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COHOMOLOGY GROUPS OF CIRCULAR UNITS

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ABSTRACT. Let k be a real abelian field of conductor f and $k_\infty = \cup_{n\geq 0} k_n$ be its $\mathbb{Z}_p$-extension for an odd prime p such that $p \nmid f\varphi(f)$. The aim of this paper is to compute the cohomology groups of circular units. For $m > n \geq 0$, let $G_{m,n}$ be the Galois group $\mathrm{Gal}(k_m/k_n)$ and C_m be the group of circular units of k_m. Let l be the number of prime ideals of k above p. Then, for $m > n \geq 0$, we have

(1) $C_m^{G_{m,n}} = C_n$,
(2) $\widehat{H}^i(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l-1}$ if i is even,
(3) $\widehat{H}^i(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l}$ if i is odd.

1. Introduction

Let k be a real abelian field of conductor f. For each prime p, let $k_\infty = \bigcup_{n\geq 0} k_n$ be the (basic) $\mathbb{Z}_p$-extension of k. For technical reasons, we will assume that p is an odd prime such that $p \nmid f\varphi(f)$. Under this assumption, primes of k above p totally ramify in k_n for all $n \geq 0$. Let l be the number of prime ideals of k above p.

For $m > n \geq 0$, we denote the Galois group $\mathrm{Gal}(k_m/k_n)$ by $G_{m,n}$ and the norm map from k_m to k_n by $N_{m,n}$. Let C_n be the group of circular units of k_n as was defined by Sinnott([7]). The aim of this paper is to compute the following cohomology groups of C_m:

THEOREM. *For $m > n \geq 0$, we have*

(1) $C_m^{G_{m,n}} = C_n$,
(2) $\widehat{H}^i(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l-1}$ *if i is even,*
(3) $\widehat{H}^i(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l}$ *if i is odd.*

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Since $G_{m,n}$ is cyclic, $\widehat{H}^i(G_{m,n}, C_m) \simeq \widehat{H}^0(G_{m,n}, C_m) \simeq C_m^{G_{m,n}}/N_{m,n}C_m$ if i is even, and $\widehat{H}^i(G_{m,n}, C_m) \simeq \widehat{H}^{-1}(G_{m,n}, C_m) \simeq {}_{N_{m,n}}C_m/C_m^{\sigma_{m,n}-1}$ if i is odd, where $\sigma_{m,n}$ is a generator of $G_{m,n}$. So it suffices to show (2) and (3) for $i = 0$ and -1, respectively.

Special cases such as when the conductor f is a prime or divisible by two primes were studied in [5] and [1], respectively. The structure of circular units becomes much more complicated as f has more distinct prime divisors. Thus the above theorem not only generalizes previous results but also shows that the cohomology groups of circular units are as simple as one can expect when compared to the cohomology groups of full units([3]).

The outline and the basic ideas of this paper are taken from [4], where cohomology groups of cyclotomic units are studied. And this paper generalizes the results of [4] about cyclotomic units in cyclotomic fields to circular units in abelian fields. The proof of (3) in the above theorem is especially similar to the corresponding one in [4], and so it could have been omitted. However we decided to include it for completeness.

2. Preliminaries and notations

In this section, we briefly review the definitions of cyclotomic units and circular units. Index theorems discovered by Sinnott are also introduced. Then we set up notations which will be used throughout this paper.

2.1. Cyclotomic units and circular units

Let $n \not\equiv 2 \mod 4$, ζ_n be an nth primitive root of 1, and V be the multiplicative subgroup of $\mathbb{Q}(\zeta_n)^\times$ generated by

$$\{\pm\zeta_n, 1 - \zeta_n^a \mid 1 \leq a \leq n-1\}.$$

Let E be the group of units of $\mathbb{Q}(\zeta_n)$ and define $C = V \cap E$. C is called the group of cyclotomic units of $\mathbb{Q}(\zeta_n)$. For $k = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$, we define the group of cyclotomic units of k by $C^+ = E^+ \cap C$, where E^+ is the group of units of k.

The group of cyclotomic units carries important information of k. To be precise, let h^+ be the class number of k. Then the index theorem of Sinnott([8]) says that $[E^+ : C^+] = 2^b h^+$ for some nonnegative integer b. This can be thought of as an algebraic version the analytic class number formula.

