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HYERS-ULAM STABILITY OF THE QUADRATIC EQUATION OF PEXIDER TYPE

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Abstract. In this paper, we will prove the Hyers-Ulam stability of the quadratic functional equation of Pexider type, $f_1(x+y)+f_2(x-y)=f_3(x)+f_4(y)$.

1. Introduction

Given an operator T and a solution class $\{u\}$ with the property that $T(u)=0$, when does $\|T(v)\| \le \varepsilon$ for an $\varepsilon > 0$ imply that $\|u-v\| \le \delta(\varepsilon)$ for some u and for some $\delta > 0$? This problem is called the stability of the functional transformation. A great deal of work has been done in connection with the ordinary and partial differential equations. In 1940 S. M. Ulam [23] asked the following problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist." Further, in 1968 S. M. Ulam [24] proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In 1978 P. M. Gruber [10] proposed the following Ulam type problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects, satisfying the property exactly?" According to P. M. Gruber this kind of stability problems is of particular interest in probability theory.

If f is a function from a normed vector space into a Banach space and satisfies $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$, D. H. Hyers [11] proved that there exists an additive function A such that $\|f(x)-A(x)\| \le \varepsilon$. If $f(x)$ is a real continuous function of x over $\mathbf{R}$, and $|f(x+y)-f(x)-f(y)| \le \varepsilon$, it was shown by D. H. Hyers and S. M. Ulam [14] that there exists a constant k such that $|f(x)-kx| \le 2\varepsilon$. Taking these results into account, we say that the additive Cauchy equation $f(x+y)=f(x)+f(y)$ is stable

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in the sense of Hyers and Ulam. The interested reader should refer to [7, 8, 9, 12, 13, 15, 16, 20] for an indepth account on the subject of stability of functional equations.

The quadratic function $f(x) = cx^2$ ($x \in \mathbf{R}$) satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1}$$

Hence, the above equation is called the quadratic functional equation, and every solution of the quadratic equation (1) is called a quadratic function. It is well known that a function $f : E_1 \to E_2$ between vector spaces is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_1 \to E_2$, which is additive in x for each fixed y, such that $f(x) = B(x,x)$ for any $x \in E_1$ (see [1]).

In Section 2, we will introduce some theorems which are indispensable for proving our main result of this paper. In the last section, we will apply some ideas from [17] and [18] to the proof of Hyers-Ulam stability of the quadratic functional equation of Pexider type, $f_1(x+y) + f_2(x-y) = f_3(x) + f_4(y)$.

2. Preliminaries

In 1941, D. H. Hyers [11] presented a celebrated theorem concerning the stability of additive functions between Banach spaces. Later in 1987, G. L. Forti [6] extended the domain of the relevant functions of Hyers's theorem to an arbitrary amenable semigroup. Indeed, Forti proved the following theorem:

THEOREM 1. *Let G be an amenable semigroup and let E be a Banach space. If a function $f : G \to E$ satisfies the inequality*

$$\|f(xy) - f(x) - f(y)\| \le \varepsilon$$

for some $\varepsilon \ge 0$ and for all $x, y \in G$, then there exists a unique homomorphism $H : G \to E$ such that

$$\|f(x) - H(x)\| \le \varepsilon$$

for all $x \in G$.

The Hyers-Ulam stability of the quadratic functional equation (1) was first proved by F. Skof [22] for functions from a normed space into a Banach space. P. W. Cholewa [3] demonstrated that Skof's theorem is also valid if the relevant domain is replaced by an abelian group. Later,

S. Czerwik [4] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation which includes the following theorem as a special case:

THEOREM 2. *Let E_1 be a normed space and let E_2 be a Banach space. If a function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \;\leq\; \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \;\leq\; \frac{1}{3}(\varepsilon + \|f(0)\|)$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then $Q(tx) = t^2 Q(x)$ for all $t \in \mathbf{R}$ and $x \in E_1$.

