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THE RANDERS CHANGES OF FINSLER SPACES WITH (α,β)-METRICS OF DOUGLAS TYPE

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ABSTRACT. A change of Finsler metric $L(x,y) \longrightarrow \overline{L}(x,y)$ is called a Randers change of L, if $\overline{L}(x,y) = L(x,y)+\rho(x,y)$, where $\rho(x,y) = \rho_i(x)y^i$ is a 1-form on a smooth manifold M^n. Let us consider the special Randers change of Finsler metric $L \longrightarrow \overline{L} = L+\beta$ by β. On the basis of this special Randers change, the purpose of the present paper is devoted to studying the conditions for Finsler space $\overline{F}^n$ which are transformed by a special Randers change of Finsler spaces F^n with (α,β)-metrics of Douglas type to be also of Douglas type, and vice versa.

1. Introduction

An n-dimensional Finsler space F^n is a Douglas space or of Douglas type if and only if the Douglas tensor vanishes identically. Recently R. Bácsó and M. Matsumoto ([2]) have introduced the notion of Douglas space as a generalization of Berwald space from the viewpoint of geodesic equations. The conditions for some Finsler spaces with an (α,β)-metric to be Douglas space are obtained by M. Matsumoto ([8]).

A change of Finsler metric $L(x,y) \longrightarrow \overline{L}$ is called a Randers change of L, if $\overline{L}(x,y) = L(x,y) + \rho(x,y)$, where $\rho(x,y) = \rho_i(x)y^i$ is a 1-form on a smooth manifold M^n. The notion of a Randers change has been proposed by M. Matsumoto ([5]). If $L(x,y)$ is a Riemannian metric, then $\overline{L}(x,y)$ becomes the Randers metric.

The purpose of the present paper is to study the Randers change of the Finsler space which is Douglas type. After the section 4, we consider

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a special Randers change of certain Finsler spaces with an (α,β)-metric L by β. The 1-form β of modification is coincided with 1-form β of (α,β)-metric L. We are devoted to finding the conditions for Finsler spaces changed by a special Randers change to be of Douglas type.

2. Preliminaries

The geodesics of an n-dimensional Finsler space $F^n = (M^n, L)$ are given by the system of the differential equations ([1]):

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2\{G^i(x.y)y^j - G^j(x.y)y^i\} = 0, \quad y^i = \frac{dx^i}{dt}$$

in a parameter t. The function $G^i(x,y)$ are given by

$$2G^i(x,y) = g^{ij}(y^r\dot{\partial}_j\partial_r F - \partial_j F),$$

where $\dot{\partial}_i = \partial/\partial y^i$, $\partial_i = \partial/\partial x^i$, $F = L^2/2$ and $g^{ij}(x,y)$ are the inverse of Finsler metric tensor $g_{ij}(x,y)$. According to [2], F^n is of Douglas type if

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i \tag{2.1}$$

are homogeneous polynomials in (y^i) of degree three. We shall denote the homogeneous polynomials in (y^i) of degree r by $hp\ (r)$ for brevity.

Let $L_i = \dot{\partial}_i L, \quad L_{ij} = \dot{\partial}_i\dot{\partial}_i L, \quad L_{ijk} = \dot{\partial}_k\dot{\partial}_j\dot{\partial}_i L.$ Then we have

$$L_i = l_i, \quad LL_{ij} = h_{ij}, \quad L^2L_{ijk} = h_{ij}l_k + h_{jk}l_i + h_{ki}l_j.$$

And we put

$$2E_{ij} = \rho_{i|j} + \rho_{j|i}, \quad 2F_{ij} = \rho_{i|j} - \rho_{j|i}, \tag{2.2}$$

where $(_|)$ denotes the h-covariant derivative with respect to the Cartan connection $C\Gamma = (F_k{}^i{}_j, G^i{}_j, C_k{}^i{}_j)$.

On the other hand, a Finsler metric $L(x,y)$ is called an (α,β)-metric, when L is a positively homogeneous function $L(\alpha,\beta)$ of degree one in two variables $\alpha(x,y) = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta(x,y) = b_i(x)y^i$. The space $R^n = (M^n, \alpha)$ is called the *associated Riemannian space* with F^n ([1], [7]).

We have the covariant differentiation (;) with respect to the Christoffel symbols $\gamma_j{}^i{}_k(x)$ in R^n. We shall use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \qquad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),$$
$$s^i{}_j = a^{ir} s_{rj}, \qquad s_j = b_r s^r{}_j.$$

Now we consider the functions $G^i(x,y)$ of F^n with an (α,β)-metric. According to [8], $G^i(x,y)$ are written in the form

$$2G^i = \gamma_0{}^i{}_0 + 2B^i,$$
$$B^i = \frac{\alpha L_\beta}{L_\alpha} s^i{}_0 + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right\}, \tag{2.3}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$, $L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha$, the subscript 0 means contraction by y^i and

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \qquad \gamma^2 = b^2\alpha^2 - \beta^2,$$
$$b^i = a^{ij} b_j, \qquad b^2 = a^{ij} b_i b_j.$$

Since $\gamma_0{}^i{}_0(x)$ are $hp(2)$, F^n with an (α,β)-metric is Douglas space, if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ are $hp(3)$. Form (2.1) and (2.3) we have

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{2.4}$$

The following lemma ([9]) is used for latter:

LEMMA. *A system of linear equations* $L_{ir}X^r = Y_i$, $(l_r + \rho_r)X^r = Y$ *and* $(Y_i y^i = \alpha^2)$ *in* X^i *has the unique solution* $X^i = LY^i + \frac{1}{\tau}(Y - LY^r\rho_r)l^i$, *where* $Y^i = g^{ir}Y_r$ *and* $\tau = \overline{L}/L$.