In general, for an abelian field F, Sinnott([8]) defines the group of circular units of F as follows. For each $n > 2$, let

$$C_{F_n} = N_{\mathbb{Q}(\zeta_n)/F\cap\mathbb{Q}(\zeta_n)}(C_{\mathbb{Q}(\zeta_n)}),$$

where $C_{\mathbb{Q}(\zeta_n)}$ is the group of cyclotomic units of $\mathbb{Q}(\zeta_n)$. Then the group C_F of circular units of F is defined to be the multiplicative subgroup of $F^\times$ generated by C_{F_n} for all $n > 2$ together with -1. Strictly speaking, the above definition is slightly different from the one given by Sinnott in [7]. But, since these two agree, we take the above one as the definition of circular units. Note that if $F \subset L$, then $N_{L/F}(C_L) \subset C_F \subset C_L^{\mathrm{Gal}(L/F)}$, where $N_{L/F}$ is the norm map from L to F. We will use this remark in the subsequent sections without comment. The index theorem of Sinnott([7]) says:

INDEX THEOREM. *Let E_F be the unit group of F and h be the class number of the maximal real subfield of F. Then $[E_F : C_F] = c_F h$ for some integer c_F.*

2.2. Notations

For each integer $s \geq 1$, we choose a primitive sth root ζ_s of 1 so that $\zeta_t^{\frac{t}{s}} = \zeta_s$ if $s|t$. Let k be a real abelian field of conductor f and let $k_\infty = \bigcup_{n\geq 0} k_n$ be its $\mathbb{Z}_p$-extension. Note that k admits a unique $\mathbb{Z}_p$-extension for each prime p, namely the basic $\mathbb{Z}_p$-extension. Throughout this paper, we assume that p is an odd prime such that $p \nmid f\varphi(f)$, where φ is the Euler φ function. For each $n \geq 0$, we denote the group of circular units of k_n by C_n. Let $K = \mathbb{Q}(\zeta_f)$, $F = \mathbb{Q}(\zeta_p)$ and $K' = \mathbb{Q}(\zeta_{pf})$. We denote their basic $\mathbb{Z}_p$-extensions by K_∞, F_∞, and K'_∞, respectively. The nth layers of these $\mathbb{Z}_p$-extensions are denoted by K_n, F_n and K'_n as usual. Let σ be the topological generator of the Galois group $\Gamma = \mathrm{Gal}(K'_\infty/K')$ which sends ζ_{p^n} to $\zeta_{p^n}^{1+p}$ for all $n \geq 1$. Restrictions of σ to various subfields are also denoted by σ. We abbreviate σ^{p^n} by σ_n. Thus σ_n is a topological generator of $\mathrm{Gal}(K'_\infty/K'_n)$.

Let $k_{(p)}$ be the decomposition subfield of k for p and let $\Delta = \mathrm{Gal}(K/k)$, $\bar{\Delta} = \mathrm{Gal}(K/\mathbb{Q})$, $\Delta_p = \mathrm{Gal}(K/k_{(p)})$, $\Delta_k = \mathrm{Gal}(k/\mathbb{Q})$ and $\Delta_{k,p} = \mathrm{Gal}(k_{(p)}/\mathbb{Q})$. Let $[k_{(p)} : \mathbb{Q}] = l$, so there are l prime ideals in k above p. Elements of Δ, $\bar{\Delta}$ or Δ_p will be denoted by τ's and those of Δ_k and $\Delta_{k,p}$ by ρ's. The Frobenius automorphism of K for p or its restriction to k is denoted by τ_p. For each d such that $d|f$, let $k^{(d)} = \mathbb{Q}(\zeta_d) \cap k$ and $k_{(p)}^{(d)}$ be the decomposition subfield of $k^{(d)}$ for p. Let $\Delta^{(d)} = \mathrm{Gal}(\mathbb{Q}(\zeta_d)/k^{(d)})$,

$\bar{\Delta}^{(d)} = \mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$, $\Delta_p^{(d)} = \mathrm{Gal}(\mathbb{Q}(\zeta_d)/k_{(p)}^{(d)})$, $\Delta_k^{(d)} = \mathrm{Gal}(k^{(d)}/\mathbb{Q})$ and $\Delta_{k,p}^{(d)} = \mathrm{Gal}(k_{(p)}^{(d)}/\mathbb{Q})$.

