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MICROLOCAL ANALYSIS IN THE DENJOY-CARLEMAN CLASS

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Abstract. Making use of the singular spectrum in the Denjoy-Carleman class we prove the microlocal decomposition theorem and quasianalytic versions of Holmgren's uniqueness theorem and watermelon theorem.

0. Introduction

Microlocal analysis means local analysis on the cotangent bundle and emphasizes the importance of localization of singularities in cotangent bundle. Around 1970 Sato introduced and studied the analytic singular spectrum for the hyperfunction and Hörmander defined the wave front set $WF(u)$ for the distributions in [4, 6], and Hörmander also introduced the wave front set $WF_M(u)$ with respect to the Denjoy-Carleman class $\mathcal{C}^M$ in [5, 6], which includes the analytic wave front set $WF_A(u)$ as a special case. On the other hand, making use of the FBI (Fourier-Bros-Iagolnitzer) transforms Bros and Iagolnitzer introduced the essential spectrum, which was shown to be equal to the analytic singular spectrum of Sato and the analytic wave front set of Hörmander by Bony in [1].

Recently Chung and Kim have unified the singular spectra for the $\mathcal{C}^\infty$ class, the analytic class and the Denjoy-Carleman class, both quasianalytic and non quasianalytic, in the category of the Fourier hyperfunctions in the spirit of the essential spectrum of Bros and Iagolnitzer in [2]. Applying their singular spectrum with respect to the Denjoy-Carleman class they give a very simple and direct proof of the following

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fundamental theorem for the Fourier hyperfunctions

$$WF_M(u) \subset \operatorname{Char} P \cup WF_M(P(x,D)u).$$

The purpose of this paper is to study several microlocal properties of the Fourier hyperfunctions by using a singular spectrum with respect to the Denjoy-Carleman class $WF_M(u)$ introduced in [2]. Section 1 is devoted to providing the necessary definitions and preliminaries. Making use of the above singular spectrum we prove the decomposition theorem in Section 2. Also, we prove quasianalytic versions of Watermelon theorem and Holmgren's uniqueness theorem in Section 3.

When we write up this paper we learn that Hörmander already has published a preprint [7] which contains a quasianalytic version of Holmgren's uniqueness theorem. But, in our opinion our proof is simpler and direct.

1. Preliminaries

Let $E(x,t)$ be the n-dimensional heat kernel;

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0 \\ 0, & t \le 0. \end{cases}$$

Let $u \in \mathcal{F}'(\mathbb{R}^n)$, i.e., let u be a Fourier hyperfunction in $\mathbb{R}^n$ (see below Definition 1.2 for definition). Then $U(x,t) = u_y(E(x-y,t))$, which is called the *defining function* of u, is a $\mathcal{C}^\infty$ function in the half space $\mathbb{R}^n_+$. Furthermore, $U(\cdot,t)$ can be extended as an entire function in $\mathbb{C}^n$ for each $t > 0$.

Note that

$$(1.1) \qquad U(x+i\xi,t) = (4\pi t)^{-n/2} \exp[\xi^2/4t - i\langle x,\xi\rangle/2t] \times u_y(\exp[-(x-y)^2/4t + i\langle y,\xi\rangle/2t]),$$

which is necessary in Section 3.

The following is the local regularity theorem for the Denjoy-Carleman class $\mathcal{C}^M$ by Chung-Kim as in **[2]** for the Fourier hyperfunctions.

THEOREM 1.1 ([2]). *Let u be a Fourier hyperfunction. Then u is $\mathcal{C}^M$ near x_0 if and only if there are positive constants C, γ, N and a neighborhood U of x_0 such that*

$$|u_y(\exp[-|\xi|(x-y)^2/2 - i\langle y,\xi\rangle]\ )| \le C \exp[-M(\gamma|\xi|)]$$

for all $x \in U$ and $|\xi| \ge N$, where $M(t)$ is the associated function of M_p (see (2.1) for definition).

REMARK. Since every analytic functional in $\mathbb{R}^n$ can be regarded as a Fourier hyperfunction, the above theorem is true for each analytic functional. From now on we denote by $\mathcal{A}'(\mathbb{R}^n)$ the space of analytic functionals in $\mathbb{R}^n$.