3. Randers change of Douglas type

For a Randers change: $L \longrightarrow \overline{L} = L(x,y) + \rho(x,y), \quad \rho(x,y) = \rho(x)_i y^i$,
we may put

$$\overline{G}^i = G^i + D^i. \tag{3.1}$$

Then $\overline{G}^i{}_j = G^i{}_j + D^i{}_j$ and $\overline{G}_j{}^i{}_k = G_j{}^i{}_k + D_j{}^i{}_k$, where $D^i{}_j = \dot{\partial}_j D^i$ and $D_j{}^i{}_k = \dot{\partial}_k D^i{}_j$. The tensors D^i, $D^i{}_j$ and $D_j{}^i{}_k$ are positively homogeneous in y^i of degree two, one and zero respectively. In the following the explicit form of D^i is necessary. To find this, we deal with equation $L_{ij|k} = 0$, where $L_{ij|k}$ is the h-covariant derivative of $L_{ij} = h_{ij}/L$ in $C\Gamma$. Then

$$\partial_k L_{ij} = L_{ijr}G^r{}_k + L_{rj}F_i{}^r{}_k + L_{ir}F_j{}^r{}_k.$$

Since $\overline{L}_{ij} = L_{ij}$ and $\overline{L}_{ijk} = L_{ijk}$ hold,

$$\overline{L}_{ijk} = L_{ijr}(G^r{}_k + D^r{}_k) + L_{rj}(F_i{}^r{}_k - D_i{}^r{}_k) + L_{ir}(F_j{}^r{}_k + D_j{}^r{}_k),$$

which imply

$$L_{ijr}D^r{}_k + L_{rj}D_i{}^r{}_k + L_{ir}D_j{}^r{}_k = 0.$$

Thus transvection of this equation by y^k yields

$$2L_{ijr}D^r + L_{rj}D^r{}_i + L_{ir}D^r{}_j = 0. \tag{3.2}$$

Next, we deal with $L_{i|j} = 0$, that is,

$$\partial_j L_i = L_{ir}G^r{}_j + L_r F_i{}^r{}_j,$$
$$\partial_j \overline{L}_i = L_{ir}(G^r{}_j + D^r{}_j) + (L_r + \rho_r)(F_i{}^r{}_j + {}^cD_i{}^r{}_j),$$

where ${}^cD_i{}^r{}_j = \overline{F}_i{}^r{}_k - F_i{}^r{}_k$. Substitution of the equations above in $\partial_j \overline{L}_i = \partial_j L_i + \partial_j \rho_i$ leads to

$$\partial_j \rho_i - \rho_r F_i{}^r{}_j = L_{ir}D^r{}_j + (l_r + \rho_r)\ {}^cD_i{}^r{}_j.$$

Then we have

$$2E_{ij} = L_{ir}D^r{}_j + L_{jr}D^r{}_i + 2(l_r + \rho_r)\ {}^cD_i{}^r{}_j, \tag{3.3}$$

$$2F_{ij} = L_{ir}D^r{}_j - L_{jr}D^r{}_i. \tag{3.4}$$

Therefore (3.2) and (3.4) give

$$L_{ir}D^r{}_j = F_{ij} - L_{ijr}D^r \tag{3.5}$$

and transvection of (3.3) by y^i shows

$$(l_r + \rho_r)D^r{}_j = E_{ij}y^i - L_{jr}D^r. \tag{3.6}$$

Furthermore transvection of (3.5) and (3.6) by y^j leads to

$$(3.7) \qquad (a) \quad L_{ir}D^r = F_{ij}y^j, \qquad (b) \quad (l_r + \rho_r)D^r = \frac{1}{2}E_{ij}y^iy^j.$$

The equations (3.7)(a)(b) constitute a system of linear equations respectively. Applying Lemma to (3.7), we have

$$(3.8) \qquad D^i = LF^i{}_0 + \frac{1}{\overline{L}}(\frac{1}{2}E_{00} - LF_0)y^i,$$

where $F^i{}_j = g^{ir}F_{rj}$ and $F_j = \rho_r F^r{}_j$. Thus we have the following

PROPOSITION 3.1. ([9]) *The tensor D^i of (3.1) arising from a Randers change are given by (3.8).*

From (3.1) and (3.8) we have

$$\overline{G}^i y^j - \overline{G}^j y^i = G^i y^j - G^j y^i + L(F^i{}_0 y^j - F^j{}_0 y^i).$$

Suppose F^n is a Douglas space, that is, $G^iy^j - G^jy^i$ are *hp* (3). Thus we have

PROPOSITION 3.2. *Let F^n be a Douglas space and $\overline{F}^n$ a Finsler space which is obtained by Randers change by ρ. $\overline{F}^n$ is also a Douglas space if and only if $L(F^i{}_0y^j - F^j{}_0y^i)$ are hp (3).*

The Randers changes is called *projective Randers changes* if all the geodesic curves are preserved under the Randers changes. According to Hashiguchi-Ichijyō ([4]), *a Randers change is projective, if and only if ρ_i are gradient vector fields.* In this case (3.8) is reduced to $D^i = E_{00}y^i/2\overline{L}$. Therefore $D^iy^j - D^jy^i = 0$. Thus we have $\overline{G}^iy^j - \overline{G}^jy^i = G^iy^j - G^jy^i$.

On the other hand, it is well-known that the Douglas tensor is projectively invariant. Hence, if a Finsler space is projectively related to a Douglas space, then it is also a Douglas space. Thus, from Hashiguchi-Ichijyō's theorem, we have the following

THEOREM 3.3. *Let $F^n(M^n, L) \longrightarrow \overline{F}^n(M^n, L+\rho_i)$ be a projective Randers change. If F^n is a Douglas space, then $\overline{F}^n$ is also a Douglas space, and vice versa.*

4. Generalized Kropina spaces

Hereafter we consider a special Randers change of certain (α, β) -metric as follows: $L(\alpha, \beta) \longrightarrow \overline{L} = L(\alpha, \beta) + \beta$, that is, the 1-form β of modification coincides with 1-form β of (α, β)-metric. In this section we deal with a Finsler space F^n $(n > 2)$ with a generalized Kropina metric. The metric of F^n is $L = \alpha^{1+m}\beta^{-m}$, where m is a constant $\neq 0, -1$. We consider the condition for a Finsler space $\overline{F}^n = (M^n, L + \beta)$ which is obtained by a special Randers change of a generalized Kropina space $F^n = (M^n, L = \alpha^{1+m}\beta^{-m})$ to be of Douglas type. It has been known ([8]) that a generalized Kropina space is of Douglas space, where $\alpha^2 \not\equiv 0$ (mod. β), if and only if $b_{i;j}$ are given by

$$s_{ij} = \frac{1}{b^2}(b^i s_j - b_j s_i), \tag{4.1}$$

$$r_{ij} = \frac{k}{m(1+m)}\{(1-m)b_i b_j + mb^2 a_{ij}\} + \frac{1-m}{(1+m)b^2}(s_i b_j - s_j b_i). \tag{4.2}$$