Theorem of Czerwik was further generalized by J. M. Rassias [19], and also by C. Borelli and G. L. Forti [2]. In [5], S. Czerwik investigated the stability problem of the 'partially pexiderized' quadratic functional equation, $f_1(x+y)+f_1(x-y) = f_2(x)+f_2(y)$. For more information on the stability of the quadratic functional equation, one can refer to [21].

3. Hyers-Ulam stability

In this section, let $(E_1, \|\cdot\|)$ be a real normed space and $(E_2, \|\cdot\|)$ a Banach space.

In the following theorem, we will apply some ideas from [17] and [18] to the proof of Hyers-Ulam stability of the quadratic functional equation of Pexider type.

THEOREM 3. *If functions $f_1, f_2, f_3, f_4 : E_1 \to E_2$ satisfies the inequality*

$$\|f_1(x+y) + f_2(x-y) - f_3(x) - f_4(y)\| \;\leq\; \varepsilon \tag{2}$$

for some $\varepsilon \geq 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ and exactly two additive functions A_1, $A_2 : E_1 \to$

E_2 such that

$$
(3)\qquad \begin{cases} \|f_1(x) - Q(x) - A_1(x) - A_2(x) - f_1(0)\| \le \frac{137}{3}\,\varepsilon, \\ \|f_2(x) - Q(x) - A_1(x) + A_2(x) - f_2(0)\| \le \frac{125}{3}\,\varepsilon, \\ \|f_3(x) - 2Q(x) - 2A_1(x) - f_3(0)\| \le \frac{136}{3}\,\varepsilon, \\ \|f_4(x) - 2Q(x) - 2A_2(x) - f_4(0)\| \le \frac{124}{3}\,\varepsilon \end{cases}
$$

for all $x \in E_1$. Moreover, if $f_3(tx)$ and $f_4(tx)$ are continuous in $t \in \mathbf{R}$ for each $x \in E_1$, then the Q satisfies $Q(tx) = t^2 Q(x)$ for all $x \in E_1$ and A_1, A_2 are linear.

Proof. Let us define $F_i(x) = f_i(x) - f_i(0)$, and by F_i^e and F_i^o denote the even part and the odd part of F_i for $i = 1, 2, 3, 4$. Then, we get $F_i(0) = F_i^e(0) = F_i^o(0) = 0$ for $i = 1, 2, 3, 4$.

By putting $x = y = 0$ in (2) and using the resulting inequality and (2), we have

$$
(4)\qquad \|F_1(x+y) + F_2(x-y) - F_3(x) - F_4(y)\| \le 2\varepsilon
$$

for all $x, y \in E_1$. First we replace x and y in (4) by $-x$ and $-y$, respectively, to get

$$
(5)\qquad \|F_1(-x-y) + F_2(-x+y) - F_3(-x) - F_4(-y)\| \le 2\varepsilon.
$$

Next we add (subtract) the argument of the norm of the inequality (5) to (from) that of the inequality (4) and then taking the norm and manipulating the resulting expression, we obtain

$$
(6)\qquad \|F_1^e(x+y) + F_2^e(x-y) - F_3^e(x) - F_4^e(y)\| \le 2\varepsilon,
$$

$$
(7)\qquad \|F_1^o(x+y) + F_2^o(x-y) - F_3^o(x) - F_4^o(y)\| \le 2\varepsilon
$$

for all $x, y \in E_1$.

If we put $y = 0$, $x = 0$ (and replace y by x), $y = x$, and $y = -x$ in (6) respectively, then we get

$$\|F_1^e(x) + F_2^e(x) - F_3^e(x)\| \leq 2\varepsilon, \tag{8}$$

$$\|F_1^e(x) + F_2^e(x) - F_4^e(x)\| \leq 2\varepsilon, \tag{9}$$

$$\|F_1^e(2x) - F_3^e(x) - F_4^e(x)\| \leq 2\varepsilon, \tag{10}$$

$$\|F_2^e(2x) - F_3^e(x) - F_4^e(x)\| \leq 2\varepsilon \tag{11}$$

for all $x \in E_1$, respectively.

