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QUOTIENTS OF THETA SERIES AS RATIONAL FUNCTIONS OF $j_{1,8}$

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ABSTRACT. Let $Q(n,1)$ be the set of even unimodular positive definite integral quadratic forms in n-variables. Then n is divisible by 8. For $A[X]$ in $Q(n,1)$, the theta series $\theta_A(z) = \sum_{X\in\mathbb{Z}^n} e^{\pi i z A[X]}$ ($z \in \mathfrak{H}$ the complex upper half plane) is a modular form of weight $n/2$ for the congruence group $\Gamma_1(8) = \{\delta \in SL_2(\mathbb{Z}) \mid \delta \equiv \left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \ mod\ 8\}$. If $n \geq 24$ and $A[X]$, $B[X]$ are two quadratic forms in $Q(n,1)$, the quotient $\theta_A(z)/\theta_B(z)$ is a modular function for $\Gamma_1(8)$. Since we identify the field of modular functions for $\Gamma_1(8)$ with the function field $K(X_1(8))$ of the modular curve $X_1(8) = \Gamma_1(8)\backslash\mathfrak{H}^*$ ($\mathfrak{H}^*$ the extended plane of $\mathfrak{H}$) with genus 0, we can express it as a rational function of $j_{1,8}$ over $\mathbb{C}$ which is a field generator of $K(X_1(8))$ and defined by $j_{1,8}(z) = \theta_3(2z)/\theta_3(4z)$. Here, θ_3 is the classical Jacobi theta series.

1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and let $\Gamma_1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$ mod N ($N = 1, 2, 3, \ldots$). Since the group $\Gamma_1(N)$ acts on $\mathfrak{H}$ by linear fractional transformations, we get the modular curve $X_1(N) = \Gamma_1(N)\backslash\mathfrak{H}^*$, as the projective closure of a smooth affine curve $\Gamma_1(N)\backslash\mathfrak{H}$, with genus $g_{1,N}$. Here, $\mathfrak{H}^*$ denotes the union of $\mathfrak{H}$ and $\mathbb{P}^1(\mathbb{Q})$. We identify the function field $K(X_1(N))$ of the curve $X_1(N)$ with the field of modular functions for $\Gamma_1(N)$. Since $g_{1,N} = 0$ only for the eleven cases $1 \leq N \leq 10$ and $N = 12$ ([6]), $K(X_1(8))$ becomes a rational function field $\mathbb{C}(j_{1,8})$ where

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$j_{1,8}(z) := \theta_3(2z)/\theta_3(4z)$ for $z \in \mathfrak{H}$ and θ_3 is the classical Jacobi theta series ([5]).

This article is a continuation of our previous works ([2], [9]). Let $A = (a_{ij})$ be a symmetric, positive definite and integral $n \times n$ matrix for which $a_{ii} \equiv 0 \pmod 2$ and $\det A = 1$. We associate to A a quadratic form $A[X] = X^t AX$, $X = (x_1, \cdots, x_n)$ which we call a positive definite integral even unimodular quadratic form in n variables. Then $n \equiv 0$ mod 8 ([10], [13]). Let $Q(n,1)$ be the set of even unimodular positive definite integral quadratic forms in n-variables. Two forms $A[X]$ and $B[X]$ are called equivalent (write $A[X] \sim B[X]$) if $B = C^t AC$ for some $C \in GL_n(\mathbb{Z})$. Set $\tilde{Q}(n,1) = Q(n,1)/\sim$. The cardinality $|\tilde{Q}(n,1)|$ is finite, so we can speak of the class number $h(Q(n,1)) = |\tilde{Q}(n,1)|$. It is well-known that $h(Q(8,1)) = 1$, $h(Q(16,1)) = 2$ and $h(Q(24,1)) = 24$. The class number $h(Q(n,1))$ has not been determined yet for $n \geq 32$. Instead, we see that the class number grows remarkably fast (Chap V, [13]). For $A[X] \in Q(n,1)$, the theta series

$$\theta_A(z) := \sum_{X \in \mathbb{Z}^n} e^{\pi i z A[X]} = 1 + \sum_{m=1}^{\infty} r_A(m) e^{2\pi i m z} \quad (z \in \mathfrak{H}),$$

where $r_A(m)$ is the cardinality of the solution set $\{X \in \mathbb{Z}^n |\ A[X] = 2m\}$ $(m \geq 1)$, is a modular form of weight $\frac{n}{2}$ for $\Gamma(1)(= SL_2(\mathbb{Z}))$ and hence for $\Gamma_1(8)$. In cases $n = 8$ and 16, the quotients of theta series are trivial, that is,

$$\theta_A(z)/\theta_B(z) = 1 \quad \text{for} \quad A[X], B[X] \in Q(n,1)$$

([13], p110). If $n \geq 24$ and $A[X], B[X]$ are two quadratic forms in $Q(n,1)$ then the quotient $\theta_A(z)/\theta_B(z)$ is a rational function of $J(z)$ ([9], Theorem 1). Meanwhile, it is theoretically natural to reduce the study of modular forms with respect to a congruence subgroup to that of type $\Gamma_1(N)$, and hence it is interesting to express the quotient as a rational function of $j_{1,8}$, too. Since $\mathbb{C}(j)$ $(j = 1728J)$ is a subfield of $\mathbb{C}(j_{1,8})$, we can express $j(z)$ as a rational function of $j_{1,8}(z)$ (Corollary 11). Therefore we are able to write $\theta_A(z)/\theta_B(z)$ as a rational funtion of $j_{1,8}$ (Theorem 8) as desired. Unlike the previous ones ([2], [9]), however, we don't have enough cusps to estimate the normalized Eisenstein series $E_4(z)$ in the process. To overcome such obstacle and finish calculations, we shall take an additional point $\rho = e^{2\pi i/3}$ from the upper half plane. In particular when $n = 24$ we shall completely determine in Appendix all the theta series $\theta_A(z)$ as polynomials over $\mathbb{Q}$ in $\theta_3(2z)$ and $\theta_3(4z)$. Throughout the paper we adopt the following notations:

- $q_h = e^{2\pi iz/h}$
- $\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) |\ \gamma \equiv I \ mod\ N\}$
- $M_k(\Gamma(N))$ the space of modular forms of weight k for the group $\Gamma(N)$
- $M_{\frac{k}{2}}(\tilde{\Gamma_1}(N))$ the space of modular forms of half integral weight for the group $\Gamma_1(N)$
- $M_k(\Gamma_1(N))$ the space of modular forms of weight k for the group $\Gamma_1(N)$
- x^t the transpose of an integral column vector x
- $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ two generators of $\Gamma(1)$

2. Preliminaries

For $\mu, \nu \in \mathbb{R}$ and $z \in \mathfrak{H}$, put

$$\Theta_{\mu,\nu}(z) := \sum_{n \in \mathbb{Z}} exp\{\pi i(n + \frac{1}{2}\mu)^2 z + \pi i n \nu\}.$$

This series converges uniformly for Im(z)$\geq \eta > 0$, and hence defines a holomorphic function on $\mathfrak{H}$. Then the Jacobi theta functions θ_2, θ_3 and θ_4 are defined by

$$\theta_2(z) := \Theta_{1,0}(z) = \sum_{n \in \mathbb{Z}} q_2^{(n+\frac{1}{2})^2}$$
$$\theta_3(z) := \Theta_{0,0}(z) = \sum_{n \in \mathbb{Z}} q_2^{n^2}$$
$$\theta_4(z) := \Theta_{0,1}(z) = \sum_{n \in \mathbb{Z}} (-1)^n q_2^{n^2}.$$

And we have the following transformation formulas ([12], p218-219)

$$\theta_2(z+1) = e^{\frac{1}{4}\pi i}\theta_2(z) \tag{1}$$
$$\theta_3(z+1) = \theta_4(z) \tag{2}$$
$$\theta_4(z+1) = \theta_3(z) \tag{3}$$
$$\theta_2(-\frac{1}{z}) = (-iz)^{\frac{1}{2}}\theta_4(z) \tag{4}$$
$$\theta_3(-\frac{1}{z}) = (-iz)^{\frac{1}{2}}\theta_3(z) \tag{5}$$
$$\theta_4(-\frac{1}{z}) = (-iz)^{\frac{1}{2}}\theta_2(z) \tag{6}$$

Furthermore, we have the following theorem at hand.

THEOREM 1. (i) $\theta_3(2z) \in M_{\frac{1}{2}}(\tilde{\Gamma_0}(4))$ *and* $\theta_3(4z) \in M_{\frac{1}{2}}(\tilde{\Gamma_0}(8), \chi_2)$, *where* $\chi_2(d) = (\frac{2}{d})$ *and* $(2,d) = 1$.
(ii) $K(X_1(8)) = \mathbb{C}(j_{1,8})$ *and* $j_{1,8}(\infty) = 1$, $j_{1,8}(0) = \sqrt{2}$, $j_{1,8}(\frac{1}{4}) = \infty$ *(simple pole),* $j_{1,8}(\frac{3}{8}) = -1$, $j_{1,8}(\frac{1}{3}) = -\sqrt{2}$, $j_{1,8}(\frac{1}{2}) = 0$ *(simple zero) where* $\infty, 0, \frac{1}{4}, \frac{3}{8}, \frac{1}{3}$ *and* $\frac{1}{2}$ *are the six cusps of* $X_1(8)$.

Proof. [5], Theorem 9. □

3. Structure of $M_{2k}(\Gamma_1(8))$

From the dimension formula for a congruence subgroup of $\Gamma(1)$ ([10] §2.5 or [14] §2.6), we get

PROPOSITION 2. *For* $k \geq 1$,

$$\dim_{\mathbb{C}} M_{2k}(\Gamma_1(8)) = 4k+1.$$

Proof. We see from [6] that $g = 0$, $\sigma_\infty = 6$ and $\Gamma_1(8)$ has no elliptic elements. Thus the result follows. □

For any positive integer $\frac{k}{2}$, $M_{\frac{k}{2}}(\tilde{\Gamma_1}(8)) = M_{\frac{k}{2}}(\Gamma_1(8))$. Indeed, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(8)$, $j(\gamma, z) = (\frac{c}{d})\sqrt{cz+d}$ since $d \equiv 1 \bmod 4$. Since k is even, $j(\gamma,z)^k = (cz+d)^{\frac{k}{2}}$, that is, $M_{\frac{k}{2}}(\tilde{\Gamma_1}(8))$ has the same automorphy factor as that of $M_{\frac{k}{2}}(\Gamma_1(8))$. For convenience, let us put $s(z) = \theta_3(2z)$ and $t(z) = \theta_3(4z)$.

PROPOSITION 3. *The vector space* $M_{2k}(\Gamma_1(8))$ *is generated by* s^{4k}, $s^{4k-1}t, \cdots, t^{4k}$ *over* $\mathbb{C}$.

Proof. It follows from Proposition 2 that the dimension of $M_{2k}(\Gamma_1(8))$ is $4k+1 (k \geq 1)$. We then observe by Theorem 1 and previous remark that $s^{4k-i}t^i$ $(0 \leq i \leq 4k)$ are members of $M_{2k}(\Gamma_1(8))$. Thus it is enough to show that the functions listed above are in fact linearly independent over $\mathbb{C}$. Suppose that

$$\sum_{i=0}^{4k} c_i s(z)^{4k-i} t(z)^i = 0 \quad \text{for } c_i \in \mathbb{C}.$$

Since $F(z) := (2\pi)^{-12}\Delta(z) = \frac{1}{2^8}\{\theta_2(z)\theta_3(z)\theta_4(z)\}^8$ ([12], p.222) and $F(z)$ has no zeros on $\mathfrak{H}$ as is well known ([7], [12]), $t(z)$ never vanishes on $\mathfrak{H}$. If we divide the above by $t(z)^{4k}$, we obtain that

$$\sum_{i=0}^{4k} c_i j_{1,8}^{4k-i} = 0 \quad \text{for } c_i \in \mathbb{C}.$$

Here it is necessary to show that $j_{1,8}$ is transcendental over $\mathbb{C}$. Choose any $c \in \mathbb{C}$ and consider $j_{1,8} - c$. Since $j_{1,8} - c$ is a nonconstant modular function, it has at least one zero. This guarantees that the image of $j_{1,8}$ is all of $\mathbb{C}$. But if we had an algebraic equation satisfied by $j_{1,8}$, then the image of $j_{1,8}$ would be mapped into the set of solutions of the algebraic equation which is at most finite. This is impossible, which concludes the Proposition. □

In order to express a modular form as a polynomial in two variables $s(z)$ and $t(z)$, we have to be certain that they are algebraically independent. To this end we are in need of the following.