For $\overline{F}^n$, (2.3) gives

$$\begin{aligned} &2\{(1-m)\beta^2 + mb^2\alpha^2\}\{(1+m)\beta\overline{B}^{ij} \\ &+(m\alpha^2 - \alpha^{1-m}\beta^{m+1})(s^i{}_0 y^j - s^j{}_0 y^i)\} - m\alpha^2\{(1+m)r_{00}\beta \\ &+2s_0(m\alpha^2 - \alpha^{1-m}\beta^{m+1})\}(b^i y^j - b^j y^i) = 0, \end{aligned} \tag{4.3}$$

which are equivalent to

(4.4)

$$\begin{aligned} &2\{(1-m)\beta^2 + mb^2\alpha^2\}\{(1+m)\beta\overline{B}^{ij} \\ &\quad +m\alpha^2(s^i{}_0 y^j - s^j{}_0 y^i)\} - m\alpha^2\{(1+m)r_{00}\beta + 2ms_0\alpha^2\}(b^i y^j - b^j y^i) \\ &\quad -2\alpha^{1-m}\beta^{m+1}[\{(1-m)\beta^2 + mb^2\alpha^2\}(s^i{}_0 y^j - s^j{}_0 y^i) \\ &\quad -ms_0\alpha^2(b^i y^j - b^j y^i)] = 0. \end{aligned}$$

Then it will be better to divide our consideration into two cases as follows:

(I) $\alpha^{1-m}\beta^{m+1}$: rational in (y^i), that is, m : odd integer,

(II) $\alpha^{1-m}\beta^{m+1}$: irrational in (y^i), that is, m : the others.

The case (I) : First we are concerned with $m \leq 1$, where m is an odd integer. Multiplication of (4.1) by β^{-m-1} leads to

$$
\begin{aligned}
(4.5)\qquad & 2\{(1-m^2)\beta^2+mb^2\alpha^2\}\{(1+m)\beta^{-m}\overline{B}^{ij}+(m\alpha^2\beta^{-1-m}\\
& -\alpha^{1-m})(s^i{}_0y^j-s^j{}_0y^i)\}-m\alpha^2\{(1+m)r_{00}\beta^{-m}\\
& +2s_0(m\alpha^2\beta^{-1-m}-\alpha^{1-m})\}(b^iy^j-b^jy^i)=0.
\end{aligned}
$$

Since $\overline{B}^{ij}$ are supposed to be $hp(3)$, the term in (4.5) which seemingly does not contain α^2 is $2(1-m^2)\beta^{2-m}\overline{B}^{ij}$ only, and hence we must have $hp(3-m)$ u^{ij}_{3-m} such that

$$
(4.6)\qquad 2(1-m^2)\beta^{2-m}\overline{B}^{ij}=\alpha^2u^{ij}_{3-m}.
$$

We treat of the general case $\alpha^2 \not\equiv 0$ (mod. β). (4.6) shows that there exist $hp(1)$ u^{ij} satisfying $u^{ij}_{3-m}=\beta^{2-m}u^{ij}$. Then (3.4) is reduced to

$$
(4.7)\qquad 2(1-m^2)\overline{B}^{ij}=\alpha^2u^{ij}.
$$

If $m \neq 1$, that is, F^n is not a Kropina space, then (4.7) gives $\overline{B}^{ij}$ and (4.5) can be rewritten in the form

(4.8)

$$
\begin{aligned}
& \{(1-m)\beta^2+mb^2\alpha^2\}\left\{\frac{\beta^{-m}u^{ij}}{1-m}+2(m\beta^{-1-m}-\alpha^{-1-m})(s^i{}_0y^j-s^j{}_0y^i)\right\}\\
& -m\{(1+m)r_{00}\beta^{-m}+2s_0(m\alpha^2\beta^{-1-m}-\alpha^{1-m})\}(b^iy^j-b^jy^i)=0.
\end{aligned}
$$

Collecting the terms of (4.8) which seemingly do not contain β, we can put

$$
2m\alpha^{1-m}\{b^2(s^i{}_0y^j-s^j{}_0y^i)-s_0(b^iy^j-b^jy^i)\}=\beta v^{ij}_{2-m},
$$

where v^{ij}_{2-m} are $hp(2-m)$. Consequently we have

$$
(4.9)\qquad b^2(s^i{}_0y^j-s^j{}_0y^i)-s_0(b^iy^j-b^jy^i)=\beta v^{ij}
$$

and $v^{ij}_{2-m}=2m\alpha^{1-m}v^{ij}$ with hp (1) v^{ij}. Thus (4.8) is reduced to

(4.10)

$$
\begin{aligned}
& \frac{\{(1-m)\beta^2+mb^2\alpha^2\}}{1-m}\beta^{-m}u^{ij}+2m^2\alpha^2\beta^{-m}v^{ij}\\
& +2[m(1-m)\beta^{1-m}-\alpha^{-1-m}\{(1-m)\beta^2+mb^2\alpha^2\}](s^i{}_0y^j-s^j{}_0y^i)\\
& -m\{(1+m)r_{00}\beta^{-m}-2s_0\alpha^{1-m}\}(b^iy^j-b^jy^i)=0.
\end{aligned}
$$

Consequently (4.9) is obtained as follows:

$$b^2 s_{ij} = b_i s_j + b_j s_i, \quad \text{provided that} \quad b^2 \neq 0. \tag{4.11}$$

That is, (4.1). From (4.11), (4.9) is reduced to $v^{ij} = y^i s^j - y^j s^i$ and (4.10) is rewritten in the form