With this characterization of C^M functions we can introduce a singular spectrum $WF_M(u)$ with respect to the Denjoy-Carleman class for the Fourier hyperfunction u as follows:

DEFINITION 1.2 ([2]). Let u be a Fourier hyperfunction. Then we denote by $WF_M(u)$ the complement of the set of (x_0, ξ_0), $\xi_0 \neq 0$ such that there exist a neighborhood U of x_0 and a conic neighborhood Γ of ξ_0 such that for some positive constants C, γ and N,

$$|u_y(\exp[-|\xi|(x-y)^2/2 - i\langle y, \xi\rangle])| \leq C \exp[-M(\gamma|\xi|)] \tag{1.2}$$

for all $x \in U$ and $\xi \in \Gamma \cap \{\xi \in \mathbb{R}^n;\ |\xi| \geq N\}$.

For the above definitions we first recall the Fourier hyperfunctions. Let us denote by $\mathbb{D}^n$ the compactification $\mathbb{R}^n \cup S_\infty^{n-1}$ of $\mathbb{R}^n$ where S_∞^{n-1} is an $(n-1)$-dimensional sphere at infinity. When x is a vector in $\mathbb{R}^n \backslash \{0\}$, we denote by $x\infty$ the point on S^{n-1} which is represented by x, where we identify S^{n-1} with $\mathbb{R}^n \backslash \{0\}/\mathbb{R}^+$. The space $\mathbb{D}^n$ is given the natural topology, that is:

(i) If a point $x \in \mathbb{D}^n$ belongs to $\mathbb{R}^n$, a fundamental system of neighborhoods of x is the set of all open balls containing the point x.
(ii) If a point $x \in \mathbb{D}^n$ belongs to S_∞^{n-1}, a fundamental system of neighborhoods of x $(= y\infty)$ is given by the following family

$$U_{\tilde{\Delta},A}(y\infty) = \{x \in \mathbb{R}^n; x/|x| \in \tilde{\Delta},\ |x| > A\} \cup \{a\infty; a \in \tilde{\Delta}\},$$

where $\tilde{\Delta}$ is a neighborhood of y in the $(n-1)$-dimensional sphere S^{n-1} in $\mathbb{R}^n$.

DEFINITION 1.3. Let K be a compact set in $\mathbb{D}^n$. We say that ϕ is in $\mathcal{F}(K)$ if $\phi \in C^\infty(\Omega \cap \mathbb{R}^n)$ for some neighborhood Ω of K and if there are positive constants h and k such that

$$|\phi|_{k,h} = \sup_{\substack{x \in \Omega \cap \mathbb{R}^n \\ \alpha}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}\alpha!} \exp k|x| < \infty, \tag{1.3}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j$ for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$, with $\mathbb{N}_0$ the set of nonnegative integers.

We denote by $\mathcal{F}'(K)$ the strong dual space of $\mathcal{F}(K)$ and call its elements *Fourier hyperfunctions* carried by K. Especially if $K = \mathbb{D}^n$ then we often use the notation $\mathcal{F}'$ simply instead of $\mathcal{F}'(\mathbb{D}^n)$.

In fact, the space $\mathcal{F}(K)$ is shown to be topologically equivalent to the space $\mathcal{P}_*(K)$ of holomorphic functions in a complex neighborhood $\Omega \cap \mathbb{R}^n + i\{|y| < r\}$ of K in $\mathbb{C}^n$ satisfying the estimate

$$\sup_{z\in\Omega\cap\mathbb{R}^n+i\{|y|<r\}} |\phi(z)| \exp k|z| < \infty \tag{1.4}$$

for some k, which was originally defined by Sato-Kawai. Here $z = x + iy$ and Ω is a neighborhood of K in $\mathbb{D}^n$.

2. Decomposition theorem

We now introduce the Denjoy-Carleman class which lies between C^∞ class and the analytic class.

Let M_p, $p = 0, 1, 2, 3, \dots$, be a sequence of positive numbers and Ω be an open set in $\mathbb{R}^n$. We impose the following conditions on M_p;

(M.0) There is a constant $h > 0$ such that

$$p! \le Ch^p M_p, \quad p = 0, 1, 2, 3, \dots.$$

(M.1) $M_p^2 \le M_{p-1}M_{p+1}$, $p = 1, 2, 3, \dots$.

(M.2)$'$ There is a constant $H > 0$ such that

$$M_{p+1} \le CH^p M_p, \quad p = 0, 1, 2, 3, \dots.$$

For each sequence (M_p), its associated function $M(t)$ on $[0, \infty)$ is defined as follows;

$$M(t) = \sup_p \log \frac{M_0 t^p}{M_p}. \tag{2.1}$$

Then (M.0) gives

$$\alpha \log(t) \le M(t) \le \beta t, \quad t > 0 \tag{2.2}$$

for some $\alpha,\ \beta > 0$.