LEMMA 4. *If* $f_k + f_{k-1} + \cdots + f_0 = 0$ *where* $k \in \mathbb{N}$ *and* $f_i \in M_{\frac{i}{2}}(\tilde{\Gamma_1}(8))$ *for all* $i = 0, 1, \cdots, k$, *then* $f_i = 0$ *for all* i.

Proof. The argument is almost the same as that of Lemma 11 in [3]. □

Assume that there is a polynomial $F \in \mathbb{C}[X,Y]$ satisfied by $s(z)$ and $t(z)$. By Theorem 1 and Lemma 4, we may suppose that F is homogeneous. Let $deg(F) = n$. Then

$$\frac{F(s,t)}{t^n} = \sum_{k=0}^{n} a_k j_{1,8}^k = 0$$

for $a_k \in \mathbb{C}$. Since $j_{1,8}$ is transcendental over $\mathbb{C}$, $a_k = 0$ for all k; hence $F = 0$. This proves the algebraic independency of $s(z)$ and $t(z)$.
For later use, we will derive the following identities. To begin with, let us set $\theta_i = \theta_i(z)$ $(i = 2, 3, 4)$, $t = t(z)$ and $s = s(z)$ for convenience in writing.

LEMMA 5. *$\theta_3^2\theta_4^2 = 4t^4 - 4s^2t^2 + s^4$ and $\theta_3^4 + \theta_4^4 = -8t^4 + 8s^2t^2 + 2s^4$.*

Proof. Recall from [3], Theorem 12 that

$$\theta_3^4 = \frac{1}{4}(\theta_3(\frac{z}{2})^4 + 2\theta_3(\frac{z}{2})^2\theta_4(\frac{z}{2})^2 + \theta_4(\frac{z}{2})^4).$$

Then we have

$$\begin{aligned} s^4 &= \theta_3(2z)^4 = \frac{1}{4}(\theta_3^4 + 2\theta_3^2\theta_4^2 + \theta_4^4) \\ t^4 &= \theta_3(4z)^4 = \frac{1}{4}(\theta_3(2z)^4 + 2\theta_3(2z)^2\theta_4(2z)^2 + \theta_4(2z)^4). \end{aligned} \tag{7}$$

On the other hand, since $2\theta_3(2z)^2 = \theta_3^2 + \theta_4^2$ and $\theta_4(2z)^2 = \theta_3\theta_4$ ([12]), we get

$$\begin{aligned} 4t^4 &= \theta_3(2z)^4 + 2\theta_3(2z)^2\theta_4(2z)^2 + \theta_4(2z)^4 \\ &= \frac{1}{4}(\theta_3^2 + \theta_4^2)^2 + 2\theta_3(2z)^2\theta_4(2z)^2 + \theta_3^2\theta_4^2 \\ &= \frac{1}{4}(\theta_3^4 + 6\theta_3^2\theta_4^2 + \theta_4^4) + 2s^2(2t^2 - s^2) \\ &\quad \text{because } \theta_4^2 = 2\theta_3(2z)^2 - \theta_3^2. \end{aligned}$$

Hence

$$\theta_3^4 + 6\theta_3^2\theta_4^2 + \theta_4^4 = 16t^4 - 16t^2s^2 + 8s^4. \tag{8}$$

The result is immediate from (7) and (8). □

PROPOSITION 6. *Let $f \in M_{2k}(\Gamma(1))$. Then f is a homogeneous polynomial over $\mathbb{C}$ in $s(z)^2$ and $t(z)^2$ whose degree is $2k$.*

Proof. Since $f \in M_{2k}(\Gamma_1(8)) \cap M_{2k}(\Gamma_1(4))$, by Proposition 5 in [2] and Proposition 3, we can write

$$\begin{aligned} f(z) &= p(\alpha(z), \beta(z)) \\ &= q(s(z), t(z)) \end{aligned}$$

where p is a symmetric homogeneous polynomial over $\mathbb{C}$ in $\alpha(z) = \theta_2(2z)^4$ and $\beta(z) = \theta_3(2z)^4$ whose degree is k and q is a homogeneous polynomial over $\mathbb{C}$ in $s(z)$ and $t(z)$ whose degree is $4k$. On the other hand, we have the following identities on $\alpha(z)$, $\beta(z)$, $s(z)$ and $t(z)$:

$$\begin{aligned}\alpha(z) &= \theta_2(2z)^4\\ &= \frac{1}{4}(\theta_3^2-\theta_4^2)^2 \quad \text{because } 2\theta_2(2z)^2 = \frac{1}{2}(\theta_3^2-\theta_4^2)\\ &= \frac{1}{4}(\theta_3^4+\theta_4^4-2\theta_3^2\theta_4^2)\\ &= 4(s^2t^2-t^4) \text{ by Lemma 5}\\ \beta(z) &= s(z)^4.\end{aligned}$$

Thus, substituting $-s$ for s and $-t$ for t we see that α and β remain unchanged. This implies that $q(s,-t) = q(s,t)$ and $q(-s,t) = q(s,t)$, that is, q involves the terms whose degrees of s and t are even. $\square$

We readily get from Proposition 6

COROLLARY 7. *Let $f_1, f_2 \in M_{2k}(\Gamma(1))$. Then*

$$\frac{f_1(z)}{f_2(z)} = \frac{p(j_{1,8}(z)^2)}{q(j_{1,8}(z)^2)}$$

where p,q are polynomials in one variable whose degrees are less than or equal to 2k.