Let R be the set of all roots of 1 in $\mathbb{Z}_p$, i.e., $R = \{\omega \in \mathbb{Z}_p | \omega^{p-1} = 1\}$. Then R can be regarded as the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ or as any Galois group isomorphic to it such as $\mathrm{Gal}(F_n/\mathbb{Q}_n)$, where $\mathbb{Q}_n$ is the subfield of F_n of degree p^n over $\mathbb{Q}$. For $m > n$, let $G_{m,n}$ be the Galois group $\mathrm{Gal}(K'_m/K'_n)$ and $N_{m,n}$ the norm map $N_{K'_m/K'_n}$ from K'_m to K'_n. $G_{m,n}$ will also mean the Galois groups $\mathrm{Gal}(k_m/k_n)$, $\mathrm{Gal}(F_m/F_n)$ and $\mathrm{Gal}(\mathbb{Q}_m/BQ_n)$. Similarly $N_{m,n}$ will have various meanings.

3. Computation of $C_m^{G_{m,n}}$

We keep all the previous notations. In this section, we will prove $C_m^{G_{m,n}} = C_n$.

Proof of Theorem (1). Obviously, $C_n \subset C_m^{G_{m,n}}$. To prove $C_m^{G_{m,n}} \subset C_n$, take $u \in C_m^{G_{m,n}}$. We will show that $u^d \in C_n$ and $u^{p^{m-n}} \in C_n$, where $d = [\mathbb{Q}(\zeta_{pf}) : k]$. Then, since $(d, p^{m-n}) = 1$, we will have $u \in C_n$. First, we view u as an element in $\overline{C}_m^{G_{m,n}}$, where $\overline{C}_s$ is the group of cyclotomic units of K'_s for each integer $s \geq 0$. Since $\overline{C}_m^{G_{m,n}} = \overline{C}_n([2])$, u is a cyclotomic unit in K'_n. Thus $N_{K'_n/k_n}(u) \in C_n$. But since $u \in K'_n \cap k_m \subset k_n$,

$$N_{K'_n/k_n}(u) = u^d \in C_n.$$

Next, note that $u^{p^{m-n}} = N_{m,n}(u)$ since u is fixed under $G_{m,n}$. Thus

$$u^{p^{m-n}} = N_{k_m/k_n}(u) \in C_n.$$

This finishes the proof. □

4. Computation of $\widehat{H}^0(G_{m,n}, C_m)$

In this section we will prove

$$\widehat{H}^0(G_{m,n}, C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l-1}.$$

First we need two lemmas.

LEMMA 1. *For* $m > n \geq 0$, $C_n = C_0 N_{m,n} C_m$.

Proof. Clearly, $C_n \supset C_0 N_{m,n} C_m$. To prove the converse, note that an element u of C_n can be written as $u = u_0 u_1 \cdots u_n$, where $u_0 \in C_0$, and for each $s, 1 \leq s \leq n$, u_s is of the form

$$u_s = \prod_{d|f} \prod_{\substack{1 \leq i \leq p^s \\ \rho_j \in \Delta_k^{(d)}}} \prod_{\substack{\omega \in R \\ \tau \in \Delta^{(d)}}} (\zeta_{p^{s+1}}^{\omega} - \zeta_d^{\tau})^{a_{i,j,d} \sigma^i \rho_j}$$

for some integers $a_{i,j,d}$. Since

$$N_{m,s}(\prod_{\omega,\tau}(\zeta_{p^{m+1}}^{\omega} - \zeta_d^{\tau})) = \prod_{\omega,\tau}(\zeta_{p^{s+1}}^{\omega} - \zeta_d^{\tau p^{m-s}}),$$

we have

$$\begin{aligned}\prod_{\omega,\tau}(\zeta_{p^{s+1}}^{\omega} - \zeta_d^{\tau}) &= \prod_{\omega,\tau}(\zeta_{p^{s+1}}^{\omega} - \zeta_d^{\tau p^{m-s}})^{\tau_p^{s-m}} \\ &= N_{m,s}(\prod_{\omega,\tau}(\zeta_{p^{m+1}}^{\omega} - \zeta_d^{\tau}))^{\tau_p^{s-m}} \\ &= N_{m,n}(\prod_{\omega,\tau}(\zeta_{p^{m+1}}^{\omega} - \zeta_d^{\tau}))^{\tau_p^{s-m} \sum_{0 \leq i < p^{n-s}} \sigma^{ip^s}}.\end{aligned}$$

So for each $s \geq 1$, $u_s \in N_{m,n} C_m$, and thus $u \in C_0 N_{m,n} C_m$. This proves Lemma 1. □

By *rank* A for a finitely generated abelian group A, we mean the $\mathbb{Z}$-rank of the free part of A.