DEFINITION 2.1. We denote by the Denjoy-Carleman class $\mathcal{C}^M(\Omega)$ the set of all $\phi \in \mathcal{C}^\infty(\Omega)$ such that on each compact set $K \subset \Omega$ its derivatives satisfy the estimates

$$\sup_{x \in K} |\partial^\alpha \phi(x)| \le C h^{|\alpha|} M_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n$$

for some constants $C > 0$ and $h > 0$.

LEMMA 2.2. *Let $u \in \mathcal{A}'(\mathbb{R}^n)$. Then $WF_M(u) \subset WF_A(u)$.*

Proof. Let $(x_0, \xi_0) \notin WF_A(u)$. Then there exist a neighborhood U of x_0 and a conic neighborhood Γ of ξ_0 such that

$$|u_y(\exp[-|\xi|(x-y)^2/2 + i\langle y, \xi\rangle]\ )| \le C \exp[-c|\xi|].$$

Now (2.2) implies that

$$\exp[-M(\gamma|\xi|)] \ge \exp[-\gamma\beta|\xi|] \ge C \exp[-c|\xi|]$$

if γ is sufficiently small. Hence $(x_0, \xi_0) \notin WF_M(u)$. □

We now state and prove the main theorem in this section, which generalizes the microlocal decomposition theorem in the category of *analytic class* given in [6].

THEOREM 2.3. *Let $u \in \mathcal{A}'(\mathbb{R}^n)$ and $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$. Then $(x_0, \xi_0) \notin WF_M(u)$ if and only if there are open convex cones Γ_1, Γ_2, $\cdots, \Gamma_N$ in $\{\xi \in \mathbb{R}^n \,;\, \langle \xi, \xi_0\rangle < 0\}$, bounded open neighborhood Z of x_0 in $\mathbb{C}^n$, $u_0 \in \mathcal{C}^M$ in Z, f_k holomorphic in $Z \cap \{\mathbb{R}^n + i\Gamma_k\}, k = 1, 2, 3, \cdots, N$, so that*

$$u = u_0 + \sum_1^N b_{\Gamma_k}(f_k) \quad \text{on} \quad Z \cap \mathbb{R}^n, \tag{2.3}$$

where $b_{\Gamma_k}(f_k)$ is the boundary value of f_k from Γ_k (see [6, p.342] for more details of this boundary value).

Proof. Let $u \in \mathcal{A}'(\mathbb{R}^n)$. Then for $|x - x_0| < \epsilon$ with $\epsilon > 0$ sufficiently small we have

$$u(x) = (2\pi)^{-n} \iint_{\substack{|\beta - x_0| \le 2\epsilon \\ |\xi| \ge 1}} u_y(\exp[-|\xi|(\beta - y)^2/2 - i\langle x - y, \xi\rangle)) \times (|\xi|/2\pi)^{n/2} d\xi \, d\beta + w_\epsilon(x),$$

where $w_\epsilon(x)$ is analytic in $\{x \in \mathbb{R}^n \; ; \; |x - x_0| < \epsilon\}$.

We now assume that $(x_0, \xi_0) \notin WF_M(u)$. Then there is a small conic neighborhood V of ξ_0 in $\mathbb{R}^n \setminus \{0\}$ so that the following estimation holds;

$$\begin{aligned} &\Big| \int_{|\beta - x_0| \le \epsilon} u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle x - y, \xi\rangle]) \, d\beta \Big| \\ &\qquad \le C \exp[-M(\gamma|\xi|)], \quad \xi \in V, \; |\xi| \ge N, \end{aligned}$$

where C and c are positive constants independent of $x \in \mathbb{R}^n$, $|x - x_0| < \epsilon$ and $\xi \in V$.