4. Proof of Theorem 8

Now we consider the theta series associated to quadratic forms. Let $Q(n,1), A[X]$ and $\theta_A(z)$ be as in the introduction. In cases $n=8$ and 16, the quotients $\theta_A(z)/\theta_B(z)$ are 1 for $A[X], B[X] \in Q(n,1)$. For $n \geq 24$, we shall prove the following theorem.

THEOREM 8. *For any two quadratic forms A, B in $Q(n,1)$ and for $n \geq 24$,*

$$\frac{\theta_A(z)}{\theta_B(z)} = \frac{p(j_{1,8}^2(z))}{q(j_{1,8}^2(z))}$$

where p, q are polynomials over $\mathbb{Q}$ in $j_{1,8}^2$ of degree $\frac{1}{2}(n - n(\text{mod } 24))$.

Since θ_A is an element of $M_{\frac{n}{2}}(\Gamma(1))$, we note that the quotient $\theta_A(z)/\theta_B(z)$ can be written as the form in Corollary 7 with p, q defined over $\mathbb{C}$. The following lemma, however, claims that p and q are in fact defined over $\mathbb{Q}$.

LEMMA 9. *For $n \equiv 0$ (mod 2), let $f \in M_{\frac{n}{2}}(\Gamma(1))$. If f has a Fourier expansion with rational coefficients, then it can be written as a homogeneous polynomial over $\mathbb{Q}$ in $s(z)^2$ and $t(z)^2$ whose degree is $\frac{n}{2}$.*

Proof. The proof goes almost in the same manner as that of Lemma 8 in [2]. □

LEMMA 10. *Let $E_4(z)$ be the normalized Eisenstein series of weight 4 and level 1 and $F(z) = (2\pi)^{-12}\Delta(z)$, where $\Delta(z)$ is the modular discriminant. Then*

$$\begin{aligned} E_4(z) =& s(z)^8 + 56s(z)^6t(z)^2 - 40s(z)^4t(z)^4 - 32s(z)^2t(z)^6 + 16t(z)^8, \\ F(z) =& \frac{1}{4}s(z)^{22}t(z)^2 - \frac{17}{4}s(z)^{20}t(z)^4 + 32s(z)^{18}t(z)^6 - 140s(z)^{16}t(z)^8 \\ &+ 392s(z)^{14}t(z)^{10} - 728s(z)^{12}t(z)^{12} + 896s(z)^{10}t(z)^{14} \\ &- 704s(z)^8t(z)^{16} + 320s(z)^6t(z)^{18} - 64s(z)^4t(z)^{20}. \end{aligned}$$

Proof. Since $E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$ with $\sigma_l(n) := \sum_{d|n} d^l$, again by Proposition 6 and Lemma 9, E_4 can be written as

$$E_4(z) = a_1s(z)^8 + a_2s(z)^6t(z)^2 + a_3s(z)^4t(z)^4 + a_4s(z)^2t(z)^6 + a_5t(z)^8$$

for $a_i \in \mathbb{Q}$. Evaluating both sides at some cusps of $\Gamma_1(8)$ and $\rho = e^{\frac{2\pi i}{3}}$, we shall determine all the $a_i's$. Dividing the above by $t(z)^8$, we come up with

$$\frac{E_4(z)}{t(z)^8} = a_1j_{1,8}(z)^8 + a_2j_{1,8}(z)^6 + a_3j_{1,8}(z)^4 + a_4j_{1,8}(z)^2 + a_5.$$

In the following, we'll use the notation: $f(z)|\gamma = f(\gamma z)$ for $\gamma \in \Gamma(1)$.
(i) $s = 0$; Observe that $S \cdot \infty = 0$.

$$\begin{aligned} E_4(z)|_S &= z^4 \cdot \{E_4(z)|_{[S]_4}\}, \\ t(z)^8|_S &= \theta_3(4z)^8|_S \\ &= \theta_3(-\frac{1}{\frac{z}{4}})^8 \\ &= \{(-i\frac{z}{4})^{\frac{1}{2}}\theta_3(\frac{z}{4})\}^8 \quad \text{by (5)} \\ &= (\frac{z}{4})^4\theta_3(\frac{z}{4})^8. \end{aligned}$$

Therefore we get

$$\lim_{z\to i\infty}\frac{E_4(z)}{t(z)^8}|_S = \lim_{z\to i\infty}\frac{z^4 \cdot \{E_4(z)|_{[S]_4}\}}{(\frac{z}{4})^4\theta_3(\frac{z}{4})^8} = 2^8,$$

from which we derive

(9) $\quad 2^8 = 2^4 a_1 + 2^3 a_2 + 2^2 a_3 + 2a_4 + a_5$ because $j_{1,8}(0) = \sqrt{2}$.

(ii) $s = \frac{1}{2}$; Observe that $(ST^{-2}S)\cdot\infty = \frac{1}{2}$.

$$\begin{aligned} t(z)^8|_S &= (\frac{z}{4})^4\theta_3(\frac{z}{4})^8 \quad \text{by (i)}, \\ t(z)^8|_{ST^{-2}} &= (\frac{z}{4})^4\theta_3(\frac{z}{4})^8|_{T^{-2}} \\ &= \{\frac{1}{4}(z-2)\}^4\theta_3(\frac{1}{4}(z-2))^8 \\ &= \frac{1}{2^8}(z-2)^4\{\theta_3(z) - i\theta_2(z)\}^8. \end{aligned}$$

Here the last equality can be justified as follows:

We recall from [1], p104 that $\theta_2(2z) = \frac{1}{2}\{\theta_3(\frac{z}{2}) - \theta_4(\frac{z}{2})\}$ and $\theta_3(2z) = \frac{1}{2}\{\theta_3(\frac{z}{2}) + \theta_4(\frac{z}{2})\}$. Summing up the above equations and replacing z by $\frac{1}{2}(z-2)$ yields that $\theta_3(\frac{1}{4}(z-2)) = \theta_2(z-2) + \theta_3(z-2)$.
Then it is easily checked by making use of the transformation formulas of theta functions in (1), (2) and (3).
And

$$\begin{aligned} t(z)^8|_{ST^{-2}S} &= \frac{1}{2^8}(z-2)^4\{\theta_3(z) - i\theta_2(z)\}^8|_S \\ &= \frac{1}{2^8}(-\frac{1}{z} - 2)^4\{\theta_3(-\frac{1}{z}) - i\theta_2(-\frac{1}{z})\}^8 \\ &= \frac{1}{2^8}\frac{(2z+1)^4}{z^4}\{(-iz)^{\frac{1}{2}}\theta_3(z) - i(-iz)^{\frac{1}{2}}\theta_4(z)\}^8 \\ &\quad \text{by (4) and (5)} \\ &= \frac{1}{2^8}(2z+1)^4\{\theta_3(z) - i\theta_4(z)\}^8. \end{aligned}$$

