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ON THE EXTENSION PROBLEM IN THE ADAMS SPECTRAL SEQUENCE CONVERGING TO $BP_*(\Omega^2 S^{2n+1})$

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Abstract. Ravenel computed the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$ and got the E_∞–term. Then he gave the conjecture about the extension. Here we prove that there should be non–trivial extension. We also study the BP_*BP comodule structures on the polynomial algebras which are related with $BP_*(\Omega^2 S^{2n+1})$.

1. Introduction

The generalized cohomology theory complex cobordism is defined by the unitary Thom spectrum MU. The spectrum MU for complex bordism has played an important role in stable homotopy. Localized the spectrum MU at a prime p, it splits as wedges of suspensions of the similar spectra BP which we call the Brown-Peterson spectrum. The corresponding homology theory for this spectrum is called the Brown-Peterson homology, the BP-homology for short. The BP theory was also proven to be very useful in stable homotopy, especially Adams Novikov spectral sequence. But practically it is never easy to compute the BP theory. Like the ordinary homology, it is essential to understand the BP–homology of $\Omega^2 S^{2n+1}$. Ravenel computed the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$ and got the E_∞–term. Then he gave the conjecture about the extension [7].

In this paper we prove that there should be non–trivial extension in the spectral sequence. We also study the BP_*BP comodule structures on the polynomial algebras which are related with $BP_*(\Omega^2 S^{2n+1})$.

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2. Preliminaries

In this section we will give some basic facts about the spectra MU, BP. The general reference for these spectra is the book by Ravenel[5]. According to Brown's representation theorem, every homology theory has its corresponding spectrum, a collections of spaces with structure maps. The spectrum for the complex bordism is the sequences of the Thom space $MU(n)$ of the classifying space $BU(n)$ for the unitary group $U(n)$ with the structure maps $\Sigma^2 MU(n-1) \to MU(n)$ induced by the map from $BU(n-1)$ with the universal bundle $\xi_{n-1} \bigoplus C$ into $BU(n)$ with ξ_n. Exploiting the map $CP^\infty \simeq MU(1) \to MU$ and the fact that $H_*(CP^\infty; Z)$ is free on $\beta_i \in H_{2i}(CP^\infty; Z)$, $i \geq 0$, we get

$$H_*(MU; Z) = Z[b_1, b_2, \dots].$$

For $p = 2$ the dual Steenrod algebra is $A_* = Z/(2)[\xi_1, \xi_2, \cdots]$ with $\xi_i = 2^i - 1$, and for p odd primes $A_* = Z/(p)[\xi_1, \xi_2, \cdots] \otimes E(\tau_0, \tau_1, \cdots)$ with $\xi_i = 2(p^i - 1)$ and dim $\tau_i = 2p^i - 1$. Using the Adams spectral sequence with

$$E_2 = \mathrm{Ext}_{A_*}(Z/(p), H_*(MU; Z/(p)))$$

converging to p-primary part of $\pi_*(MU)$ and the nice A_* comodule structure of $H_*(MU)$, Milnor[3] computed

$$\pi_*(MU) = MU_* = Z[x_2, x_4, \dots]$$

where dim $x_{2i} = 2i$. Localized the spectrum MU at a prime p, Quillen[4] constructed a multiplicative idempotent map ϵ of ring spectra:

$$\epsilon : MU_{(p)} \to MU_{(p)}.$$

For any space X consider the map $\epsilon \wedge 1 : MU_{(p)} \wedge X \to MU_{(p)} \wedge X$. Then the image of ϵ_* becomes a natural direct summand of $MU_*(X)_{(p)}$ and it satisfies all the axioms for the generalized homology theory, so by the Brown's representation theorem it has its representing spectrum. We denote it by BP and the homology theory by $BP_*(X)$ with

$$\pi_*(BP) = BP_* = Z_{(p)}[v_1, v_2, \dots]$$

where $Z_{(p)}$ denotes the integers localized at p and dim $v_i = 2(p^i - 1)$. There are new polynomial generators m_n for $H_*(MU)$ satisfying $m_n = [CP^n]/(n+1) \in \pi_{2n}(MU) \otimes Q$ such that