(4.12)

$$\begin{aligned}
&\{(1-m)\beta^2 + mb^2\alpha^2\}\left\{\frac{\beta^{-m}u^{ij}}{1-m} - \frac{2(m\beta^{-m} - \alpha^{-1-m}\beta)}{b^2}(s^i y^j - s^j y^i)\right\}\\
&+\Big\{\left[2m(1-m)\beta^{1-m} - 2\alpha^{-1-m}\{(1-m)\beta^2 + mb^2\alpha^2\}\right]\frac{s_0}{b^2}\\
&- m\{(1+m)r_{00}\beta^{-m} - 2s_0\alpha^{1-m}\}\Big\}(b^i y^j - b^j y^i) = 0.
\end{aligned}$$

Multiplying (4.12) by β^m, we obtain

(4.13)

$$\begin{aligned}
&\{(1-m)\beta^2 + mb^2\alpha^2\}\left\{\frac{u^{ij}}{1-m} - \frac{2(m - \alpha^{-1-m}\beta^{1+m})}{b^2}(s^i y^j - s^j y^i)\right\}\\
&+\Big\{\left[2m(1-m)\beta - 2\alpha^{-1-m}\beta^m\{(1-m)\beta^2 + mb^2\alpha^2\}\right]\frac{s_0}{b^2}\\
&- m\{(1+m)r_{00} - 2s_0\alpha^{1-m}\beta^m\}\Big\}(b^i y^j - b^j y^i) = 0.
\end{aligned}$$

Transvecting (4.13) by $b_i s_j$, we have

(4.14)

$$\begin{aligned}
&\{(1-m)\beta^2 + mb^2\alpha^2\}\left\{\frac{1}{1-m}u^{ij}b_i s_j + \frac{2}{b^2}(m - \alpha^{-1-m}\beta^{1+m})s^j s_j \beta\right\}\\
&= \Big\{m\{(1+m)r_{00} - 2s_0\alpha^{1-m}\beta^m\}\\
&- 2\left[m(1-m)\beta - \alpha^{-1-m}\beta^m\{(1-m)\beta^2 + mb^2\alpha^2\}\right]\frac{s_0}{b^2}\Big\}b^2 s_0.
\end{aligned}$$

Suppose that there exists $u = u_i(x)y^i$ such that $(1-m)\beta^2 + mb^2\alpha^2 = b^2 s_0 u$. Then this is written in the form

$$2\{(1-m)b_i b_j + mb^2 a_{ij}\} = b^2(s_i u_j + s_j u_i).$$

Transvection by $b^i b^j$ leads to the contradiction $b^2 = 0$. Therefore (4.14) shows that we have a function $h_1(x)$ satisfying

$$\frac{1}{1-m}u^{ij}b_i s_j + \frac{2}{b^2}(m - \alpha^{-1-m}\beta^{1+m})s^j s_j \beta = h_1(x) b^2 s_0,$$

$$\Big\{\{m(1+m)r_{00} - 2ms_0\alpha^{1-m}\beta^m\} - 2\left[m(1-m)\beta - \alpha^{-1-m}\beta^m\cdot\right.$$

$$\left.\{(1-m)\beta^2 + mb^2\alpha^2\}\right]\frac{s_0}{b^2}\Big\}s_0 = \{(1-m)\beta^2 + mb^2\alpha^2\}h_1(x)s_0.$$

If $s_0 \neq 0$, then we get from the latter

(4.15)

$$r_{00} = \frac{h_1(x)}{m(1+m)}\{(1-m)\beta^2 + mb^2\alpha^2\} + \frac{2(1-m)s_0\beta}{m(1+m)b^2}(m - \alpha^{-1-m}\beta^{1+m}).$$

Thus (4.13) gives u^{ij} of the form

(4.16)

$$u^{ij} = \frac{2(1-m)}{b^2}(m - \alpha^{-1-m}\beta^{1+m})(s^iy^j - s^jy^i) + h_1(x)(1-m)(b^iy^j - b^jy^i).$$

Since r_{00} is *hp* (2) from (4.15), $\alpha^{-1-m}\beta^{1+m}$ must be *hp* (0). The condition for $\alpha^{-1-m}\beta^{1+m}$ to be *hp* (0) is $m = -3$ alone. Thus substituting $m = -3$ in (4.15), we have

$$\text{(4.17)} \qquad r_{00} = \frac{h_1(x)}{6}(4\beta^2 - 3b^2\alpha^2) - \frac{4s_0}{3b^2\beta}(\alpha^2 + 3\beta^2).$$

(4.17) shows that there exists $h_2(x)$ satisfying $s_0 = h_2(x)\beta$. Then (4.17) is reduced to

$$\text{(4.18)} \qquad r_{ij} = \left(\frac{2h_1(x)}{3} - \frac{4h_2(x)}{b^2}\right)b_ib_j - \left(\frac{b^2h_1(x)}{2} + \frac{4h_2(x)}{3b^2}\right)a_{ij}.$$

That is, (4.2). If s_0 is assumed to vanish, then (4.11) gives $s_{ij} = 0$ and (4.13) is reduced to

$$\{(1-m)\beta^2 + mb^2\alpha^2\}u^{ij} = m(1-m^2)r_{00}(b^iy^j - b^jy^i).$$

Transvection by $b_iy_j (y_j = a_{jr}y^r)$ leads to

$$\{(1-m)\beta^2 + mb^2\alpha^2\}u^{ij}b_iy_j = m(1-m^2)r_{00}(b^2\alpha^2 - \beta^2).$$

It is easy to show that $(1-m)\beta^2 + mb^2\alpha^2 \quad (= m\gamma^2 + \beta^2)$ is not contained in $b^2\alpha^2 - \beta^2 \quad (= \gamma^2)$. Consequently it is contained in r_{00} ; there exists a function $h_3(x)$ such that $r_{00} = h_3(x)\{(1-m)\beta^2 + mb^2\alpha^2\}$. Therefore (4.18) holds in this case, too.

Next, we deal with $m > 1$. Multiplication of (4.3) by α^{-1+m} leads to $s_0 = 0$ and $s_{ij} = 0$. Thus we obtain $r_{00} = h_3(x)\{(1-m)\beta^2 + mb^2\alpha^2\}$ in common with $s_0 = 0$.