On the other hand, since $E_4(z)|_{ST^{-2}S} = (2z+1)^4 \cdot \{E_4(z)|_{[ST^{-2}S]_4}\}$, we have

$$\begin{aligned} \lim_{z\to i\infty} \frac{E_4(z)}{t(z)^8}|_{ST^{-2}S} &= \lim_{z\to i\infty} \frac{(2z+1)^4 \cdot \{E_4(z)|_{[ST^{-2}S]_4}\}}{\frac{1}{2^8}\cdot(2z+1)^4\{\theta_3(z) - i\theta_4(z)\}^8} \\ &= \frac{2^8}{(1-i)^8} = 2^4. \end{aligned}$$

Hence we obtain

(10) $$a_5 = 2^4 \text{ because } j_{1,8}(\frac{1}{2}) = 0.$$

Now, dividing $E_4(z)$ by $s(z)^8$ this time, we work with

$$\frac{E_4(z)}{s(z)^8} = a_1 + \frac{a_2}{j_{1,8}^2} + \frac{a_3}{j_{1,8}^4} + \frac{a_4}{j_{1,8}^6} + \frac{a_5}{j_{1,8}^8}.$$

(iii)$s = \frac{1}{4}$; Observe that $(ST^{-4}S)\cdot\infty = \frac{1}{4}$.
We have

$$s(z)^8|_S = (\frac{z}{2})^4\theta_3(\frac{z}{2})^8 \quad \text{by (5) and so}$$

$$s(z)^8|_{ST^{-4}} = \frac{1}{2^4}(z-4)^4\theta_3(\frac{1}{2}z)^8 \quad \text{by (2) and (3).}$$

Thus we derive

$$\begin{aligned} s(z)^8|_{ST^{-4}S} &= \frac{1}{2^4}(z-4)^4\theta_3(\frac{1}{2}z)^8|_S \\ &= \frac{1}{2^4}(-\frac{1}{z}-4)^4\theta_3(-\frac{1}{2z})^8 \\ &= \frac{1}{2^4}\frac{(4z+1)^4}{z^4}\{(-i2z)^{\frac{1}{2}}\theta_3(2z)\}^8 \quad \text{by (5)} \\ &= \frac{1}{2^4}\frac{(4z+1)^4}{z^4}(-i2z)^4\theta_3(2z)^8 \\ &= (4z+1)^4\theta_3(2z)^8. \end{aligned}$$

On the other hand, since $E_4(z)|_{ST^{-4}S} = (4z+1)^4\cdot\{E_4(z)|_{[ST^{-4}S]_4}\}$,

$$\lim_{z\to i\infty}\frac{E_4(z)}{s(z)^8}|_{ST^{-4}S} = \lim_{z\to i\infty}\frac{(4z+1)^4\cdot\{E_4(z)|_{[ST^{-4}S]_4}\}}{(4z+1)^4\cdot\theta_3(2z)^8} = 1$$

Hence we get

(11) $$a_1 = 1 \text{ because } j_{1,8}(\frac{1}{4}) = \infty.$$

(iv) $s = \infty$;
Since $j_{1,8}(\infty) = 1$, we can easily get

(12) $$a_1 + a_2 + a_3 + a_4 + a_5 = 1.$$

Finally, we have to estimate the value $j_{1,8}(\rho)^2$ to find out a_2, a_3 and a_4 because we can not use other cusps of $X_1(8)$ any more. In fact, $j_{1,8}(\frac{3}{8}) = -1$, $j_{1,8}(\frac{1}{3}) = -\sqrt{2}$ and the expression for E_4 involves only the square terms of $s(z)$ and $t(z)$; hence the evaluation at the cusps $\frac{3}{8}$ and $\frac{1}{3}$ will be the same as that of the cases at $s = \infty$, 0. Therefore we must work out at some other point in $\mathfrak{H}$. We'll take ρ as such a point $z \in \mathfrak{H}$ because we already know the value of $E_4(z)$ at ρ and, moreover, can calculate the value $j_{1,8}(\rho)^2$ in the following.

(v)$z = \rho$; Observe that

$a_1 s(\rho)^8 + a_2 s(\rho)^6 t(\rho)^2 + a_3 s(\rho)^4 t(\rho)^4 + a_4 s(\rho)^2 t(\rho)^6 + a_5 t(\rho)^8 = E_4(\rho) = 0$ ([7], p. 115 or [13], p. 85). Dividing the above by $t(\rho)^8 (\neq 0)$, we get

$$(13) \qquad a_1 j_{1,8}(\rho)^8 + a_2 j_{1,8}(\rho)^6 + a_3 j_{1,8}(\rho)^4 + a_4 j_{1,8}(\rho)^2 + a_5 = 0.$$

At this stage we are required to compute the value of $j_{1,8}(z)^2$ at $z = \rho$.

$$\begin{aligned} j_{1,8}(z)^2 &= \frac{\theta_3(2z)^2}{\theta_3(4z)^2} \\ &= \frac{2\theta_3(2z)^2}{\theta_3(2z)^2 + \theta_4(2z)^2} \quad \text{because } 2\theta_3(2z)^2 = \theta_3(z)^2 + \theta_4(z)^2 \\ &= \frac{2\{\frac{\theta_3(2z)}{\theta_4(2z)}\}^2}{\{\frac{\theta_3(2z)}{\theta_4(2z)}\}^2 + 1} \\ &= \frac{2j_4(4z)^2}{j_4(4z)^2 + 1} \quad \text{because } j_4(z) = \frac{\theta_3(\frac{z}{2})}{\theta_4(\frac{z}{2})} \ ([3]) \\ &= \frac{j_4(2z) + j_4(2z)^{-1}}{\frac{1}{2}(j_4(2z) + j_4(2z)^{-1}) + 1} \\ &\quad \text{because } j_4(2z)^2 = \frac{1}{2}(j_4(z) + j_4(z)^{-1}) \quad ([3], \text{Lemma 15}) \\ &= \frac{2(j_4(2z)^2 + 1)}{(j_4(2z) + 1)^2}. \end{aligned}$$