$$H_*(BP) = Z_{(p)}[m_{p-1}, m_{p^2-1}, \dots].$$

Let $\ell_i \in BP_* \otimes Q$ denote the image of m_{p^i-1} under the Quillen idempotent $\epsilon : MU_{(p)} \to MU_{(p)}$ where $\ell_0 = 1$. The polynomial generator $v_i \in BP_*$ are related to the ℓ_i recursively by the formula of Araki [5],

$$p\ell_n = \sum_{0 \le i \le n} \ell_i v_{n-i}^{p^i}.$$

Quillen found some strong connection between the bordism theory and the formal group law. There is a formal group law $F(x,y) \in BP_*[[x,y]]$ associated with BP given by

$$F(x,y) = \exp\ (\log x + \log y)$$

where $\log x = \sum_{i \ge 0} \ell_i x^{p^i}$ and $\exp\ (\log x) = x$. We will denote exp $(\sum_{i \ge 0} \log a_i)$ by $\sum_i^F a_i$. We recall the following result due to Quillen.

THEOREM 1. [4,6] *As a ring,*

$$BP_*BP \cong BP_*[t_1, t_2, \cdots]$$

where $t_i \in BP_{2(p^i-1)}BP$.
(i) The left unit η_L is the standard inclusion $BP_* \to BP_*BP$ while the right unit η_R is given by

$$\eta_R(\ell_k) = \sum_{i=0}^{k} \ell_i t_{k-i}^{p^i}.$$

(ii) The counit ε has $\varepsilon(1) = 1$, $\varepsilon(t_i) = 0$, $i > 0$.
(iii) The coproduct ψ is computed by

$$\sum_{i=0}^{k} \ell_i(\psi(t_{k-i}))^{p^i} = \sum_{h+i+j=k} \ell_h t_i^{p^h} \otimes t_j^{p^{h+i}}$$

or

$$\sum_{i \ge 0}^{F} \psi(t_i) = \sum_{i,j \ge 0}^{F} t_i \otimes t_j^{p^i}.$$

2. BP_*BP comodule structure

We recall the following definition in [6].

DEFINITION 2. *A left comodule over a Hopf algebroid (S, Σ) is a left S–module M together a left S–linear map $\psi : M \to \Sigma \otimes_S M$ which is counitary and coassociative, i.e., the composite*

$$M \xrightarrow{\psi} \Sigma \otimes_S M \xrightarrow{\varepsilon \otimes 1_M} S \otimes_S M \cong M$$

is the identity on M and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & \Sigma \otimes_S M \\ \Big\downarrow \psi & & \Big\downarrow 1_\Sigma \otimes \psi \\ \Sigma \otimes_S M & \xrightarrow{\Delta \otimes 1_M} & \Sigma \otimes_S \Sigma \otimes_S M \end{array}$$

commute. In this case, ψ is called a left Σ–comodule map. A left comodule M over a Hopf algebroid (S, Σ) is usually called a left Σ–comodule M. An element $m \in M$ is primitive if $\psi(m) = 1 \otimes m$.

We need to recall the following well–known fact.

THEOREM 3. [1] *There are choices of generators x_i, y_i such that*
(a) *For $p = 2$, $H_*(\Omega^2 S^{2n+1}; Z/(2)) = Z/(2)[x_i : i \geq 0]$*

$$\beta x_{i+1} = x_i^2 \quad \text{for } i \geq 1$$

(b) *For p odd primes,*

$$H_*(\Omega^2 S^{2n+1}; Z/(p)) = E(x_i : i \geq 0) \otimes Z/(p)[y_i : i > 0]$$
$$\beta x_{i+1} = y_i \quad \text{for } i \geq 1$$
$$\mathcal{P} y_{i+1} = y_i^p \quad \text{for } i \geq 1$$

where dim $x_i = 2np^i - 1$ and dim $y_i = 2np^i - 2$.