LEMMA 2. *rank* $N_{k/k_{(p)}} C_0 = l - 1$.

Proof. Let $C_{(p)}$ be the group of circular units of $k_{(p)}$. Note that $k^{(d)} \cap k_{(p)} = k_{(p)}^{(d)}$. Hence for $\alpha \in k^{(d)}$,

$$N_{k/k_{(p)}}(\alpha) = N_{k^{(d)}/k_{(p)}^{(d)}}(\alpha)^{[k:k^{(d)} k_{(p)}]}.$$

Therefore

$$C_{(p)}^{\varphi(f)} \subset N_{k/k_{(p)}}(C_0) \subset C_{(p)} \subset E_{k_{(p)}},$$

where $E_{k_{(p)}}$ is the group of units of $k_{(p)}$. Note that $[E_{k_{(p)}} : C_{(p)}]$ is finite by the index theorem of Sinnott. Since $C_{(p)}$ is a finitely generated abelian group, $[C_{(p)} : C_{(p)}^{\varphi(f)}]$ is also finite. Thus $N_{k/k_{(p)}}C_0$ is of finite index in the full unit group $E_{k_{(p)}}$. Therefore,

$$\text{rank } N_{k/k_{(p)}}C_0 = \text{rank } E_{k_{(p)}} = [k_{(p)} : \mathbb{Q}] - 1 = l - 1.$$

This finishes the proof of Lemma 2. □

Now we compute $\widehat{H}^0(G_{m,n}, C_m)$.

Proof of Theorem (2). Since $C_n = C_0 N_{m,n} C_m$ by Lemma 1, the natural map

$$C_0 \to C_n \to C_n/N_{m,n}C_m$$

is surjective. Thus

$$\widehat{H}^0(G_{m,n}, C_m) \simeq C_m^{G_{m,n}}/N_{m,n}C_m \simeq C_n/N_{m,n}C_m \simeq C_0/C_0 \cap N_{m,n}C_m.$$

Let C'_m be the subgroup of C_m generated by circular units of the form

$$\prod_{\omega\in R, \tau\in\Delta^{(d)}} (\zeta_{p^{m+1}}^{a\omega} - \zeta_d^{b\tau}),$$

where $p^{m+1} \nmid a$, and $d \mid f$. Then clearly $C_m = C_0 C'_m$ and $N_{m,n}C'_m = C'_n$. Hence $N_{m,n}C_m = C_0^{p^{m-n}} C'_n$. Therefore

$$\widehat{H}^0(G_{m,n}, C_m) \simeq C_0/C_0 \cap C_0^{p^{m-n}} C'_n = C_0/C_0^{p^{m-n}}(C_0 \cap C'_n).$$

Next we claim that $C_0^{\tau_p - 1} \subset C_0 \cap C'_n \subset {}_N C_0$, where ${}_N C_0 = \{u \in C_0 \mid N_{k/k_{(p)}} u = 1\}$. The first inclusion follows from the identity

$$(1 - \zeta_d)^{\tau_p - 1} = \prod_{\omega\in R} (\zeta_p^{\omega} - \zeta_d).$$

To check the second one, take $u \in C_0 \cap C'_n$ and write u as

$$u = \prod_{d|f} \prod_{\substack{a,b \\ p^{n+1}\nmid a}} \prod_{\omega\in R, \tau\in\Delta^{(d)}} (\zeta_{p^{n+1}}^{a\omega} - \zeta_d^{b\tau})^{f(a,b,d)}$$

for some integers $f(a,b,d)$. By taking $N_{n,0}$, we have

$$u^{p^n} = \prod_{c,d,\omega,\tau} (\zeta_p^{\omega} - \zeta_d^{c\tau})^{g(c,d)} = \prod_{c,d,\tau} (1 - \zeta_d^{c\tau})^{g(c,d)(\tau_p - 1)}$$

for some integers $g(c,d)$. Therefore $N_{k/k_{(p)}}u^{p^n} = 1$ and the second inclusion follows. Since ${}_NC_0/C_0^{\tau_p-1}$ is annihilated by $[k:\mathbb{Q}]$, which is prime to p, we obtain

$$\widehat{H}^0(G_{m,n},C_m) \simeq C_0/C_0^{p^{m-n}}(C_0\cap C_n') = C_0/C_0^{p^{m-n}}{}_NC_0.$$