For $\xi_i \in \mathbb{R}^n \setminus \{0\}$ with $\langle \xi_i, \xi_0 \rangle < 0$, let Γ_{ξ_i} be a closed conic neighborhood of ξ_i such that

$$\langle \eta, \xi_i \rangle > 0, \quad \langle \eta, \xi_0 \rangle < 0 \;\text{ for all }\; \eta \in \Gamma_{\xi_i}.$$

Also let $\Gamma^o_{\xi_i}$ be the dual cone of Γ_{ξ_i}. Then we can choose $\xi_1, \xi_2, \ldots, \xi_N$ in $\{\xi \in \mathbb{R}^n \; ; \; \langle \xi, \xi_0 \rangle < 0\}$ so that

$$\mathbb{R}^n \setminus \{0\} = V \cup \mathrm{Int}(\Gamma^o_{\xi_1}) \cup \mathrm{Int}(\Gamma^o_{\xi_2}) \cup \cdots \cup \mathrm{Int}(\Gamma^o_{\xi_N}).$$

Let $\chi_0, \chi_1, \chi_2, \ldots, \chi_N$ be a continuous partition of unity of $\mathcal{S}^{n-1}$ subordinate to the covering $\{\mathcal{S}^{n-1} \cap V, \; \mathcal{S}^{n-1} \cap \mathrm{Int}(\Gamma^o_{\xi_1}), \ldots, \; \mathcal{S}^{n-1} \cap \mathrm{Int}(\Gamma^o_{\xi_N})\}$.

Then we have

$$\begin{aligned} u(x) = (2\pi)^{-n} \iint_{\substack{|\beta - x_0| \le 2\epsilon \\ |\xi| \ge 1}} & u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle x - y, \xi\rangle]) \\ & \times (|\xi|/2\pi)^{n/2} d\xi \; d\beta + w_\epsilon(x) \\ = (2\pi)^{-n} \iint_{\substack{|\beta - x_0| \le 2\epsilon \\ 1 \le |\xi| \le N}} & u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle x - y, \xi\rangle]) \\ & \times (|\xi|/2\pi)^{n/2} d\xi \; d\beta \\ + \sum_0^N (2\pi)^{-n} \iint_{\substack{|\beta - x_0| \le 2\epsilon \\ |\xi| \ge N}} & u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle x - y, \xi\rangle]) \\ & \times |\xi|/2\pi)^{n/2} \chi_j(\xi/|\xi|) \, d\xi \; d\beta + w_\epsilon(x) \\ = (\mathrm{I}) + (\mathrm{II}) + w_\epsilon(x). & \end{aligned}$$

Observe that (I) and $w_\epsilon(x)$ are analytic functions of x near x_0.

We also have

$$\begin{aligned} (\mathrm{II}) = (2\pi)^{-n} \iint & u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle x - y, \xi\rangle]) \\ & \times (|\xi|/2\pi)^{n/2} \chi_0(\xi/|\xi|) \, d\xi d\beta \\ + \sum_1^N (2\pi)^{-n} \iint_{\substack{|\beta - x_0| \le 2\epsilon \\ |\xi| \ge N}} & u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle x - y, \xi\rangle]) \\ & \times (|\xi|/2\pi)^{n/2} \chi_j(\xi/|\xi|) \, d\xi d\beta \\ = u_0(x) + \sum_1^N u_j. & \end{aligned}$$

Here we write $u_j = b_{\Gamma_j}(f_j)$, and

$$\begin{aligned} f_j(z) = (2\pi)^{-n} \iint_{\substack{|\beta - x_0| \le 2\epsilon \\ |\xi| \ge 1}} & u_y(\exp[-|\xi|(\beta - y)^2/2 + i\langle z - y, \xi\rangle]) \\ & \times (|\xi|/2\pi)^{n/2} \chi_j(\xi/|\xi|) d\xi \; d\beta. \end{aligned}$$

Then $f_j(z)$ is an analytic function in $\tilde{Z} \cap (\mathbb{R}^n + i\Gamma_{\xi_j})$ with a small complex neighborhood $\tilde{Z}$ of x_0. Therefore it remains only to show that f_0 is in $\mathcal{C}^M$ near x_0. In fact,

$$\begin{aligned}
|\partial^\alpha f_0(x)| &= \Big|\frac{\partial^\alpha}{\partial x^\alpha}(2\pi)^{-n} \iint_{\substack{|\beta - x_0|\le 2\epsilon\\ |\xi|\ge N}} u_y(\exp[-|\xi|(\beta-y)^2/2 + i\langle y-x,\xi\rangle]) \\
&\qquad \times (|\xi|/2\pi)^{n/2}\chi_0(\xi/|\xi|)\, d\xi\, d\beta\Big| \\
&\le C \iint_{\substack{|\beta - x_0|\le 2\epsilon\\ \xi\in V, |\xi|\ge N}} (|\xi|/2\pi)^{n/2} \exp[-M(\gamma|\xi|)]\ |\xi|^{|\alpha|}\, d\beta d\xi \\
&= \iint_{\substack{|\beta - x_0|\le \epsilon\\ \xi\in V, |\xi|\ge N}} (|\xi|/2\pi)^{n/2}\ |\xi|^{-2n}\ \exp[-M(\gamma|\xi|)]\ |\xi|^{|\alpha|+2n}\, d\beta d\xi \\
&\le C'(1/\gamma)^{|\alpha|+2n}\ M_{|\alpha|+2n} \\
&\le C(H/\gamma)^{|\alpha|}\ M_{|\alpha|}
\end{aligned}$$