Using the Adams spectral sequence converging to $\pi_*(BP \wedge \Omega^2 S^{2n+1}) = BP_*(\Omega^2 S^{2n+1})$ with

$$\begin{aligned} E_2 &= \mathrm{Ext}_{A_*}(Z/(p), H_*(BP \wedge \Omega^2 S^{2n+1}; Z/(p))) \\ &= \mathrm{Ext}_{A_*}(Z/(p), H_*(BP; Z/(p))) \otimes H_*(\Omega^2 S^{2n+1}; Z/(p)), \end{aligned}$$

Ravenel showed that the spectral sequence collapses at the E_2–term and got the following result.

THEOREM 4. [7] *For each prime p and each integer $n > 0$, the E_∞–term of the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$ is*

$$E(x_0) \otimes BP_*[y_i : i > 0]/(r_1, r_2, \cdots) \text{ where } r_i = \sum_{0 \le j < i} v_j y_{i-j}^{p^j}.$$

The E_∞ of the Adams spectral sequence is the associated bigraded module for I–adic filtration of $BP_*(\Omega^2 S^{2n+1})$, that is, $E_\infty^{s,*} = I^s BP_* (\Omega^2 S^{2n+1}) / I^{s+1}$ where $v_0 = p$ and $I = (p, v_1, v_2, \cdots)$. Since $y_i \in E_\infty^{0,*}$ and $v_j \in E_\infty^{1,*}$ for each i, j, we have that in $BP_*(\Omega^2 S^{2n+1})$

$$\sum_{0 \le j < i} v_j y_{i-j}^{p^j} = 0 \bmod I^2 .$$

For the extension problems arising from above relations, Ravenel gave the following conjecture.

CONJECTURE 5. [7]

$$BP_*(\Omega^2 S^{2n+1}) = E(x_0) \otimes BP_*[y_i : i > 0]/L,$$

where L is generated by the homogeneous components of the formal group law sum expression $\sum_{0 \le j < i}^F v_j y_{i-j}^{p^j}$.

Let $M_0 = BP_*[y_i : i > 0]$. Then we can define the left BP_*BP–comodule map on M_0 using the coproduct of BP_*BP.

THEOREM 6. *M_0 is a left BP_*–module with a left BP_*–linear map $\psi : M_0 \to BP_*BP \otimes_{BP_*} M_0$,*

$$\sum_{j>0}^F y_j \xrightarrow{\psi} \sum_{i \ge 0, j > 0}^F t_i \otimes y_j^{p^i}$$

which is counitary and coassociative, that is, M_0 is a left BP_*BP comodule.

Proof. First we show the counitarity, $(\varepsilon \otimes 1_{M_0}) \circ \psi \cong i_{M_0}$ using Theorem 1. Taking log for $\psi : \sum_{j>0}^F y_j \to \sum_{i\geq 0, j>0}^F t_i \otimes y_j^{p^i}$, we have

$$\begin{aligned}\psi(\sum_{i\geq 0, j>0} \ell_i y_j^{p^i}) &= \sum_{i,j\geq 0, k>0} \ell_i t_j^{p^i} \otimes y_k^{p^{i+j}} \\ &= \sum_{i\geq 0, j>0} \eta_R(\ell_i) \otimes y_j^{p^i} .\end{aligned}$$

Then we have

$$\begin{aligned}(\varepsilon \otimes 1_{M_0}) \circ \psi(\sum_{i\geq 0, j>0} \ell_i y_j^{p^i}) &= (\varepsilon \otimes 1_{M_0})(\sum_{i\geq 0, j>0} \eta_R(\ell_i) \otimes y_j^{p^i}) \\ &= (\varepsilon \otimes 1_{M_0})(\sum_{i\geq 0, j>0} 1 \otimes \ell_i y_j^{p^i}) \\ &= \sum_{i\geq 0, j>0} 1 \otimes \ell_i y_j^{p^i} \\ &\cong \sum_{i\geq 0, j>0} \ell_i y_j^{p^i} .\end{aligned}$$

Hence we have $(\varepsilon \otimes 1_{M_0}) \circ \psi \cong i_{M_0}$. Next we show the coassociativity, $(1_{BP_*BP} \otimes \psi) \circ \psi = (\triangle \otimes 1_{M_0}) \circ \psi$.