On the other hand, since $j_4(2\rho) = \zeta_{24}^{-1}$ with $\zeta_n = e^{\frac{2\pi i}{n}}$ (shown in the proof of Proposition 23, [4]), we get

$$j_{1,8}(\rho)^2 = \frac{2(\zeta_{24}^{-2} + 1)}{(\zeta_{24}^{-1} + 1)^2}.$$

Due to (9), (10), (11), (12) and (13) we are able to summarize what we have done so far as follows:

$$a_1 = 1, a_5 = 16, a_2 + a_3 + a_4 = -16, 4a_2 + 2a_3 + a_4 = 112,$$
$$a_1 j_{1,8}(\rho)^8 + a_2 j_{1,8}(\rho)^6 + a_3 j_{1,8}(\rho)^4 + a_4 j_{1,8}(\rho)^2 + a_5 = 0.$$

Plugging $a_3 = -3a_2 + 128$, $a_4 = 2a_2 - 144$ and $j_{1,8}(\rho)^2 = \frac{2(\zeta_{24}^{-2}+1)}{(\zeta_{24}^{-1}+1)^2}$ into the last yields that

$$\begin{aligned} a_2 &= \frac{-4(\zeta_{24}^{-2}+1)^4 - 128(\zeta_{24}^{-2}+1)^2(\zeta_{24}^{-1}+1)^4 + 72(\zeta_{24}^{-2}+1)(\zeta_{24}^{-1}+1)^6 - 4(\zeta_{24}^{-1}+1)^8}{2(\zeta_{24}^{-2}+1)^3(\zeta_{24}^{-1}+1)^2 - 3(\zeta_{24}^{-2}+1)^2(\zeta_{24}^{-1}+1)^4 + (\zeta_{24}^{-2}+1)(\zeta_{24}^{-1}+1)^6} \\ &= \frac{28\{(-\sqrt{2}+\sqrt{6}) + (-3\sqrt{2}+\sqrt{6})i\}}{\frac{1}{2}\{(-\sqrt{2}+\sqrt{6}) + (-3\sqrt{2}+\sqrt{6})i\}} \end{aligned}$$

$= 56.$

Therefore $a_3 = -40$ and $a_4 = -32$. Consequently, we obtain

$$E_4(z) = s(z)^8 + 56s(z)^6t(z)^2 - 40s(z)^4t(z)^4 - 32s(z)^2t(z)^6 + 16t(z)^8.$$

Next, we consider the case of F.

$$\begin{aligned}
F &= \frac{1}{2^8}\theta_2^8\theta_3^8\theta_4^8 \\
&= \frac{1}{2^8}(\theta_3^2\theta_4^2)^4\{\theta_3^4 - \theta_4^4\}^2 \quad \text{by the relation } \theta_3^4 = \theta_2^4 + \theta_4^4 \ \ ([12]) \\
&= \frac{1}{2^8}(\theta_3^2\theta_4^2)^4\{(\theta_3^4 + \theta_4^4)^2 - 4\theta_3^4\theta_4^4\} \\
&= \frac{1}{2^8}(4t^4 - 4s^2t^2 + s^4)^4\{(-8t^4 + 8s^2t^2 + 2s^4)^2 - 4(4t^4 - 4s^2t^2 + s^4)^2\} \\
&\quad \text{by Lemma 5} \\
&= \frac{1}{2^8}(64s^{22}t^2 - 1088s^{20}t^4 + 8192s^{18}t^6 - 35840s^{16}t^8 \\
&\qquad + 100352s^{14}t^{10} - 186368s^{12}t^{12} + 229376s^{10}t^{14} \\
&\qquad - 180224s^8t^{16} + 81920s^6t^{18} - 16384s^4t^{20}) \\
&= \frac{1}{4}s^{22}t^2 - \frac{17}{4}s^{20}t^4 + 32s^{18}t^6 - 140s^{16}t^8 + 392s^{14}t^{10} - 728s^{12}t^{12} \\
&\quad + 896s^{10}t^{14} - 704s^8t^{16} + 320s^6t^{18} - 64s^4t^{20}. \qquad \square
\end{aligned}$$

Since $\mathbb{C}(j)$ is a subfield of $\mathbb{C}(j_{1,8})$ and $j(z) = E_4(z)^3/F(z)$, we get the following corollary by Lemma 10.

Corollary 11.

$$j(z) = \frac{\alpha(z)}{\beta(z)},$$

where $\alpha(z) = 4 \cdot (j_{1,8}^{24} + 168j_{1,8}^{22} + 9288j_{1,8}^{20} + 162080j_{1,8}^{18} - 382224j_{1,8}^{16} - 19200j_{1,8}^{14} + 515840j_{1,8}^{12} - 199680j_{1,8}^{10} - 217344j_{1,8}^{8} + 133120j_{1,8}^{6} + 18432j_{1,8}^{4} - 24576j_{1,8}^{2} + 4096)$ *and* $\beta(z) = j_{1,8}^{22} - 17j_{1,8}^{20} + 128j_{1,8}^{18} - 560j_{1,8}^{16} + 1568j_{1,8}^{14} - 2912j_{1,8}^{12} + 3584j_{1,8}^{10} - 2816j_{1,8}^{8} + 1280j_{1,8}^{6} - 256j_{1,8}^{4}$.

We are now ready to prove Theorem 8. In [9], Theorem 1, we showed that for $n \geq 24$ and for any two quadratic forms $A[X]$ and $B[X]$ in $Q(n,1)$,

$$\frac{\theta_A(z)}{\theta_B(z)} = \frac{f(J(z))}{g(J(z))}, \quad z \in \mathfrak{H}$$

where f and g are polynomials over $\mathbb{Q}$ in $J = \frac{j}{1728}$ of degree $\left[\frac{n}{24}\right]$.