The case (II): Since $\alpha^{1-m}\beta^{m+1}$ is irrational in (y^i), (4.4) is divided into two equations as follows:

$$(4.19)\quad \begin{aligned} &2\{(1-m)\beta^2 + mb^2\alpha^2\}\{(1+m)\beta\overline{B}^{ij} + m\alpha^2(s^i{}_0y^j - s^j{}_0y^i)\} \\ &- m\alpha^2\{(1+m)r_{00}\beta + 2ms_0\alpha^2\}(b^iy^j - b^jy^i) = 0, \end{aligned}$$

$$(4.20)\quad \{(1-m)\beta^2 + mb^2\alpha^2\}(s^i{}_0y^j - s^j{}_0y^i) - ms_0\alpha^2(b^iy^j - b^jy^i) = 0.$$

Transvecting (4.20) by b_iy_j, we get

$$s_0\alpha^2\{(1-m)\beta^2 + mb^2\alpha^2\} - ms_0\alpha^2(b^2\alpha^2 - \beta^2) = 0,$$

which implies $s_0\alpha^2\beta = 0$. Hence we get $s_0 = 0$, that is, $s_i = 0$. (4.20) is reduced to $s^i{}_0y^j - s^j{}_0y^i = 0$. Transvection of this by y_i leads to $s^i{}_0 = 0$. Therefore $s_{ij} = 0$. Substituting $s_{ij} = 0$ in (4.19), we obtain

$$(4.21)\quad 2\{(1-m)\beta^2 + mb^2\alpha^2\}\overline{B}^{ij} - m\alpha^2r_{00}(b^iy^j - b^jy^i) = 0.$$

The term in (4.21) which seemingly does not contain α^2 is $2(1-m)\beta^2\overline{B}^{ij}$ only, and hence we must have $hp(3)$ u_3^{ij} satisfying

$$(4.22)\quad 2(1-m)\beta^2\overline{B}^{ij} = \alpha^2u_3^{ij}.$$

Suppose $\alpha^2 \not\equiv 0$ (mod. β). Then (4.22) is reduced to $\overline{B}^{ij} = \alpha^2u^{ij}$, where u^{ij} are $hp(1)$. Hence (4.21) leads to

$$(4.23)\quad 2\{(1-m)\beta^2 + mb^2\alpha^2\}u^{ij} - r_{00}(b^iy^j - b^jy^i) = 0.$$

Transvecting (4.23) by b_iy_j, we obtain

$$2\{(1-m)\beta^2 + mb^2\alpha^2\}u^{ij}b_iy_j - r_{00}(b^2\alpha^2 - \beta^2) = 0.$$

Thus there exists a function $h_4(x)$ such that

$$2(m-1)u^{ij}b_iy_j - r_{00} = h_4(x)\alpha^2, \qquad 2mb^2u^{ij}b_iy_j - b^2r_{00} = h_4(x)\beta^2.$$

Eliminating $u^{ij}b_iy_j$ from the above equations, we have

$$b^2r_{00} = h_4(x)\{(m-1)\beta^2 - mb^2\alpha^2\},$$

which implies

$$r_{ij} = \frac{h_4(x)}{b^2}\{(m-1)b_ib_j - mb^2a_{ij}\}. \tag{4.24}$$

From $s_{ij} = 0$ and (4.24) we obtain

$$b_{i;j} = h_5(x)\{(m-1)b_ib_j - mb^2a_{ij}\}, \tag{4.25}$$

where $h_5(x) = h_4(x)/b^2$.

Consequently, if (4.25) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = h_5(x)\{(m-1)\beta^2 - mb^2\alpha^2\},$$

from which $\overline{B}^{ij}$ of (4.4) are $hp(3)$. Hence (4.18) holds in this case, too.

In any case we obtain $b_{i;j}$ by (4.11) and (4.18), then $\overline{B}^{ij}$ are given by (4.7) together with (4.16). Consequently a Finsler space $\overline{F}^n = (M^n, L+\beta)$ $(n > 2)$ with non zero b^2 which is obtained by Randers change of a generalized Kropina space $F^n = (M^n, L = \alpha^{1+m}\beta^{-m}, m \neq \pm 1, 0)$ is a Douglas space, if and only if $b_{i;j}$ are given (4.11) and (4.18). That is, (4.1) and (4.2) hold.

On the other hand, it has been known ([8]) that *a generalized Kropina space F^n $(n > 2)$ with non zero b^2 is a Douglas space, if and only if $b_{i;j}$ are given by* (4.1) *and* (4.2). That is to say, the case $s_0 \neq 0$ for F^n to be a Douglas space corresponds to the case $m = -3$ for $\overline{F}^n$ to be a Douglas space and the case $s_0 = 0$ for F^n to be of Douglas type corresponds to the case $m \neq -3$, $m \in \mathbb{R}$ for $\overline{F}^n$ to be of Douglas type. Thus we obtain the following

THEOREM 4.1. *Let F^n $(n > 2)$ be a generalized Kropina space with $L = \alpha^{1+m}\beta^{-m}$, m being a constant $\neq \pm 1$, 0. A Finsler space $\overline{F}^n$ which is obtained by a special Randers change of F^n with non zero b^2 of Douglas type is also of Douglas type, and vice versa.*

5. Kropina space

Let F^n be a Kropina space with $L = \alpha^2/\beta$ and $\overline{F}^n = (M^n, \overline{L})$ a Finsler space which is obtained by Randers change of $F^n = (M^n, L)$. From (2.4), $\overline{B}^{ij} = \overline{B}^i y^j - \overline{B}^j y^i$ in $\overline{F}^n$ are written as

$$(5.1)\quad \overline{B}^{ij} = B^{ij} + \frac{\alpha}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^i y^j - b^j y^i).$$

Suppose F^n is a Douglas space. Since B^{ij} are $hp(3)$, the necessary and sufficient condition for $\overline{F}^n$ to be also a Douglas space is that

$$\frac{\alpha}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^i y^j - b^j y^i)$$

are $hp(3)$. Thus we have the following

PROPOSITION 5.1. *Let $F^n = (M^n, L)$ be a Finsler space with an (α,β)-metric of Douglas type. Then $\overline{F}^n = (M^n, L+\beta)$ which is obtained by a special Randers change of F^n is also a Douglas space, if and only if*

$$(5.2)\quad W^{ij} = \frac{\alpha}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^i y^j - b^j y^i)$$

are hp (3).