$$\begin{aligned}(\triangle \otimes 1_{M_0}) \circ \psi(\sum_{i\geq 0, j>0} \ell_i y_j^{p^i}) &= (\triangle \otimes 1_{M_0})(\sum_{i,j\geq 0, k>0} \ell_i t_j^{p^i} \otimes y_k^{p^{i+j}}) \\ &= \sum_{i,j,k\geq 0, m>0} \ell_i t_j^{p^i} \otimes t_k^{p^{i+j}} \otimes y_m^{p^{i+j+k}} \\ &= \sum_{i,j\geq 0, k>0} 1 \otimes \ell_i t_j^{p^i} \otimes y_k^{p^{i+j}}\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(1_{BP_*BP} \otimes \psi) \circ \psi(\sum_{i\geq 0, j>0} \ell_i y_j^{p^i}) &= (1_{BP_*BP} \otimes \psi)(\sum_{i,j\geq 0, k>0} \ell_i t_j^{p^i} \otimes y_k^{p^{i+j}}) \\ &= (1_{BP_*BP} \otimes \psi)(\sum_{i\geq 0, j>0} \eta_R(\ell_i) \otimes y_j^{p^i}) \\ &= (1_{BP_*BP} \otimes \psi)(\sum_{i\geq 0, j>0} 1 \otimes \ell_i y_j^{p^i}) \\ &= \sum_{i,j\geq 0, k>0} 1 \otimes \ell_i t_j^{p^i} \otimes y_k^{p^{i+j}} .\end{aligned}$$

Then we have $(1_{BP_*BP}\otimes\psi)\circ\psi=(\triangle\otimes 1_{M_0})\circ\psi$. Hence M_0 is a left BP_*BP comodule. □

THEOREM 7. *$I=(r_1,r_2,\cdots)$ is an invariant ideal, that is,*

$$\psi(I)\subset BP_*BP\otimes_{BP_*}I$$

Proof. We have that

$$\sum_{n>0}^{F} r_n=\sum_{i\geq 0,j>0}^{F} v_i y_j^{p^i}.$$

From the Araki's formula, $pl_n=\sum_{0\leq i\leq n}\ell_i v_{n-i}^{p^i}$, we have

$$\sum_{i,j\geq 0,k>0}\ell_i v_j^{p^i} y_k^{p^{i+j}}=\sum_{i\geq 0,j>0} p\ell_i y_j^{p^i}.$$

Taking exp to both sides, we have

$$\sum_{i\geq 0,j>0}^{F} v_i y_j^{p^i}=\exp\Big(\sum_{i\geq 0,j>0} p\ell_i y_j^{p^i}\Big).$$

Taking log both sides, we have $\sum_{i\geq 0,j>0}\ell_i r_j^{p^i}=\sum_{i\geq 0,j>0}p\ell_i y_j^{p^i}$. In the proof of Theorem 6, we know that $\psi(\ell_i y_j^{p^i})=1\otimes\ell_i y_j^{p^i}$, that is, $\ell_i y_j^{p^i}$ is comodule primitive. Hence

$$\psi\Big(\sum_{i\geq 0,j>0}\ell_i r_j^{p^i}\Big)=\psi\Big(\sum_{i\geq 0,j>0}p\ell_i y_j^{p^i}\Big)=1\otimes\sum_{i\geq 0,j>0}p\ell_i y_j^{p^i}=1\otimes\sum_{i\geq 0,j>0}\ell_i r_j^{p^i}.$$

Therefore $\psi(r_j^{p^i})\in BP_*BP\otimes I$ for all $i\geq 0, j>0$. Taking $i=0$, we have $\psi(r_j)\in BP_*BP\otimes I$ for all $j>0$, so that $\psi(I)\subset BP_*BP\otimes I$. Therefore $I=(r_1,r_2,\cdots)$ is an invariant ideal. □

COROLLARY 8. *M_n which is defined by $BP_*[y_i : i > 0]/(r_1, \cdots, r_n)$ is also a left BP_*BP comodule for each n.*

Here we confront the following two questions.