for some constants $C = C(n) > 0$ and $H > 0$. Therefore, there exist constants $C > 0$, $h > 0$ such that

$$|\partial^\alpha f_0(x)| \le Ch^{|\alpha|}\ M_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n,$$

for all x in the region $|x - x_0| < \epsilon$, which means that u belongs to $\mathcal{C}^M$ in $|x - x_0|$.

Conversely, suppose the condition (2.3) holds. Then $WF_A(u_j) \subset \Gamma_j^o$, $j = 1, 2, \ldots, N$. Hence $(x_0, \xi_0) \notin \cup_{j=1}^N WF_A(u_j)$. Therefore, $(x_0, \xi_0) \notin WF_A(u - u_0)$ and hence $(x_0, \xi_0) \notin WF_M(u - u_0)$ by Lemma 2.2. Note that by Theorem 1.1, u_0 is $\mathcal{C}^M$ near x_0, which implies that

$$|u_{0y}(\exp[-|\xi|(x-y)^2/2 - i\langle y,\xi\rangle]\ )| \le C \exp[-M(\gamma|\xi|)]$$

for all x near x_0 and $|\xi| \ge N$. Therefore, we must have

$$|u_y(\exp[-|\xi|(x-y)^2/2 - i\langle y,\xi\rangle]\ )| \le C' \exp[-M(\gamma'|\xi|)]$$

for all x near x_0 and $|\xi| \ge N$. Hence $(x_0, \xi_0) \notin WF_M(u)$, which completes the proof. □

REMARK. In the above theorem we do not exclude the case when $\mathcal{C}^M$ is the C^∞ class or the analytic class. Therefore, our approach is very natural, since this unifies all the cases of the $\mathcal{C}^M$, differentiable and analytic classes.

3. Quasianalytic Holmgren's uniqueness theorem and Watermelon theorem

In this section we will prove the quasianalytic versions of Watermelon theorem, and Holmgren's uniqueness theorem consequently. We closely follow the method of Sjöstrand as in [10, 11, 8].

We first note that (1.2) is equivalent to the following condition.

$$|U(x+i\xi,t)| \leq C\,\exp\left[1/4t - M\left(\gamma/2t\right)\right] \tag{3.1}$$

for all $x \in \Omega,\ \xi \in (-\Gamma) \cap S^{n-1}$ and sufficiently small $t > 0$.

Also, for every $\varepsilon > 0$ we always obtain from (1.1) the estimate of form

$$|U(x+i\xi,t)| \leq C_\varepsilon \exp[\left(\varepsilon(1+|\xi|) + \xi^2 - \operatorname{dist}(x, \operatorname{supp} u)^2\right)/4t] \tag{3.2}$$

for $t > 0,\ z = x + i\xi \in \mathbb{C}^n$ and $u \in \mathcal{A}'(\mathbb{R}^n)$.

Lemma 3.1. *Let $u \in \mathcal{A}'(\mathbb{R}^n)$ and $\operatorname{supp} u \subset \{x_n \geq 0\}$. Also, let Ω be a neighborhood of 0 and let $\Gamma \subset \mathbb{R}^n \setminus 0$ be an open cone. Suppose that for some $\gamma > 0$,*

$$|U(x+i\xi,t)| \leq C\exp[1/4t - M(\gamma/2t)]$$

for $x \in \Omega,\ \xi \in (-\Gamma) \cap S^{n-1}$ and for sufficiently small $t > 0$. Then for each small $r > 0$ there exist a neighborhood ω of 0 and a positive constant α such that

$$|U(x+i(\xi', \xi_n+\mu),t)| \leq C_r \exp[(r+\xi'^2+(\xi_n+\mu)^2)/4t - M(\alpha/t)] \tag{3.3}$$

for $x \in \omega,\ \mu \in \mathbb{R}$ and for all sufficiently small $t > 0$.

