  $E^*$ .*

*Proof.* For the proof of the first part, refer to Hiriart-Urruty [6, Proposition 1.3.8, p. 46, Chapter X]. For the proof of the second part, see Phelps [7, Proposition 3.3].

A function  $f : E \rightarrow \mathbf{R}$  is said to be *strongly convex* with modulus  $a$  ( $a > 0$ ) if for all  $x, y \in E$ , and  $\alpha \in [0, 1]$ , we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) - \frac{1}{2}a\alpha(1 - \alpha)\|x - y\|^2.$$

For the sake of completeness, we provide a proof of the following.

**PROPOSITION 4.2.** *A continuous and strongly convex function with modulus  $a$  ( $a > 0$ )  $f : E \rightarrow \mathbf{R}$  is coercive.*

*Proof.* For  $x, y \in E$  and  $\alpha \in (0, 1)$ , we have

$$(4.3) \quad f(y) \geq f(x) + \langle x^*, y - x \rangle + \frac{1}{2}a\|y - x\|^2 \quad \text{for all } x^* \in \partial f(x).$$

Indeed,

$$\frac{f((1 - \alpha)x + \alpha y) - f(x)}{\alpha} \leq f(y) - f(x) - \frac{1}{2}a(1 - \alpha)\|y - x\|^2.$$

Hence,

$$\lim_{\alpha \downarrow 0} \frac{f(x + \alpha(y - x)) - f(x)}{\alpha} \leq f(y) - f(x) - \frac{1}{2}a\|y - x\|^2.$$

Since

$$\langle x^*, y - x \rangle \leq \lim_{\alpha \downarrow 0} \frac{f(x + \alpha(y - x)) - f(x)}{\alpha} \quad \text{for all } x^* \in \partial f(x),$$

we get

$$\langle x^*, y - x \rangle \leq f(y) - f(x) - \frac{1}{2}a\|y - x\|^2,$$

as desired. Dividing (4.3) by  $\|y\|$  yields

$$\frac{f(y)}{\|y\|} \geq \frac{f(x)}{\|y\|} - \frac{\|x^*\|(\|y\| + \|x\|)}{\|y\|} + \frac{1}{2}a \frac{(\|y\| - \|x\|)^2}{\|y\|}.$$

Thus

$$\lim_{\|y\| \rightarrow +\infty} \frac{f(y)}{\|y\|} \geq +\infty.$$

This completes the proof.  $\square$

Now we are in a position to establish the second case in which  $\Omega \equiv 0$ .

**THEOREM 4.2.** *Let  $E$  be a real Banach space and  $E^*$  be its dual space equipped with the norm topology. Let  $F : C \rightarrow E^*$  be a continuous operator, and  $f : E \rightarrow \mathbf{R}$  be a continuous and strongly convex function with modulus  $a$  ( $a > 0$ ). Then  $\phi(x) = \sup_{y \in C} \{\langle F(x), x - y \rangle + f(x) - f(y)\}$ , is a continuous gap function for EVI (2.1).*

*Proof.* Since  $\Omega \equiv 0$  in (2.2),  $\phi$  is obviously a gap function for EVI (2.1) by Theorem 3.1. Thus we only need to verify the continuity of  $\phi$ . In fact,

$$\begin{aligned} \phi(x) &= \sup_{y \in C} \{\langle F(x), x - y \rangle + f(x) - f(y)\} \\ &= \langle F(x), x \rangle + f(x) + \sup_{y \in E} \{\langle -F(x), y \rangle - f(y) - \delta_C(y)\} \\ (4.4) \quad &= \langle F(x), x \rangle + f(x) + (f + \delta_C)^*(-F(x)), \end{aligned}$$

where  $\delta_C$  denotes the indicator function of  $C$ . By Proposition 4.2,  $f$  is coercive, hence  $f + \delta_C$  is so. Thus  $(f + \delta_C)^*$  is continuous by means of Proposition 4.1. Therefore  $\phi$  is clearly continuous if  $F$  is continuous. This completes the proof.  $\square$

For the differentiability of  $\phi$ , we assume that  $E$  is reflexive and  $C = E$ .

**THEOREM 4.3.** *Let  $E$  be a real reflexive Banach space and  $E^*$  be its dual space equipped with the norm topology. Let  $F : E \rightarrow E^*$  be a continuously Gâteaux differentiable operator, and  $f : E \rightarrow \mathbf{R}$  be a continuously Gâteaux differentiable, and strongly convex function with modulus  $a$  ( $a > 0$ ). Then  $\phi$  (4.4), the gap function for EVI (2.1), is continuously Gâteaux differentiable, and its gradient is given by*

$$\nabla\phi(x) = F(x) + \langle \nabla F(x)(\cdot), x \rangle + \nabla f(x) - (\nabla f)^{-1}(-Fx) \circ \nabla F(x).$$

*Proof.* Since  $C = E$ ,  $\delta_C \equiv 0$ . So (4.4) becomes

$$\phi(x) = \langle F(x), x \rangle + f(x) + f^*(-Fx).$$

To achieve our result, we have only to verify that  $f^*$  and  $h(x) = \langle F(x), x \rangle$  are continuously Gâteaux differentiable.