In view of (8) and (9), we see that

$$\|F_3^e(x) - F_4^e(x)\| \leq 4\varepsilon, \tag{12}$$

and it follows from (10) and (11) that

$$\|F_1^e(x) - F_2^e(x)\| \leq 4\varepsilon \tag{13}$$

for any x in E_1. By using (6), (12) and (13), we have

$$\begin{aligned} &\|F_2^e(x+y) + F_2^e(x-y) - F_4^e(x) - F_4^e(y)\| \\ &\quad \leq \|F_1^e(x+y) + F_2^e(x-y) - F_3^e(x) - F_4^e(y)\| \\ &\qquad + \|F_2^e(x+y) - F_1^e(x+y)\| + \|F_3^e(x) - F_4^e(x)\| \\ &\quad \leq 10\varepsilon. \end{aligned} \tag{14}$$

By putting $y = 0$ in (14), we get

$$\|2F_2^e(x) - F_4^e(x)\| \leq 10\varepsilon. \tag{15}$$

Hence, (14) and (15) imply

$$\begin{aligned} &\|F_4^e(x+y) + F_4^e(x-y) - 2F_4^e(x) - 2F_4^e(y)\| \\ &\quad \leq 2\|F_2^e(x+y) + F_2^e(x-y) - F_4^e(x) - F_4^e(y)\| \\ &\qquad + \|F_4^e(x+y) - 2F_2^e(x+y)\| + \|F_4^e(x-y) - 2F_2^e(x-y)\| \\ &\quad \leq 40\varepsilon \end{aligned}$$

for all $x, y \in E_1$.

By Theorem 2, there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|F_4^e(x) - 2Q(x)\| \leq \frac{40}{3}\varepsilon \tag{16}$$

for all $x \in E_1$. Furthermore, if $f_4(tx)$ is continuous in $t \in \mathbf{R}$ for each $x \in E_1$, then the quadratic function Q satisfies $Q(tx) = t^2 Q(x)$ for all $x \in E_1$.

On account of (12), (13), (15) and (16), we get

$$\begin{aligned} &\|F_1^e(x) - Q(x)\| \\ &\leq \|F_1^e(x) - F_2^e(x)\| + \left\|F_2^e(x) - \frac{1}{2}F_4^e(x)\right\| + \left\|\frac{1}{2}F_4^e(x) - Q(x)\right\| \\ (17) \quad &\leq \frac{47}{3}\,\varepsilon, \end{aligned}$$

$$\begin{aligned} &\|F_2^e(x) - Q(x)\| \\ (18) \qquad &\leq \left\|F_2^e(x) - \frac{1}{2}F_4^e(x)\right\| + \left\|\frac{1}{2}F_4^e(x) - Q(x)\right\| \leq \frac{35}{3}\,\varepsilon, \end{aligned}$$

$$(19)\ \ \|F_3^e(x) - 2Q(x)\| \leq \|F_3^e(x) - F_4^e(x)\| + \|F_4^e(x) - 2Q(x)\| \leq \frac{52}{3}\,\varepsilon$$

for any $x \in E_1$.

As before, if we put $y = 0$, $x = 0$ (and replace y by x), $y = x$, and $y = -x$ in (7) separately, then we obtain

$$(20) \qquad \|F_1^o(x) + F_2^o(x) - F_3^o(x)\| \leq 2\varepsilon,$$

$$(21) \qquad \|F_1^o(x) - F_2^o(x) - F_4^o(x)\| \leq 2\varepsilon,$$

$$(22) \qquad \|F_1^o(2x) - F_3^o(x) - F_4^o(x)\| \leq 2\varepsilon,$$

$$(23) \qquad \|F_2^o(2x) - F_3^o(x) + F_4^o(x)\| \leq 2\varepsilon$$

for all $x \in E_1$, respectively.