We suppose F^n is a Douglas space. The condition for $\overline{F}^n = (M^n, L+\beta)$ to be a Douglas space is that (5.2) is hp (3). From (5.2) we have

$$W^{ij} = \frac{\beta}{2}(s^i{}_0 y^j - s^j{}_0 y^i) - \frac{s_0\beta}{b^2}(b^i y^j - b^j y^i).$$

Since B^{ij} and W^{ij} are $hp(3)$, $\overline{B}^{ij}$ are $hp(3)$, that is, $\overline{F}^n$ is a Douglas space. Thus *a Kropina space F^n is of Douglas type, then a Finsler space $\overline{F}^n$ which is obtained by a special Randers change of F^n is of Douglas type also.* We consider the condition for a Finsler space which is obtained by a special Randers change of a Kropina space to be of Douglas type. For $\overline{F}^n = (M^n, \overline{L} = \alpha^2/\beta + \beta)$, (2.4) gives

(5.3)

$$\overline{B}^{ij} = \frac{1}{2\beta}(\beta^2 - \alpha^2)(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{1}{2b^2\beta}\{r_{00}\beta - s_0(\beta^2 - \alpha^2)\}(b^i y^j - b^j y^i).$$

Since the terms $(\beta/2)(s^i{}_0 y^j - s^j{}_0 y^i) + (1/2b^2\beta)(r_{00} - s_0\beta)(b^i y^j - b^j y^i)$ are $hp(3)$, these terms may be neglected in our discussion and we treat only of

$$\overline{W}^{ij} = \frac{\alpha^2}{2\beta}\left\{\frac{s_0}{b^2}(b^i y^j - b^j y^i) - (s^i{}_0 y^j - s^j{}_0 y^i)\right\}. \tag{5.5}$$

For $n > 2$, $\alpha^2 \not\equiv 0$ (mod.β) ([3]). Therefore there exist $hp(1)$ $v^{ij} = v^{ij}_k(x)y^k$ such that

$$\frac{s_0}{b^2}(b^i y^j - b^j y^i) - (s^i{}_0 y^j - s^j{}_0 y^i) = \beta v^{ij}. \tag{5.6}$$

This equation is written in the form

$$\begin{aligned}&\frac{1}{b^2}\{b^i(s_h\delta^j_k + s_k\delta^j_h) - b^j(s_h\delta^i_k + s_k\delta^i_h)\}\\ &\qquad -(s^i{}_h\delta^j{}_k + s^i{}_k\delta^j{}_h) + (s^j{}_h\delta^i{}_k + s^j{}_k\delta^i{}_h) = b_h v^{ij}_k + b_k v^{ij}_h.\end{aligned} \tag{5.7}$$

Transvection of (5.7) by a^{hk} leads to

$$\frac{1}{b^2}(b^i s^j - b^j s^i) - 2s^{ij} = b^r v^{ij}_r. \tag{5.8}$$

Next, transvecting (5.7) by b^h, we have

$$(s^i\delta^j_k + b^i s^j{}_k) - (s^j\delta^i_k + b^j s^i{}_k) = b^2 v^{ij}_k + b_k b^r v^{ij}_r. \tag{5.9}$$

Contraction of (5.7) with j and h leads to

$$n\left(\frac{1}{b^2}b^i s_k - s^i{}_k\right) = b_r v^{ir}_k - b_k v^{ir}_r. \tag{5.10}$$

Substituting $b^r v^{ij}_r$ of (4.8) in (4.9), we have

$$b^2 v^{ij}_k = 2s^{ij}b_k + \left\{b^i s^j{}_k - b^j s^i{}_k + s^i\delta^j_k - s^j\delta^i_k + \frac{1}{b^2}(s^i b^j b_k - s^j b^i b_k)\right\},$$

which imply

$$b^2 v^i r_r = (n-1)s^i, \qquad b^2 b_r v^{ir}_k = b^i s_k - b^2 s^i{}_k.$$

Consequently (5.10) leads to

$$(5.11)\qquad s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i).$$

Then (5.5) gives

$$\overline{W}^{ij} = \frac{\alpha^2}{2b^2}(s^i y^j - s^j y^i),$$

which are $hp(3)$. Therefore (5.11) is the necessary and sufficient condition for $\overline{F}^n$ to be of Douglas type.

On the other hand, it is known ([8]) that *a Kropina space* $F^n (n > 2)$ *with* $b^2 \neq 0$ *is of Douglas type, if and only if* (5.11) *is satisfied.* Thus we have the

THEOREM 5.2. *A Finsler space* $\overline{F}^n (n > 2)$ *which is obtained by a special Randers change of a Kropina space* F^n *with* $b^2 \neq 0$ *is of Douglas type, if and only if the Kropina space* F^n *is of Douglas type.*

6. Matsumoto space

We consider the condition for a Finsler space $\overline{F}^n = (M^n, L + \beta)$ which is obtained by a special Randers change of Matsumoto space $F^n = (M^n, L = \alpha^2/(\alpha - \beta))$ to be of Douglas type. It is known ([6]) that *a Matsumoto space* $F^n (n > 2)$ *is of Douglas type, if and only if* $b_{i;j} = 0$. Hence, for a Matsumoto space F^n of Douglas type, (2.4) leads to $\overline{W}^{ij} = 0$, that is, $\overline{B}^{ij} = B^{ij}$. Thus *if a Matsumoto space* F^n *is of Douglas type, then a Finsler space which is obtained by a special Randers change of* F^n *is also of Douglas type.* It is known ([8]) that *a Matsumoto space* $F^n (n > 2)$ *is of Douglas type, if and only if* $b_{i;j} = 0$. Hence, for a Matsumoto space F^n of Douglas type, (5.2) leads to $\overline{W}^{ij} = 0$, that is, $\overline{B}^{ij} = B^{ij}$. Thus *if a Matsumoto space* F^n *is of Douglas type, then a Finsler space which is obtained by a special Randers change of* F^n *is also of Douglas type.* For $\overline{F}^n$, (2.3) gives