For convenience, we denote $N_{k/k_{(p)}}$ simply by N. By Lemma 2, we know that NC_0 modulo $\{\pm 1\}$ is a free abelian group of rank $l-1$. Let $\xi_1,\xi_2,\cdots,\xi_{l-1}$ be elements of C_0 such that $\{N(\xi_1),N(\xi_2),\cdots,N(\xi_{l-1})\}$ generates NC_0 modulo $\{\pm 1\}$. Let C_0' be the subgroup of C_0 generated by $\{\xi_1,\xi_2,\cdots,\xi_{l-1}\}$. Then

$$[C_0 : C_0'{}_NC_0] = [NC_0 : NC_0'][{}_NC_0 : {}_NC_0] = 1 \text{ or } 2.$$

Therefore

$$\widehat{H}^0(G_{m,n},C_m) \simeq \frac{\langle\xi_1,\cdots,\xi_{l-1}\rangle_NC_0}{\langle\xi_1,\cdots,\xi_{l-1}\rangle^{p^{m-n}}{}_NC_0} \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^{l-1}$$

as desired. □

5. Computation of $\widehat{H}^{-1}(G_{m,n},C_m)$

In this section, we will prove

$$\widehat{H}^{-1}(G_{m,n},C_m) \simeq (\mathbb{Z}/p^{m-n}\mathbb{Z})^l.$$

LEMMA 3. *Let $n<m<s$. If $\delta\in C_m$ is such that $N_{m,n}\delta=1$, then $\delta\in N_{s,m}C_s$.*

Proof. Suppose $\delta\in C_m$ satisfies $N_{m,n}\delta = 1$. In Section 4, we proved that $C_0=\langle\pm1\rangle\langle\xi_1,\ldots,\xi_{l-1}\rangle_NC_0$. So $C_m = C_0C_m' = \langle\pm1\rangle\langle\xi_1,\ldots,\xi_{l-1}\rangle_N C_0C_m'$. Thus we can write $\delta = \pm\xi uv$, for some $\xi\in\langle\xi_1,\ldots,\xi_{l-1}\rangle$, $u\in{}_NC_0$ and $v\in C_m'$. Then $1 = N_{m,n}\delta = \pm\xi^{p^{m-n}}u^{p^{m-n}}N_{m,n}v$. Since $v\in C_m'$, $N_{m,n}v\in C_n'$. Hence $N_{m,n}v = \pm\xi^{-p^{m-n}}u^{-p^{m-n}}\in C_0\cap C_n'\subset {}_NC_0$. Therefore, we have $\xi^{p^{m-n}}w=\pm1$, where $w = u^{p^{m-n}}N_{m,n}v\in{}_NC_0$. This implies that $\xi^{p^{m-n}}=\pm1$, and thus we obtain $\xi=\pm1$. Hence $\delta=\pm uv$.

Note that $u^t\in C_0\cap C_m'\subset C_m'$, where $t=[{}_NC_0 : C_0\cap C_m']$, which is finite and prime to p. Therefore $\delta^{2t}=u^{2t}v^{2t}\in C_m'\subset N_{s,m}C_s$. Since $\delta^{p^{s-m}} = N_{s,m}(\delta)\in N_{s,m}C_s$, we have $\delta\in N_{s,m}C_s$, as desired. □

Proof of Theorem (3). We prove the theorem by induction on $m \geq n+1$. Let $m = n+1$. Since the Herbrand quotient for C_m is $1/p$([6]) and since $p\widehat{H}^1(G_{m,n}, C_m) = 0$, $\widehat{H}^1(G_{m,n}, C_m)$ must be $(\mathbb{Z}/p\mathbb{Z})^l$.