Proof. First of all, it follows from (3.2) that

$$\begin{aligned} &|U(x', x_n) + i(\xi', \xi_n+\mu), t)| \\ &\leq \begin{cases} C_\varepsilon \exp\left[\frac{(\varepsilon(1+|\xi'|+|\xi_n+\mu|)+\xi'^2+(\xi_n+\mu)^2)}{4t}\right], & x_n \geq 0, \\ C_\varepsilon \exp\left[\frac{(\varepsilon(1+|\xi'|+|\xi_n+\mu|)+\xi'^2+(\xi_n+\mu)^2-x_n^2)}{4t}\right], & x_n \leq 0 \end{cases} \\ &\leq C_r \exp\left[\frac{(r+\xi'^2)}{4t}\right] \exp[\lambda\varphi(x_n+i\mu)], \end{aligned} \tag{3.4}$$

where

$$\lambda = 1/2t > 0,$$

$$\varphi(x_n + i\mu) = \begin{cases} (\xi_n + \mu)^2/2, & x_n \geq 0, \\ ((\xi_n + \mu)^2 - x_n^2)/2, & x_n \leq 0. \end{cases}$$

From now on we may assume that $\mu > 0$. Consider now the function

$$v(x_n + i\mu) = \frac{1}{C_r} \exp[-(r + \xi'^2)/4t] U((x', x_n) + i(\xi', \xi_n + \mu), t)$$

for fixed (x', ξ'). Let $R = \{(x_n, \mu); |x_n| \leq a,\ 0 \leq \mu \leq b\}$. Then v is holomorphic in R and (3.4) implies that

$$|v(x_n + i\mu)| \leq \exp[\lambda\varphi(x_n + i\mu)], \quad (x_n, \mu) \in R. \tag{3.5}$$

By hypothesis we can write

$$\begin{aligned} |v(x_n)| &= \frac{1}{C_r} \exp[-(r + \xi'^2)/4t] |U(x + i\xi, t)| \\ &\leq \exp[\xi_n^2/4t - M(\gamma/2t)] \\ &= \exp[\lambda(\xi_n^2/2 - 2tM(\gamma/2t))] \\ &= \exp[\lambda(\varphi(x_n) - 2tM(\gamma/2t))]. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we obtain the following subharmonic function

$$(x_n, \mu) \mapsto \frac{1}{\lambda} \log |v| - \varphi(x_n + i\mu) + 2tM(\gamma/2t) f(\mu) \sin \frac{\pi}{a}(\delta - x_n) \tag{3.7}$$

on

$$R_\delta = \{(x_n, \mu); -a + \delta \leq x_n \leq \delta,\ 0 \leq \mu \leq b\},$$

where

$$f(\mu) = \left(e^{\pi(b-\mu)/a} - e^{-\pi(b-\mu)/a}\right) / \left(e^{\pi b/a} - e^{-\pi b/a}\right).$$

Observe that the right hand side of (3.7) is negative or zero on the boundary of R_δ provided that $\delta > 0$ is sufficiently small.

Define a function $\psi : R \to \mathbb{R}$ by

(3.8)

$$\psi(x_n, \mu) = \begin{cases} \varphi(x_n + i\mu) - 2tM(\gamma/2t) f(\mu) \sin(\pi(\delta - x_n)/a), & (x_n, \mu) \in R_\delta, \\ \varphi(x_n + i\mu), & (x_n, \mu) \notin R_\delta. \end{cases}$$

Then ψ is continuous on R and

$$|v(z)| \le \exp\,[\lambda\psi(z)], \quad z = x_n + i\mu \in R. \tag{3.9}$$

Note that if μ belongs to a compact subinterval of $[0, b)$ we have

$$\begin{aligned} |v(i\mu)| &\le \exp\,[\lambda\psi(i\mu)] \\ &= \exp[\lambda(\varphi(i\mu) - 2tM(\gamma/2t)d], \quad d > 0 \\ &= \exp[(\xi_n + \mu)^2/4t - dM(\gamma/2t)] \\ &\le \exp[(\xi_n + \mu)^2/4t - M(\beta/2t)] \\ &\le \exp[(\xi_n + \mu)^2/4t - c - M(\alpha/2t)]. \end{aligned} \tag{3.10}$$

Therefore

$$\begin{aligned} &|U((x', x_n) + i(\xi', \xi_n + \mu), t)| \\ \le &C_r \exp\left[\left(r + {\xi'}^2 + (\xi_n + \mu)^2)\right) /4t - M(\alpha/2t)\right] \end{aligned} \tag{3.11}$$

for sufficiently small $|x_n|$ and $|\eta|$. □

LEMMA 3.2. *Let* $\tilde{U}(x,t) = \tilde{u}_y(E(x-y,t))$, *where*

$$\tilde{u}_y = \lambda^{n/2} \exp[(1/\lambda - 1)/4t] \exp[(1-\lambda)(x-y)^2/4t] U_y.$$