On the other hand, we see from Proposition 6 and Lemma 9 that θ_A and θ_B are homogeneous polynomials over $\mathbb{Q}$ in $s(z)^2$ and $t(z)^2$ whose degree is $\frac{n}{2}$. Therefore, the theorem follows from the above result and Corollary 11.

REMARK 12. We see from Corollary 11 that $\mathbb{C}(j_{1,8})$is an algebraic extension of $\mathbb{C}(j)$ of degree $[\overline{\Gamma}(1) : \overline{\Gamma}_1(8)] = 24$ ([10], Theorem 4.2.5) and the irreducible polynomial of $j_{1,8}$ over $\mathbb{C}(j)$ is given by
$X^{24} + (168 - \frac{1}{4}j)X^{22} + (9288 + \frac{17}{4}j)X^{20} + (162080 - 32j)X^{18} - (382224-140j)X^{16} - (19200+392j)X^{14} + (515840+728j)X^{12} - (199680+896j)X^{10} - (217344-704j)X^{8} + (133120-320j)X^{6} + (18432+64j)X^{4} - 24576X^{2} + 4096.$

5. Example

In case $n = 24$, we are able to completely determine the polynomials discussed in Theorem 8.

THEOREM 13. *For $A \in Q(24, 1)$,*

$$\begin{aligned}
&\theta_A(z)\\
&= s^{24} + (168 + \frac{g_A - 1728}{4})s^{22}t^2 + (9288 - \frac{17}{4}(g_A - 1728))s^{20}t^4\\
&\quad + (162080 + 32(g_A - 1728))s^{18}t^6 - (382224 + 140(g_A - 1728))s^{16}t^8\\
&\quad - (19200 - 392(g_A - 1728))s^{14}t^{10} + (515840 - 728(g_A - 1728))s^{12}t^{12}\\
&\quad - (199680 - 896(g_A - 1728))s^{10}t^{14} - (217344 + 704(g_A - 1728))s^{8}t^{16}\\
&\quad + (133120 + 320(g_A - 1728))s^{6}t^{18} + (18432 - 64(g_A - 1728))s^{4}t^{20}\\
&\quad - 24576s^2t^{22} + 4096t^{24},
\end{aligned}$$

where $g_A = c_A + \frac{762048}{691} = r_A(1) + 1008$ $(\in \mathbb{Z})$ depending on the Niemeier's classification ([11]). Here $c_A = r_A(1) - \frac{65520}{691}$ and $r_A(1)$ denotes the number of integral solutions $x = (x_1, \cdots, x_{24})$ of $A[x] = x^tAx = 2$.

Proof. Since E_{12} and F span $M_{12}(\Gamma(1))$ and $F = \frac{1}{1728}(E_4^3 - E_6^2)$, we can write

$$\begin{aligned}
\theta_A(z) &= E_{12}(z) + c_AF(z)\\
&= E_6(z)^2 + g_AF(z)\\
&= E_4(z)^3 + (g_A - 1728)F(z). \qquad (14)
\end{aligned}$$

Compairing the q-expansions, we get $g_A = c_A + \frac{762048}{691}$. Now, plugging the results from Lemma 10 into (14), we obtain the assertion. □

REMARK. The following list in the Appendix is first related to theta function identities ([1], p120 and p134) discovered independently by J.H. Conway and N.J.A. Sloane (in the notation of [1], θ_{D_n+} instead of θ_{D_n} should be considered), and so it is meaningful to express $\theta_A(z)$ in terms of various Jacobi theta series. Secondly, the second named author has shown in [8] that the Ramanujan number $\tau(m)$ is zero for some integer $m(> 10^{15})$ if and only if $r_A(m) = r_B(m)$ for any two quadratic forms $A[X]$ and $B[X]$ in $Q(24,1)$ whose corresponding theta series θ_A, θ_B are distinct. On the other hand, since the classical Jacobi theta series have simple $q_2(= e^{\pi i z})$ expansions, we feel that the list would be useful in the study of D.H. Lehmer's conjecture on $\tau(m)$ ([13], p98) which reads "$\tau(m) \neq 0$ for all $m \geq 1$".

Appendix

By Theorem 13, the values of $r_A(1)$ in (9) of [8] and following Niemeier's notation, we obtain the following identities.

$$\theta_{3\times E_8}(z) = \theta_{E_8 \oplus D_{16}}(z) =$$
$$s^{24} + 168s^{22}t^2 + 9288s^{20}t^4 + 162080s^{18}t^6 - 382224s^{16}t^8 - 19200s^{14}t^{10}$$
$$+515840s^{12}t^{12} - 199680s^{10}t^{14} - 217344s^8t^{16} + 133120s^6t^{18} + 18432s^4t^{20}$$
$$-24576s^2t^{22} + 4096t^{24}$$

$$\theta_{E_7 \oplus E_7 \oplus D_{10}}(z) = \theta_{E_7 \oplus A_{17}}(z) =$$
$$s^{24} + 96s^{22}t^2 + 10512s^{20}t^4 + 152864s^{18}t^6 - 347904s^{16}t^8 - 132096s^{14}t^{10}$$
$$+725504s^{12}t^{12} - 457728s^{10}t^{14} - 14592s^8t^{16} + 40960s^6t^{18} + 36864s^4t^{20}$$
$$-24576s^2t^{22} + 4096t^{24}$$

$$\theta_{D_{24}}(z) =$$
$$s^{24} + 14s^{22}t^2 + 11906s^{20}t^4 + 142368s^{18}t^6 - 301984s^{16}t^8 - 260672s^{14}t^{10}$$
$$+964288s^{12}t^{12} - 751616s^{10}t^{14} + 216320s^8t^{16} - 64000s^6t^{18} + 57856s^4t^{20}$$
$$-24576s^2t^{22} + 4096t^{24}$$

$$\theta_{D_{12} \oplus D_{12}}(z) =$$
$$s^{24} + 120s^{22}t^2 + 10104s^{20}t^4 + 155936s^{18}t^6 - 361344s^{16}t^8 - 94464s^{14}t^{10}$$
$$+655616s^{12}t^{12} - 371712s^{10}t^{14} - 82176s^8t^{16} + 71680s^6t^{18} + 30720s^4t^{20}$$
$$-24576s^2t^{22} + 4096t^{24}$$

$$\theta_{3\times D_8}(z) =$$
$$s^{24} + 72s^{22}t^2 + 10927s^{20}t^4 + 149792s^{18}t^6 - 334464s^{16}t^8 - 169728s^{14}t^{10}$$
$$+795392s^{12}t^{12} - 543744s^{10}t^{14} + 52992s^8t^{16} + 10240s^6t^{18} + 43008s^4t^{20}$$