Assuming the result for m, we will prove $\widehat{H}^1(G_{s,n}, C_s) = (\mathbb{Z}/p^{s-n}\mathbb{Z})^l$ when $s = m+1$. Since the inflation map

$$\widehat{H}^1(G_{m,n}, C_s^{G_{s,m}}) \stackrel{\text{inflation}}{\longrightarrow} \widehat{H}^1(G_{s,n}, C_s)$$

is injective, we may identify $\widehat{H}^1(G_{m,n}, C_m)$ with its image (we know that $C_s^{G_{s,m}} = C_m$). Let $[\delta]$ be an element in $\widehat{H}^{-1}(G_{m,n}, C_m)$. Since that $\delta = N_{s,m}\xi$ for some $\xi \in C_s$ by Lemma 3, we have

$$[\xi^p] = [(N_{s,m}\xi)\frac{\xi^p}{N_{s,m}\xi}] = [N_{s,m}\xi] = [\delta].$$

Hence $\widehat{H}^1(G_{m,n}, C_m) \simeq \widehat{H}^{-1}(G_{m,n}, C_m) \simeq {}_{N_{m,n}}C_m/C_m^{\sigma_n - 1}$ is the image of

$$p : \widehat{H}^{-1}(G_{s,n}, C_s) \stackrel{\times p}{\to} \widehat{H}^{-1}(G_{s,n}, C_s).$$

Namely $\widehat{H}^{-1}(G_{m,n}, C_m) = p\widehat{H}^{-1}(G_{s,n}, C_s)$. Since $p^{s-n}\widehat{H}^{-1}(G_{s,n}, C_s) = 0$ and since the Herbrand quotient for C_s is p^{s-n}, $\widehat{H}^1(G_{s,n}, C_n)$ must be $(\mathbb{Z}/p^{s-n}\mathbb{Z})^l$. □

COROLLARY. *Let $\Gamma = Gal(k_\infty/k)$ and $C_\infty = \bigcup_{n\geq 0} C_n$. Then*

(1) $H^1(\Gamma, C_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^l$
(2) $H^2(\Gamma, C_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{l-1}$.

Proof. Since $C_n = C_m^{G_{m,n}}$ by Theorem (1), we have $C_n = C_\infty^{\Gamma_n}$, where $\Gamma_n = \mathrm{Gal}(k_\infty/k_n)$. Hence, for $i = 1, 2$,

$$\begin{aligned} H^i(\Gamma, C_\infty) &= \varinjlim H^i(G_n, C_\infty^{\Gamma_n}) \\ &= \varinjlim H^i(G_n, C_n) \\ &= \begin{cases} \varinjlim (\mathbb{Z}/p^n\mathbb{Z})^l & \text{if } i = 1 \\ \varinjlim (\mathbb{Z}/p^n\mathbb{Z})^{l-1} & \text{if } i = 2 \end{cases} \\ &= \begin{cases} (\mathbb{Q}_p/\mathbb{Z}_p)^l & \text{if } i = 1 \\ (\mathbb{Q}_p/\mathbb{Z}_p)^{l-1} & \text{if } i = 2. \end{cases} \end{aligned}$$

□

References

[1] J. R. Belliard, *Sur la structure galoisienne des unites circulaires dans les $\mathbb{Z}_p$-extensions*, J. Number Theory **69** (1998), 16–49.

[2] R. Gold and J. M. Kim, *Bases for cyclotomic units*, Compositio Math. **71** (1989), 13–28.

[3] K. Iwasawa, *On cohomology groups of units for $\mathbb{Z}_p$-extensions*, Amer. J. Math. **105** (1983), no. 1, 189–200.

[4] J. M. Kim, *Cohomology groups of cyclotomic units*, J. Algebra **152** (1992), 514–519.

[5] ———, *Circular units in the $\mathbb{Z}_p$-extensions of real abelian fields of prime conductor*, Tohoku Math. J. **51** (1999), 305–313.

[6] J. P. Serre, *Local fields*, G. T. M. 67, Springer-Verlag, New York, 1979.

[7] W. Sinnott, *On the Stickelberger ideal and the circular units of an abelian field*, Invent. Math. **62** (1980), 181–234.

[8] L. Washington, *Introduction to Cyclotomic Fields*, G. T. M. 83, Springer-Verlag, New York, 1980.

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