(6.1)

$$\begin{aligned}\{\alpha(1+2b^2) - 3\beta\}\{(\alpha - 2\beta)\overline{B}^{ij} - (2\alpha^2 - 2\alpha\beta + \beta^2)(s^i{}_0 y^j - s^j{}_0 y^i)\}\\ + \alpha\{2s_0(2\alpha^2 - 2\alpha\beta + \beta^2) - r_{00}(\alpha - 2\beta)\}(b^i y^j - b^j y^i) = 0.\end{aligned}$$

Suppose that $\overline{F}^n$ be a Douglas space, that is, $\overline{B}^{ij}$ be $hp(3)$. Since α

is irrational in (y^i), (6.1) is divided as follows:
(6.2)

$$\{(1+2b^2)\alpha^2+6\beta^2\}\overline{B}^{ij}+\{2\alpha^2\beta(1+2b^2)+3\beta(2\alpha^2+\beta^2)\}(s^i{}_0y^j-s^j{}_0y^i) - (4s_0\alpha^2\beta+r_{00}\alpha^2)(b^iy^j-b^jy^i)=0,$$

$$(6.3)\qquad (5+4b^2)\beta\overline{B}^{ij}+\{(1+2b^2)(2\alpha^2+\beta^2)+6\beta^2\}(s^i{}_0y^j-s^j{}_0y^i) - 2\{s_0(2\alpha^2+\beta^2)+r_{00}\beta)\}(b^iy^j-b^jy^i)=0.$$

Eliminating $\overline{B}^{ij}$ from these equations, we have

$$(6.4)\qquad A(s^i{}_0y^j-s^j{}_0y^i)+B(b^iy^j-b^jy^i)=0,$$

where we put

$$A=\alpha^2(21\beta^2+12\beta^2b^2+12\beta^2b^4-2\alpha^2-8\alpha^2b^2-8\alpha^2b^4)-27\beta^4,$$
$$B=\alpha^2\{s_0(6\beta^2-12\beta^2b^2+4\alpha^2+8\alpha^2b^2)-3r_{00}\beta\}+12\beta^3(s_0\beta+r_{00}).$$

Transvection of (6.4) by b_iy_j leads to

$$(6.5)\qquad As_0\alpha^2+B(b^2\alpha^2-\beta^2)=0.$$

Since the terms $12(s_0\beta+r_{00})\beta^5$ of (6.5) seemingly do not contain α^2, we must have $hp(5)$ v_5 such that

$$(6.6)\qquad 12(s_0\beta+r_{00})\beta^5=\alpha^2v_5.$$

In the first case of $v_5=0$, we have $r_{00}=-s_0\beta$ from (6.6), and (6.5) is reduced to

$$\{\alpha^2(17\beta^2+13\beta^2b^2-2\alpha^2-4\alpha^2b^2)+12\beta^4(b^2-3)\}s_0=0.$$

If the coefficient of s_0 does not vanish, then

$$\alpha^2(17\beta^2+13\beta^2b^2-2\alpha^2-4\alpha^2b^2)=12\beta^4(3-b^2).$$

Since we suppose $\alpha^2\not\equiv 0$ (mod.β), the above assumption is a contradiction. Therefore we obtain $s_0=0$ and $r_{00}=0$ from (6.6). Next, in the second case of $v_5\neq 0$, (6.6) shows the existence of a function

$k_1(x)$ satisfying $v_5 = k_1(x)\beta^5$, and hence $r_{00} = k_2(x)\alpha^2 - s_0\beta$, where $k_2(x) = k_1(x)/12$. Then (6.5) is reduced to

(6.7)

$$As_0+\{s_0(9\beta^2-12\beta^2b^2+4\alpha^2+8\alpha^2b^2)-3k_2(x)\beta(\alpha^2-4\beta^2)\}(b^2\alpha^2-\beta^2) = 0.$$

Only the terms $-36s_0\beta^4 + 12\beta^4b^2s_0 - 12k_2(x)\beta^5$ of (6.7) seemingly do not contain α^2, and hence we must have $hp(3)$ v_3 such that

$$12\{s_0(b^2-3) - k_2(x)\beta\}\beta^4 = \alpha^2 v_3.$$

From $\alpha^2 \not\equiv 0$ (mod.β) it follows that v_3 must vanish, and hence $s_0(b^2 - 3) = k_2(x)\beta$, that is, $(b^2-3)s_i = k_2(x)b_i$. Then transvection by b^i gives $k_2(x)b^2 = 0$. In case of $k_2(x) = 0$, we get $b^2 = 3$ or $s_i = 0$. If $b^2 = 3$, then (6.7) is reduced to $14s_0(4\beta^2 - \alpha^2)\alpha^2 = 0$. Thus we obtain $s_0 = 0$ and $r_{00} = 0$. Next, if $s_i = 0$, then we have $s_0 = 0$ and $r_{00} = 0$, too. On the other hand, in the case of $b^2 = 0$, (6.7) is reduced to $s_0(17\alpha^2\beta^2 - 2\alpha^4 - 36\beta^4) + 3k_2(x)\beta^3(\alpha^2 - 4\beta^2) = 0$, which implies $s_0 =$ and $k_2(x) = 0$. Therefore, for $n > 2$. both the cases of $v_5 = 0$ and $v_5 \neq 0$ lead to $r_{00} = 0$ and $s_0 = 0$. Hence (6.4) is reduced to ${s^i}_0y^j - {s^j}_0y^i = 0$, and transvection by y_i gives ${s^i}_0 = 0$. Finally $r_{ij} = s_{ij} = 0$, that is, $b_{i;j} = 0$.