**Step 1.**  $f^*$  is continuously Gâteaux differentiable.

It is well known that  $\partial f^* = (\partial f)^{-1}$ . Hence for the differentiability of  $f^*$ , it suffices to show that  $\partial f^*$  is single-valued, that is,  $\partial f(x_1) \cap \partial f(x_2) = \emptyset$  whenever  $x_1 \neq x_2$ . Suppose  $x^* \in \partial f(x_1) \cap \partial f(x_2)$ , where  $x_1 \neq x_2$ . Then we have

$$(4.5) \quad f^*(x^*) + f(x_i) = \langle x^*, x_i \rangle \text{ for } i = 1, 2.$$

According to Young's inequality, it follows from (4.5) that

$$f^*(x^*) + \sum_{i=1}^2 \alpha_i f(x_i) = \langle x^*, \sum_{i=1}^2 \alpha_i x_i \rangle \leq f^*(x^*) + f(\sum_{i=1}^2 \alpha_i x_i),$$

where  $\sum_{i=1}^2 \alpha_i = 1$ ,  $\alpha_i \geq 0$  for  $i = 1, 2$ . This implies that  $\sum_{i=1}^2 \alpha_i f(x_i) = f(\sum_{i=1}^2 \alpha_i x_i)$  on the line segment  $[x_1, x_2]$ , which contradicts the strong convexity of  $f$ .

It remains to check that  $\partial f^* = \nabla f^*$  is continuous. We can easily deduce the following from (4.3):

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq a \|x_1 - x_2\|^2,$$

where  $x_i^* \in \partial f(x_i)$ , i.e.  $x_i = \nabla f^*(x_i^*)$  for  $i = 1, 2$ . Thus we have

$$\|\nabla f^*(x_1^*) - \nabla f^*(x_2^*)\| \leq \frac{1}{a} \|x_1^* - x_2^*\| \text{ for all } x_1^*, x_2^* \in E^*,$$

which implies that  $\nabla f^*$  is Lipschitzian with constant  $1/a$  on  $E^*$ .

**Step 2.**  $h(x) = \langle F(x), x \rangle$  is continuously Gâteaux differentiable.

To do this, we directly compute the Gâteaux derivative of  $h$ . For each  $x, v \in E$ ,

$$\begin{aligned} \nabla h(x)(v) &= \lim_{t \rightarrow 0} \frac{h(x + tv) - h(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\langle F(x + tv), x + tv \rangle - \langle F(x), x \rangle}{t} \\ &= \lim_{t \rightarrow 0} \left\langle \frac{F(x + tv) - F(x)}{t}, x \right\rangle + \langle F(x + tv), v \rangle \\ &= \langle \nabla F(x)(v), x \rangle + \langle F(x), v \rangle. \end{aligned}$$

Thus

$$\nabla h(x) = F(x) + \langle \nabla F(x)(\cdot), x \rangle,$$

which shows that  $h$  is continuously Gâteaux differentiable, as desired. Furthermore the gradient formula  $\nabla \phi(x)$  immediately comes from (4.4) as follows;

$$\begin{aligned} \nabla \phi(x) &= \nabla h(x) + \nabla f(x) + \nabla f^*(-Fx) \circ (-\nabla F(x)) \\ &= F(x) + \langle \nabla F(x)(\cdot), x \rangle + \nabla f(x) - (\nabla f)^{-1}(-Fx) \circ \nabla F(x). \end{aligned}$$

This completes our proof.  $\square$

**REMARK 4.2.** For the continuity of the gap function  $\phi$ , we did not need to suppose that  $C = E$ . However,  $C = E$  was to be assumed for the differentiability of  $\phi$ . Theorem 4.3 is an infinite dimensional modification of Chen et. al. [2, Theorem 3.3] in a reflexive Banach space.

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Sangho Kum

Department of Applied Mathematics

Korea Maritime University

Pusan 606-791, Korea

*E-mail:* kum@hanara.kmaritime.ac.kr

Gue Myung Lee

Department of Applied Mathematics

Pukyong National University

Pusan 608-737, Korea

*E-mail:* gmlee@dolphin.pknu.ac.kr