$-24576s^2t^{22}+4096t^{24}$

$\theta_{D_9\oplus A_{15}}(z)=$
$s^{24}+84s^{22}t^2+10716s^{20}t^4+151328s^{18}t^6-341184s^{16}t^8-150912s^{14}t^{10}$
$+760448s^{12}t^{12}-500736s^{10}t^{14}+19200s^8t^{16}+25600s^6t^{18}+39936s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{4\times E_6}(z)=\theta_{E_6\oplus D_7\oplus A_{11}}(z)=$
$s^{24}+60s^{22}t^2+11124s^{20}t^4+148256s^{18}t^6-327744s^{16}t^8-188544s^{14}t^{10}$
$+830336s^{12}t^{12}-586752s^{10}t^{14}+86784s^8t^{16}-5120s^6t^{18}+46080s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{4\times D_6}(z)=\theta_{D_6\oplus A_9\oplus A_9}(z)=$
$s^{24}+98s^{22}t^2+11328s^{20}t^4+146720s^{18}t^6-321024s^{16}t^8-207360s^{14}t^{10}$
$+865280s^{12}t^{12}-629760s^{10}t^{14}+120576s^8t^{16}-20480s^6t^{18}+49152s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{D_5\oplus D_5\oplus A_7\oplus A_7}(z)=$
$s^{24}+36s^{22}t^2+11532s^{20}t^4+145184s^{18}t^6-314304s^{16}t^8-226176s^{14}t^{10}$
$+900224s^{12}t^{12}-672768s^{10}t^{14}+154368s^8t^{16}-35840s^6t^{18}+52224s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{3\times A_8}(z)=$
$s^{24}+42s^{22}t^2+11430s^{20}t^4+145952s^{18}t^6-317664s^{16}t^8-216768s^{14}t^{10}$
$+882752s^{12}t^{12}-651264s^{10}t^{14}+137472s^8t^{16}-28160s^6t^{18}+50688s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{A_{24}}(z)=$
$s^{24}+138s^{22}t^2+9798s^{20}t^4+158240s^{18}t^6-371424s^{16}t^8-66240s^{14}t^{10}$
$+603200s^{12}t^{12}-307200s^{10}t^{14}-132864s^8t^{16}+94720s^6t^{18}+26112s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{A_{12}\oplus A_{12}}(z)=$
$s^{24}+66s^{22}t^2+11022s^{20}t^4+149024s^{18}t^6-331104s^{16}t^8-179136s^{14}t^{10}$
$+812864s^{12}t^{12}-565248s^{10}t^{14}+69888s^8t^{16}+2560s^6t^{18}+44544s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{6\times D_4}(z)=\theta_{D_4\oplus(4\times A_5)}(z)=$
$s^{24}+24s^{22}t^2+11736s^{20}t^4+143648s^{18}t^6-307584s^{16}t^8-244992s^{14}t^{10}$
$+935168s^{12}t^{12}-715776s^{10}t^{14}+188160s^8t^{16}-51200s^6t^{18}+55296s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{4\times A_6}(z)=$
$s^{24}+30s^{22}t^2+11634s^{20}t^4+144416s^{18}t^6-310944s^{16}t^8-235584s^{14}t^{10}$
$+917696s^{12}t^{12}-694272s^{10}t^{14}+171264s^8t^{16}-43520s^6t^{18}+53760s^4t^{20}$
$-24576s^2t^{22}+4096t^{24}$

$\theta_{6\times A_4}(z) =$
$s^{24} + 18s^{22}t^2 + 11838s^{20}t^4 + 142880s^{18}t^6 - 304224s^{16}t^8 - 254400s^{14}t^{10} + 952640s^{12}t^{12} - 737280s^{10}t^{14} + 205056s^8t^{16} - 58880s^6t^{18} + 56832s^4t^{20} - 24576s^2t^{22} + 4096t^{24}$

$\theta_{8\times A_3}(z) =$
$s^{24} + 37s^{22}t^2 + 11515s^{20}t^4 + 145312s^{18}t^6 - 314864s^{16}t^8 - 224608s^{14}t^{10} + 897312s^{12}t^{12} - 669184s^{10}t^{14} + 151552s^8t^{16} - 34560s^6t^{18} + 51968s^4t^{20} - 24576s^2t^{22} + 4096t^{24}$

$\theta_{12\times A_2}(z) =$
$s^{24} + 6s^{22}t^2 + 12042s^{20}t^4 + 141344s^{18}t^6 - 297504s^{16}t^8 - 273216s^{14}t^{10} + 987584s^{12}t^{12} - 780288s^{10}t^{14} + 238848s^8t^{16} - 74240s^6t^{18} + 59904s^4t^{20} - 24576s^2t^{22} + 4096t^{24}$

$\theta_{24\times A_1}(z) =$
$s^{24}+12144s^{20}t^4+140576s^{18}t^6-294144s^{16}t^8-282624s^{14}t^{10}+1005056s^{12}t^{12} - 801792s^{10}t^{14} + 255744s^8t^{16} - 81920s^6t^{18} + 61440s^4t^{20} - 24576s^2t^{22} + 4096t^{24}$

$\theta_{G_0}(z) =$
$s^{24} - 12s^{22}t^2 + 12348s^{20}t^4 + 139040s^{18}t^6 - 287424s^{16}t^8 - 301440s^{14}t^{10} + 1040000s^{12}t^{12} - 844800s^{10}t^{14} + 289536s^8t^{16} - 97280s^6t^{18} + 64512s^4t^{20} - 24576s^2t^{22} + 4096t^{24}$

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