Thus a Finsler space $\overline{F}^n = (M^n, L+\beta)$ $(n > 2)$ which is obtained by a special Randers change of a Matsumoto space $F^n = (M^n, L = \alpha^2/(\alpha-\beta))$ is Douglas space, if and only if $b_{i;j} = 0$. On the other hand, M. Matsumoto proved ([8]) that *a Matsumoto space F^n $(n > 2)$ is of Douglas type, if and only if $b_{i;j} = 0$.* Thus we have the following

THEOREM 6.1. *A Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by a special Randers change of a Matsumoto space F^n of Douglas type is also of Douglas type, and vice versa.*

On the other hand, it has been shown ([1]) that *Matsumoto space is a Berwlad space, if and only if $b_{i;j} = 0$.* Then according to Theorem 6.1 we have the following

COROLLARY 6.2. *Let $\overline{F}^n$ $(n > 2)$ be a Finsler space which is obtained by a special Randers change of a Matsumoto space F^n. If F^n is a Douglas space, then $\overline{F}^n$ is a Berwald space.*

7. Finsler space with $L = \alpha + \beta^2/\alpha$

We consider a Finsler space $F^n = (M^n, L)$ with an (α,β)-metric $L = \alpha + \beta^2/\alpha$. This metric may be regarded as constructed from α and one more Riemannian metric $\sqrt{\alpha^2+\beta^2}$, and it is thought of as desirable in the viewpoint of geometry and applications ([8]). For $\overline{F}^n = (M^n, \overline{L})$ which is obtained by a special Randers change of $F^n = (M^n, L = \alpha + \beta^2/\alpha)$, (2.3) gives

$$(7.1)\quad \begin{aligned}\overline{B}^{ij} =& \frac{\alpha^2(\alpha+2\beta)}{(\alpha^2-\beta^2)}(s^i{}_0 y^j - s^j{}_0 y^i)\\ &+ \frac{\alpha^2\{r_{00}(\alpha^2-\beta^2) - 2s_0\alpha^2(\alpha+2\beta)\}}{(\alpha^2-\beta^2)\{\alpha^2(1+2b^2)-3\beta^2\}}(b^i y^j - b^j y^i).\end{aligned}$$

Suppose that $\overline{F}^n$ be a Douglas space, that is, $\overline{B}^{ij}$ be hp (3). Separating (7.1) into the rational and irrational terms of y^i, we have

$$\begin{aligned}&\{\alpha^2(1+2b^2)-3\beta^2\}\{(\alpha^2-\beta^2)\overline{B}^{ij} - 2\alpha^2\beta(s^i{}_0 y^j - s^j{}_0 y^i)\}\\ &-\alpha^2\{r_{00}(\alpha^2-\beta^2) - 4s_0\alpha^2\beta\}(b^i y^j - b^j y^i)\\ &+\alpha[2s_0\alpha^4(b^i y^j - b^j y^i) - \alpha^2\{\alpha^2(1+2b^2)-3\beta^2\}(s^i{}_0 y^j - s^j{}_0 y^i)] = 0,\end{aligned}$$

which yield two equations as follows:

$$(7.2)\quad \begin{aligned}&\{\alpha^2(1+2b^2)-3\beta^2\}\{(\alpha^2-\beta^2)\overline{B}^{ij} - 2\alpha^2\beta(s^i{}_0 y^j - s^j{}_0 y^i)\}\\ &-\alpha^2\{r_{00}(\alpha^2-\beta^2) - 4s_0\alpha^2\beta\}(b^i y^j - b^j y^i) = 0,\end{aligned}$$

$$(7.3)\quad 2s_0\alpha^2(b^i y^j - b^j y^i) - \{\alpha^2(1+2b^2)-3\beta^2\}(s^i{}_0 y^j - s^j{}_0 y^i) = 0.$$

Transvecting (7.3) by $b_i y_j$, we obtain

$$2s_0\alpha^2(b^2\alpha^2-\beta^2) - \{\alpha^2(1+2b^2)-3\beta^2\}s_0\alpha^2 = 0,$$

which implies $s_0\alpha^2(\beta^2-\alpha^2)=0$. Therefore we get $s_i = 0$. Hence (7.3) is reduced to $s^i{}_0 y^j - s^j{}_0 y^i = 0$, and transvection by y_i gives $s^i{}_0 = 0$. Consequently $s_{ij} = 0$. On the other hand, substituting (7.3) in (7.2), we have

$$(7.4)\quad \{\alpha^2(1+2b^2)-3\beta^2\}\overline{B}^{ij} - \alpha^2\{r_{00}(b^i y^j - b^j y^i)\} = 0.$$

Only the terms $3\beta^2\overline{B}^{ij}$ of (7.4) seemingly do not contain α^2. Hence we must have $hp(3)$ v_3^{ij} satisfying

$$(7.5)\quad 3\beta^2\overline{B}^{ij} = \alpha^2 v_3^{ij}.$$

For the sake of brevity we suppose $\alpha^2 \not\equiv 0$ (mod.β). Then (7.5) is reduced to $\overline{B}^{ij} = \alpha^2 v^{ij}$, where v^{ij} are $hp(1)$. Thus (7.4) leads to

$$\{\alpha^2(1+2b^2) - 3\beta^2\}v^{ij} - r_{00}(b^i y^j - b^j y^i) = 0. \tag{7.6}$$

Transvecting (6.6) by $b_i y_j$, we get

$$\{\alpha^2(1+2b^2) - 3\beta^2\}b_i v^{ij} y_j - r_{00}(b^2\alpha^2 - \beta^2) = 0,$$

which imply

$$\alpha^2\{(1+2b^2)b_i v^{ij} y_j - b^2 r_{00}\} = \beta^2(3b_i v^{ij} y_j - r_{00}).$$

Therefore there exists a function $f_1(x)$ satisfying

$$(1+2b^2)b_i v^{ij} y_j - b^2 r_{00} = f_1(x)\beta^2, \quad 3b_i v^{ij} y_j - r_{00} = f_1(x)\alpha^2.$$

Eliminating $b_i v^{ij} y_j$ from above the equations, we obtain

$$r_{00} = f_1(x)\frac{(1+2b^2)\alpha^2 - 3\beta^2}{b^2 - 1}. \tag{7.7}$$

From (7.7) and $s_{ij} = 0$,

$$b_{i;j} = f_2(x)\{(1+2b^2)a_{ij} - 3b_i b_j\}, \tag{7.8}$$

where $f_2(x) = f_1(x)/(b^2 - 1)$.

Conversely, if (7.8) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = f_2(x)\{(1+2b^2)\alpha^2 - 3\beta^2\},$$

from which $\overline{B}^{ij}$ of (7.1) are $hp(3)$. Thus we have the following

THEOREM 7.1. *A Finsler sapce $\overline{F}^n$ $(n > 2)$ which is obtained by a special Randers change of a Finsler space F^n with an (α, β)-metric $L = \alpha + \beta^2/\alpha$ $(b^2 \neq 1)$ of Douglas type, is also a Douglas space, and vice versa.*

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