Due to (20) and (21), we have

$$\begin{aligned} &\|2F_1^o(x) - F_3^o(x) - F_4^o(x)\| \\ &\leq \|F_1^o(x) + F_2^o(x) - F_3^o(x)\| + \|F_1^o(x) - F_2^o(x) - F_4^o(x)\| \\ (24) \quad &\leq 4\varepsilon \end{aligned}$$

and

$$\begin{aligned} &\|2F_2^o(x) - F_3^o(x) + F_4^o(x)\| \\ &\leq \|F_1^o(x) + F_2^o(x) - F_3^o(x)\| + \|F_2^o(x) + F_4^o(x) - F_1^o(x)\| \\ (25) \quad &\leq 4\varepsilon \end{aligned}$$

for each $x \in E_1$.

Combining (22) with (24) yields

$$\begin{aligned} &\|F_3^o(2x) + F_4^o(2x) - 2F_3^o(x) - 2F_4^o(x)\| \\ &\quad \leq \|F_3^o(2x) + F_4^o(2x) - 2F_1^o(2x)\| \\ &\quad\quad + \|2F_1^o(2x) - 2F_3^o(x) - 2F_4^o(x)\| \\ &\quad \leq 8\varepsilon. \end{aligned} \tag{26}$$

Analogously, by (23) and (25), we get

$$\begin{aligned} &\|F_3^o(2x) - F_4^o(2x) - 2F_3^o(x) + 2F_4^o(x)\| \\ &\quad \leq \|F_3^o(2x) - F_4^o(2x) - 2F_2^o(2x)\| \\ &\quad\quad + \|2F_2^o(2x) - 2F_3^o(x) + 2F_4^o(x)\| \\ &\quad \leq 8\varepsilon \end{aligned} \tag{27}$$

for any $x \in E_1$. Now it follows from (26) and (27) that

$$\begin{aligned} \|F_3^o(2x) - 2F_3^o(x)\| &\leq \left\|\frac{1}{2}F_3^o(2x) + \frac{1}{2}F_4^o(2x) - F_3^o(x) - F_4^o(x)\right\| \\ &\quad + \left\|\frac{1}{2}F_3^o(2x) - \frac{1}{2}F_4^o(2x) - F_3^o(x) + F_4^o(x)\right\| \\ &\leq 8\varepsilon \end{aligned} \tag{28}$$

and analogously

$$\|F_4^o(2x) - 2F_4^o(x)\| \leq 8\varepsilon \tag{29}$$

for all $x \in E_1$.

In view of (7), (24), (25), (28), and (29), we have

$$\begin{aligned} &\|F_3^o(x+y) + F_4^o(x+y) + F_3^o(x-y) \\ &\qquad - F_4^o(x-y) - F_3^o(2x) - F_4^o(2y)\| \\ &\leq \|2F_1^o(x+y) + 2F_2^o(x-y) - 2F_3^o(x) - 2F_4^o(y)\| \\ &\quad + \|F_3^o(x+y) + F_4^o(x+y) - 2F_1^o(x+y)\| \\ &\quad + \|F_3^o(x-y) - F_4^o(x-y) - 2F_2^o(x-y)\| \\ &\quad + \|2F_3^o(x) - F_3^o(2x)\| + \|2F_4^o(y) - F_4^o(2y)\| \\ &\leq 28\varepsilon \end{aligned} \tag{30}$$

for all $x, y \in E_1$. If we replace y in (30) by $-y$ and then using the fact that F_4^o is an odd function, we get

$$\begin{aligned} &\|F_3^o(x-y) + F_4^o(x-y) + F_3^o(x+y) \\ &\quad -F_4^o(x+y) - F_3^o(2x) + F_4^o(2y)\| \\ &\leq 28\varepsilon \end{aligned} \tag{31}$$