Question 1. Is there only unique way to give a comodule structure on M_0?

Question 2. Given a comodule structure on M_0, is each comodule M_n uniquely determined from comodule M_{n-1}?

Those questions have the deep meaning because of following reason. Assume that there is unique way to give the comodule structure on M_0 and each M_n is uniquely determined from M_{n-1}. Then from M_0, we can construct uniquely $M = BP_*[y_i : i > 0]/(I)$. This means that there is no extension problem in the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$ in Theorem 4. Then $BP_*(\Omega^2 S^{2n+1})$ would be $E(x_0) \otimes BP_*[y_i : i > 0]/(r_1, r_2, \cdots)$, so we would solve the long standing unsolved problem, $BP_*(\Omega^2 S^{2n+1})$.

In the next section we will show that it is not true, that is, there should be nontrivial extension. In fact, it is not easy to answer above questions directly because of the following reason. Let $M_n^{'} = BP_*[y_i : i > 0]/(r_1, \cdots, r_{n-1}, r_n^{'})$ be another comodule determined from M_{n-1}. Then $e_n = r_n - r_n^{'}$ must be comodule primitive in M_{n-1}. Hence e_n must be in $\mathrm{Ext}^{0,2(p^n-1)}_{BP_*BP}(BP_*, M_{n-1})$. Therefore we should compute $\mathrm{Ext}^{0,2(p^n-1)}_{BP_*BP}$ (BP_*, M_{n-1}) for each n. But it is never easy to compute those groups for all n.

3. Some approach to the conjecture for $BP_*(\Omega^2 S^{2n+1})$

Now we show that $BP_*(\Omega^2 S^{2n+1})$ is not equal to $E(x_0) \otimes BP_*[y_i : i > 0]/(r_1, r_2, \cdots)$. Then this implies that there should be non–trivial extension in the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$. And this also implies that there exist various comodule structures on M_0 or given comodule map, each comodule M_n is not uniquely determined from M_{n-1}.

Now we consider the following fibration.

$$\Omega^3 SU/SU(n) \xrightarrow{f_n} \Omega^3 SU/SU(n+1) \xrightarrow{h_n} \Omega^2 S^{2n+1}$$

From [7], we have

$$BP_*(\Omega^3 SU/SU(n+1)) = BP_*[z_{2(n+i)} : i \geq 0].$$

where dim $z_{2(n+i)} = 2(n+i)$. From now on, we use the following notation to denote generators for each space $\Omega^3 SU/SU(n+1)$.

$$BP_*(\Omega^3 SU/SU(n+1)) = BP_*[z_{n+1,2(n+i)} : i \geq 0].$$

LEMMA 9. [7] *The polynomial generators*

$$a_i \in BP_{2(np^i-1)}(\Omega^3 SU/SU(n)) \text{ and } c_i \in BP_{2(np^i-1)}(\Omega^3 SU/SU(n+1))$$

can be chosen such that

$$(f_n)_*(a_i) \equiv \sum_{j\geq 0} v_j c_{i-j}^{p^j} \mod I^2;$$
$$(h_n)_*(c_i) = y_i .$$

Moreover the polynomial generators in other dimensions can be chosen so that $(f_n)_*(z_{n,n-1}) = 0$ and $(f_n)_*(z_{n,n+i}) = z_{n+1,n+i}$ for $i \geq 0$.

Assume that $BP_*(\Omega^2 S^{2n+1})$ is equal to $E(x_0) \otimes BP_*[y_i : i > 0] /(r_1, r_2, \cdots)$. Then we have

$$(f_n)_*(a_i) = \sum_{j\geq 0} v_j c_{i-j}^{p^j} . \tag{1}$$

By the study for the coalgebra structures through $(f_n)_*$, we will show that the relation (1) can not be happen.