Then

$$U(x + i\xi, t/\lambda) = \tilde{U}(x + i\lambda\xi, t), \quad |\lambda\xi| = 1.$$

Proof. By simple calculations we obtain that

$$
\begin{aligned}
&U(x+i\xi,t/\lambda)\\
=&\lambda^{n/2}(4\pi t)^{-n/2}\exp[\frac{\lambda\xi^2}{4t}-\frac{i\langle\lambda\xi\rangle}{2t}]u_y(\exp[-\frac{\lambda(x-y)^2}{4t}+\frac{i\langle y,\lambda\xi\rangle}{2t}])\\
=&\lambda^{n/2}\exp[\frac{(\frac{1}{\lambda}-1)}{4t}](4\pi t)^{-n/2}\exp[\frac{1}{4t}-\frac{i\langle x,\lambda\xi\rangle}{2t}]\\
&\qquad\times u_y(\exp[\frac{(1-\lambda)(x-y)^2}{4t}]\exp[-\frac{(x-y)^2}{4t}+\frac{i\langle y,\lambda\xi\rangle}{2t}])\\
=&(4\pi t)^{-n/2}\exp[\frac{1}{4t}-\frac{i\langle x,\lambda\xi\rangle}{2t}]\langle\lambda^{n/2}\exp[\frac{(\frac{1}{\lambda}-1)}{4t}]\\
&\qquad\times\exp[\frac{(1-\lambda)(x-y)^2}{4t}]u_y,\exp[-\frac{(x-y)^2}{4t}+\frac{i\langle y,\lambda\xi\rangle}{2t}]\rangle\\
=&(4\pi t)^{-n/2}\exp[\frac{1}{4t}-\frac{i\langle x,\lambda\xi\rangle}{2t}]\\
&\qquad\times\langle\tilde{u}_y,\exp[-\frac{(x-y)^2}{4t}+\frac{i\langle y,\lambda\xi\rangle}{2t}]\rangle\\
=&\tilde{U}(x+\lambda\xi,t)
\end{aligned}
$$

for every $|\lambda\xi|=1$. Thus we complete the proof. □

Observe that

$$\lambda^{n/2}\exp[(1/x-1)/4t]\exp[(1-\lambda)(x-y)^2/4t]\neq 0.$$

Hence

$$WF_M(u)=WF_M(\tilde{u}).$$

We now state and prove the main theorem of this section.

THEOREM 3.3. *Let $u\in\mathcal{A}'(\mathbb{R}^n)$ and $\operatorname{supp}u\subset\{x_n\geq 0\}$, and let $(0,\xi_0)\notin WF_M(u)$, $\xi_0\neq 0$. Then $(0,(\xi'_0,\mu))\notin WF_M(u)$ for all μ with $|\mu|\leq|\xi_{0n}|$, where $\xi_0=(\xi_0',\xi_{0n})$.*

Proof. We may assume that $|\xi_0|\neq 0$, $\xi_{0n}\geqq 0$. Since $(0,\xi_0)\notin WF_M(u)$, there exist positive constants C and γ, a neighborhood Ω of 0 and an open cone $\Gamma\subset\mathbb{R}^n\setminus 0$ containing ξ_0 such that

$$|U(x+i\xi,t)|\leqq C\exp[1/4t-M(\gamma/2t)],$$

for $x \in \Omega$, $\xi \in (-\Gamma) \cap \mathcal{S}^{n-1}$, sufficiently small t. For given $\lambda > 1$ consider all $(\xi', \xi_n + \mu)$ such that

$$\lambda(\xi', \xi_n + \mu) \in \mathcal{S}^{n-1}.$$

Choose $r > 0$ so that $r\lambda + 1/\lambda = 1$. Then Lemma 3.2 implies that

$$\begin{aligned}|\tilde{U}(x + \lambda(\xi', \xi_n + \mu), t)| &= |U(x + i(\xi', \xi_n + \mu), t/\lambda)| \\ &\leq C_r \exp[\lambda(r + \xi'^2 + (\xi_n + \mu)^2)/4t - M(\alpha/t)] \\ &= C_r \exp[(\lambda r + 1/\lambda)/4t - M(\alpha/t)] \\ &= C_r \exp[1/4t - M(\alpha/t)]\end{aligned}$$

for $\lambda(\xi', \xi_n + \mu) \in \mathcal{S}^{n-1}$.