From (30) and (31), we get

$$\begin{aligned} &\|F_3^o(x+y) + F_3^o(x-y) - F_3^o(2x)\| \\ &= \frac{1}{2}\|F_3^o(x+y) + F_4^o(x+y) + F_3^o(x-y) - F_4^o(x-y) \\ &\quad -F_3^o(2x) - F_4^o(2y) + F_3^o(x-y) + F_4^o(x-y) \\ &\quad +F_3^o(x+y) - F_4^o(x+y) - F_3^o(2x) + F_4^o(2y)\| \\ &\leq \frac{1}{2}\|F_3^o(x-y) + F_4^o(x-y) + F_3^o(x+y) - F_4^o(x+y) \\ &\quad -F_3^o(2x) + F_4^o(2y)\| + \frac{1}{2}\|F_3^o(x-y) + F_4^o(x-y) \\ &\quad +F_3^o(x+y) - F_4^o(x+y) - F_3^o(2x) + F_4^o(2y)\| \\ &\leq 28\varepsilon. \end{aligned} \tag{32}$$

Similarly, from (30) and (31), we get

$$\begin{aligned} &\|F_4^o(x+y) - F_4^o(x-y) - F_4^o(2y)\| \\ &= \frac{1}{2}\|F_3^o(x+y) + F_4^o(x+y) + F_3^o(x-y) \\ &\quad -F_4^o(x-y) - F_3^o(2x) - F_4^o(2y) \\ &\quad -\{F_3^o(x-y) + F_4^o(x-y) + F_3^o(x+y) \\ &\quad -F_4^o(x+y) - F_3^o(2x) + F_4^o(2y)\}\| \\ &\leq \frac{1}{2}\|F_3^o(x-y) + F_4^o(x-y) + F_3^o(x+y) \\ &\quad -F_4^o(x+y) - F_3^o(2x) + F_4^o(2y)\| \\ &\quad +\frac{1}{2}\|F_3^o(x-y) + F_4^o(x-y) + F_3^o(x+y) \\ &\quad -F_4^o(x+y) - F_3^o(2x) + F_4^o(2y)\| \\ &\leq 28\varepsilon. \end{aligned} \tag{33}$$

By letting $u = x + y$ and $v = x - y$ in (32), we obtain

$$\|F_3^o(u) + F_3^o(v) - F_3^o(u+v)\| \leq 28\varepsilon$$

for all $u, v \in E_1$. According to Theorem 1, there exists a unique additive function $A_1 : E_1 \to E_2$ such that

$$\|F_3^o(x) - 2A_1(x)\| \leq 28\varepsilon \tag{34}$$

for all x in E_1. If, moreover, $f_3(tx)$ is continuous in $t \in \mathbf{R}$ for every fixed $x \in E_1$, then A_1 is a linear function.

By putting $u = x - y$ and $v = 2y$ in (33), we get

$$\|F_4^o(u+v) - F_4^o(u) - F_4^o(v)\| \leq 28\varepsilon$$

for all $u, v \in E_1$. By Theorem 1 again, there exists a unique additive function $A_2 : E_1 \to E_2$ such that

$$\|F_4^o(x) - 2A_2(x)\| \leq 28\varepsilon \tag{35}$$

for any x in E_1. Furthermore, if $f_4(tx)$ is continuous in $t \in \mathbf{R}$ for all $x \in E_1$, then A_2 is also linear.

From (24), (25), (34) and (35) it follows that

$$\begin{aligned}
&\|F_1^o(x) - A_1(x) - A_2(x)\| \\
&\quad \leq \left\|F_1^o(x) - \frac{1}{2}F_3^o(x) - \frac{1}{2}F_4^o(x)\right\| + \left\|\frac{1}{2}F_3^o(x) - A_1(x)\right\| \\
&\qquad + \left\|\frac{1}{2}F_4^o(x) - A_2(x)\right\| \\
&\quad \leq 30\varepsilon
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&\|F_2^o(x) - A_1(x) + A_2(x)\| \\
&\quad \leq \left\|F_2^o(x) - \frac{1}{2}F_3^o(x) + \frac{1}{2}F_4^o(x)\right\| + \left\|\frac{1}{2}F_3^o(x) - A_1(x)\right\| \\
&\qquad + \left\|A_2(x) - \frac{1}{2}F_4^o(x)\right\| \\
&\quad \leq 30\varepsilon
\end{aligned} \tag{37}$$

for each x in E_1.