In general, $BP_*(X)$ does not have the coproduct structure for any space X since BP theory does not have a Künneth isomorphism. But for $X(n) = \Omega^3 SU/SU(n)$, $BP_*(X(n))$ is a free BP_*–module. So we have

$$BP_*(X(n) \times X(n)) = BP_*(X(n)) \otimes BP_*(X(n)) .$$

Hence $BP_*(X(n))$ has the coproduct structure. So we have the following diagram.

$$\begin{array}{ccc} BP_*(X(n)) & \xrightarrow{\triangle} & BP_*(X(n)) \otimes BP_*(X(n)) \\ {\scriptstyle (f_n)_*}\downarrow & & \downarrow{\scriptstyle (f_n)_*\otimes(f_n)_*} \\ BP_*(X(n+1)) & \xrightarrow{\triangle} & BP_*(X(n+1)) \otimes BP_*(X(n+1)) \end{array} \tag{2}$$

Here we study the case of $n = 1$. The other cases of n also follow by the same way. Note that $\Omega^3 SU = BU \times Z$. We recall that $H_*(BU)$ and $BP_*(BU)$ is bipolynomial Hopf algebra which is isomorphic as Hopf algebra to its own dual [2]. That is, $BP_*(BU) = BP_*[z_{1,2i} : i \geq 1]$ with $V(z_{1,2pi}) = z_{1,2i}$ where V is Verschiebung map and $z_{1,2i}$ is primitive if $p \nmid i$.

If there is no extension in the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$, then from Lemma 9 and the relation (1) we have

$$(3) \qquad (f_n)_*(z_{1,i}) = \begin{cases} z_{2,i} & \text{if } i \neq 2(p^k - 1) \\ \sum_{0 \leq j \leq k-1} v_j z^{p^j}_{2,2(p^{i-j}-1)} & \text{if } i = 2(p^k - 1) \end{cases}$$

Now we study the coalgebra structure of $z_{2,2(p^i-1)}$ through $(f_1)_*$. From (3), we have that

$$\begin{aligned} (f_1)_*(z_{1,2(p-1)}) &= pz_{2,2(p-1)} \\ (f_1)_*(z_{1,2(p^2-1)}) &= pz_{2,2(p^2-1)} + v_1 z^p_{2,2(p-1)} \\ (f_1)_*(z_{1,2(p^3-1)}) &= pz_{2,2(p^3-1)} + v_1 z^p_{2,2(p^2-1)} + v_2 z^{p^2}_{2,2(p-1)} . \end{aligned}$$

Since $p \nmid p^i - 1$, each $z_{1,2(p^i-1)}$ is primitive for $i \geq 1$. Then from the commutativity of the diagram (2), we get

$$\triangle(z_{2,2(p-1)}) = z_{2,2(p-1)} \otimes 1 + 1 \otimes z_{2,2(p-1)} .$$

Now we consider the coalgebra structure for $z_{2,2(p^2-1)}$. We have

$$\begin{aligned} &((f_1)_* \otimes (f_1)_*)(\triangle(z_{1,2(p^2-1)})) \\ =&((f_1)_* \otimes (f_1)_*)(z_{1,2(p^2-1)} \otimes 1 + 1 \otimes z_{1,2(p^2-1)}) \\ =&(pz_{2,2(p^2-1)} + v_1 z^p_{2,2(p-1)}) \otimes 1 + 1 \otimes (pz_{2,2(p^2-1)} + v_1 z^p_{2,2(p-1)}) \end{aligned}$$

On the other hands, we have

$$\begin{aligned} &(\triangle \circ (f_1)_*)(z_{1,2(p^2-1)}) \\ =&\triangle(pz_{2,2(p^2-1)} + v_1 z^p_{2,2(p-1)}) \\ =&\triangle(pz_{2,2(p^2-1)}) + v_1 \triangle(z_{2,2(p-1)})^p \\ =&p\triangle(z_{2,2(p^2-1)}) + v_1(z_{2,2(p-1)} \otimes 1 + 1 \otimes z_{2,2(p-1)})^p \end{aligned}$$

Since $((f_1)_* \otimes (f_1)_*) \circ \triangle = \triangle \circ (f_1)_*$ in the diagram (2), we get

$$\triangle(z_{2,2(p^2-1)}) = z_{2,2(p^2-1)} \otimes 1 + 1 \otimes z_{2,2(p^2-1)} - \frac{1}{p}\Big(\sum_{j=1}^{p-1}\binom{p}{j} z^{j}_{2,2(p-1)} \otimes z^{p-j}_{2,2(p-1)}\Big).$$

Note that the coefficient $\frac{1}{p}\binom{p}{j}$ for $1 \le j \le p-1$ is in $Z_{(p)}$.