Hence

$$\lambda(\xi', \xi_n + \mu) \notin WF_M(\tilde{u}) = WF_M(u),$$

which completes the proof. □

COROLLARY 3.4. *In addition to the hypotheses of the above theorem let* $(0, \xi) \in WF_M(u)$. *Then* $(0, (\xi', \mu)) \in WF_M(u)$ *for all* μ, $|\mu| \geq |\xi_n|$, *where* $\xi = (\xi', \xi_n)$.

In the course of the proof of the above theorem, we can also take $\lambda < 1$ provided that the associated function M satisfies the following condition: There are positive constants c and k such that

$$\text{(A)} \qquad ct \leq M(kt) - M(t)$$

for all sufficiently large t. However, we can easily show that only the analytic class satisfies the condition (A).

COROLLARY 3.5. *If M satisfies the condition* (A), *then under the condition of Theorem 3.3 we have* $(0, (\xi', \mu)) \notin WF_M(u)$ *for all* $\mu \in \mathbb{R}$.

The following Corollary 3.6 plays a crucial role in a uniqueness theorem of Holmgren type. In the case $n = 1$, Hormander proved that if $u = 0$ in $(x^0 - \epsilon, x^0)$ or $(x^0, x^0 + \epsilon)$ for some $\epsilon > 0$ and $(x^0, 1) \notin WF_L(u)$, then L is non-quasianalytic. Making use of the singular spectrum with respect to the Denjoy-Carleman class we give here a very short and direct proof of the following quasianalytic version of Watermelon theorem.

COROLLARY 3.6. *In addition to the hypotheses of Theorem 3.3, let us assume that $\mathcal{C}^M$ is quasianalytic and $0 \in \operatorname{supp} u$. Then*

$$(0,(0,\dots,0,\pm 1)) \in WF_M(u).$$

Proof. We observe that there exists $\xi \in S^{n-1}$ such that $(0,\xi) \in WF_M(u)$. Otherwise, u is $\mathcal{C}^M$ in a neighborhood of 0. But then $u = 0$ in a neighborhood of 0, since $\mathcal{C}^M$ is quasianalytic and $\operatorname{supp} u \subset \{x | \, x_n \geq 0\}$. Hence $0 \notin \operatorname{supp} u$, which is a contradiction. First, let us consider the case that $n = 1$. Decomposing u into real and complex parts we may assume that u is real. Suppose that $(0,1) \notin WF_M(u)$. In this case we have the following estimate

$$|u_y\left(\exp[-t(x-y)^2 + ity]\right)| \leq C \exp\left[-M(\gamma t)\right]$$

for sufficiently large $t > 0$ (see Definition 1.2). Therefore, if u is real we also have the following estimates

$$|u_y\left(\exp[-t(x-y)^2]\cos(ty)\right)| \leq C \exp\left[-M(\gamma t)\right]$$

$$|u_y\left(\exp[-t(x-y)^2]\sin(ty)\right)| \leq C \exp\left[-M(\gamma t)\right].$$

Therefore

$$|u_y\left(\exp[-t(x-y)^2 + ity]\right)| \leq C \exp\left[-M(\gamma t)\right].$$

Hence $(0,-1) \notin WF_M(u)$. So we must have $(0,\pm 1) \in WF_M(u)$ if $0 \in \operatorname{supp} u$. Decomposing u into real and complex parts we conclude that $(0,\pm 1) \in WF_M(u)$. Now assume that $n \geq 2$. Then, since $(0,\xi) \in WF_M(u)$ for some $\xi \in S^n$ we obtain that $(0,(0,\dots,0,\pm 1) \in WF_M(u)$ by Corollary 3.5. □

We are finally in a position to state and prove a quasianalytic version of Holmgren' uniqueness theorem.

THEOREM 3.7. *Suppose $\mathcal{C}^M$ is quasianalytic. If $u \in \mathcal{A}'(\mathbb{R}^n)$ vanishes on one side of an analytic hypersurface S and $(0,\xi) \notin WF_M(u)$, where ξ is one of the conormals to S at x, then u must vanish in some neighborhood of x.*

Proof. If u does not vanish in a neighborhood of x, then by Corollary 3.6 and an analytic change of coordinates, we must have $(0,\xi) \in WF_M(u)$, which is a contradiction. □

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