The inequalities in (3) are direct consequences of the inequalities (16), (17), (18), (19), (34), (35), (36) and (37).

Now, let Q', A_1', $A_2' : E_1 \to E_2$ be another quadratic function and additive functions, respectively, satisfying the inequalities in (3) instead of Q, A_1 and A_2. Then, we have

$$\begin{aligned}
&\|Q(x)+A_2(x)-Q'(x)-A_2'(x)\| \\
&\leq \left\|-\frac{1}{2}f_4(x)+Q(x)+A_2(x)+\frac{1}{2}f_4(0)\right\| \\
&\quad + \left\|\frac{1}{2}f_4(x)-Q'(x)-A_2'(x)-\frac{1}{2}f_4(0)\right\| \\
&\leq \frac{124}{3}\varepsilon \qquad (38)
\end{aligned}$$

for each $x \in E_1$. Replacing x by $-x$ in (38) and using the fact that quadratic functions are even and additive functions are odd, we get

$$\|Q(x)-A_2(x)-Q'(x)+A_2'(x)\| \leq \frac{124}{3}\varepsilon. \qquad (39)$$

From (38) and (39), we see that

$$\begin{aligned}
&\|Q(x)-Q'(x)\| \\
&= \frac{1}{2}\|Q(x)+A_2(x)-Q'(x)-A_2'(x) \\
&\qquad +Q(x)-A_2(x)-Q'(x)+A_2'(x)\| \\
&\leq \frac{1}{2}\|Q(x)+A_2(x)-Q'(x)-A_2'(x)\| \\
&\qquad +\frac{1}{2}\|Q(x)-A_2(x)-Q'(x)+A_2'(x)\| \\
&\leq \frac{124}{3}\varepsilon. \qquad (40)
\end{aligned}$$

Similarly, again from (38) and (39), we get

$$\begin{aligned}
&\|A_2(x)-A_2'(x)\| \\
&= \frac{1}{2}\|Q(x)+A_2(x)-Q'(x)-A_2'(x) \\
&\qquad -\{Q(x)-A_2(x)-Q'(x)+A_2'(x)\}\| \\
&\leq \frac{1}{2}\|Q(x)+A_2(x)-Q'(x)-A_2'(x)\| \\
&\qquad +\frac{1}{2}\|Q(x)-A_2(x)-Q'(x)+A_2'(x)\| \\
&\leq \frac{124}{3}\varepsilon. \qquad (41)
\end{aligned}$$

Hence (40) and (41) imply that $Q(x) = Q'(x)$ and $A_2(x) = A_2'(x)$ for every $x \in E_1$. Similarly, we can show that $A_1(x) = A_1'(x)$ for any $x \in E_1$. Now the proof of the theorem is complete. $\square$

REMARK. Let us denote by $f_i \equiv f_j$ that $f_i(x) = f_j(x)$ for all $x \in E_1$. We can easily verify the following statements:

(i) If $f_1 \equiv f_2$ in Theorem 3, then $A_2 \equiv 0$;

(ii) If $f_3 \equiv f_4$ in Theorem 3, then $A_1 \equiv A_2$;

(iii) If $2f_1 \equiv f_3$ in Theorem 3, then $A_2 \equiv 0$;

(iv) If $2f_2 \equiv f_4$ in Theorem 3, then $A_1 \equiv 2A_2$;

(v) If $2f_1 \equiv f_4$ in Theorem 3, then $A_1 \equiv 0$;

(vi) If $2f_2 \equiv f_3$ in Theorem 3, then $A_2 \equiv 0$.

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