Next consider the coalgebra structure for $z_{2,2(p^3-1)}$. We have

$$\begin{aligned}
&(f_1)_* \otimes (f_1)_* \circ \triangle(z_{1,2(p^3-1)}) \\
=&(f_1)_* \otimes (f_1)_*(z_{1,2(p^3-1)}) \otimes 1 + 1 \otimes z_{1,2(p^3-1)}) \\
=&(pz_{2,2(p^3-1)} + v_1 z^{p}_{2,2(p^2-1)} + v_2 z^{p^2}_{2,2(p-1)}) \otimes 1 \\
&+ 1 \otimes (pz_{2,2(p^3-1)} + v_1 z^{p}_{2,2(p^2-1)} + v_2 z^{p^2}_{2,2(p-1)})
\end{aligned}$$

On the other hands, we have

$$\begin{aligned}
&\triangle((f_1)_*(z_{1,2(p^3-1)})) \\
=&\triangle(pz_{2,2(p^3-1)} + v_1 z^{p}_{2,2(p^2-1)} + v_2 z^{p^2}_{2,2(p-1)}) \\
=&p\triangle(z_{2,2(p^3-1)}) + v_1\triangle(z^{p}_{2,2(p^2-1)}) + v_2\triangle(z^{p^2}_{2,2(p-1)}) \\
=&p\triangle(z_{2,2(p^3-1)}) + v_1[z_{2,2(p^2-1)} \otimes 1 + 1 \otimes z_{2,2(p^2-1)} \\
&- \frac{1}{p}\sum_{j=1}^{p-1}\binom{p}{j} z^{j}_{2,2(p^2-1)} \otimes z^{p-j}_{2,2(p^2-1)}]^p \\
&+ v_2(z_{2,2(p-1)} \otimes 1 + 1 \otimes z_{2,2(p-1)})^{p^2}.
\end{aligned}$$

Since $(f_1)_* \otimes (f_1)_* \circ \triangle = \triangle \circ (f_1)_*$ in the diagram (2), we get

$$\begin{aligned}
&\triangle(z_{2,2(p^3-1)}) \\
=&z_{2,2(p^3-1)} \otimes 1 + 1 \otimes z_{2,2(p^3-1)} + \cdots \\
&+ \frac{1}{p}(v_1 z^{p}_{2,2(p^2-1)} \otimes z^{(p-1)p}_{2,2(p^2-1)} + v_1 z^{(p-1)p}_{2,2(p^2-1)} \otimes z^{p}_{2,2(p^2-1)}) + \cdots
\end{aligned}$$

Here we have that the coefficient of $z^{p}_{2,2(p^2-1)} \otimes z^{(p-1)p}_{2,2(p^2-1)}$ and $z^{(p-1)p}_{2,2(p^2-1)} \otimes z^{p}_{2,2(p^2-1)}$ in $\triangle(z_{2,2(p^3-1)})$ is $\frac{1}{p}v_1$. But this is a contradiction because the coefficient groups of the BP homology are $Z_{(p)}[v_1, v_2, \ldots]$. Hence $BP_*(\Omega^2 S^{2n+1})$ can not be $E(x_0) \otimes BP_*[y_i : i > 0]/(r_1, r_2, \cdots)$, that is, there must be non-trivial extensions in the Adams spectral sequence.

Therefore we have the following results.

THEOREM 10. *There exist non-trivial extensions in the Adams spectral sequence converging to $BP_*(\Omega^2 S^{2n+1})$. Furthermore there exist various comodule structures on M_0 or given comodule map, each comodule M_n is not uniquely determined from M_{n-1} where $M_n = BP_*[y_i : i > 0]/(r_1, \cdots, r_n)$.*

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