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RECENT DEVELOPMENTS IN NONLINEAR HYPERBOLIC PDE

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ABSTRACT. In this lecture I shall discuss some recent progress in the development of methods for attacking the central questions of the formation and structure of singularities and of global regularity for solutions of the Cauchy problem for nonlinear systems of partial differential equations of hyperbolic type. Applications to the Einstein equations of general relativity and to the equations of compressible fluid flow shall be particularly emphasized and detailed.

Einstein has remarked that Newton's most important contribution to science was the discovery that Nature's fundamental laws take the form of differential equations, which in principle determine the evolution of a physical system once the initial conditions are specified. Off course since the time of Newton there has been a transition, initiated by Euler, from ordinary differential equations describing the motion of a finite number of point particles to partial differential equations describing the evolution of a material continuum, such as a fluid or an elastic solid, the electromagnetic field in Maxwell's theory, and finally, in Einstein's general theory of relativity, the spacetime geometry itself. Nevertheless, Newton's basic insight has remained the central tenet of classical physics. Thus, the mathematical problem embodying this tenet, the initial value problem, is the mathematical problem par excellence in fluid mechanics and general relativity no less than in Newtonian celestial mechanics.

The theory of compressible perfect fluids, is formulated in the nonrelativistic Galilean spacetime framework as follows. A fluid is described by the fluid velocity v, the mass density ρ and the entropy per unit mass s; v is a vectorfield on Euclidean space, while ρ and s are functions, all depending on the time t. These satisfy a first order system of partial

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differential equations, the *compressible Euler equations*:

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)=0\\
&\frac{\partial(\rho v)}{\partial t}+\nabla\cdot(\rho v\otimes v+pI)=0\\
&\frac{\partial(\rho(\frac{1}{2}|v|^2+u))}{\partial t}+\nabla\cdot((\rho(\frac{1}{2}|v|^2+u)+p)v)=0
\end{aligned}$$

These are the differential laws of conservation of mass, momentum and energy, respectively. Here p is the pressure and u the internal energy per unit mass. The expression $\rho v\otimes v+pI$ in the momentum equation is a 2-contravariant symmetric tensorfield and I is the Euclidean inner product of covectors. The equations assume a definite form once an *equation of state* is specified which gives u as a function of ρ and s. According to the laws of thermodynamics p and the temperature θ are also functions of ρ and s, given by:

$$\begin{aligned}
p&=\rho\frac{\partial u}{\partial\rho}-u\\
\theta&=\frac{1}{\rho}\frac{\partial u}{\partial s}
\end{aligned}$$

Initial data for the compressible Euler equations consists of the specification of v, ρ and s at $t=0$.

According to general relativity the spacetime manifold is a four dimensional oriented differentiable manifold $\mathcal{M}$ which is endowed with a Lorentzian metric g, that is, a continuous assignment of g_x, a symmetric bilinear form of index 1 in $T_x\mathcal{M}$, at each point $x\in\mathcal{M}$. The connection of g corresponds to the Newtonian gravitational force, while the curvature of g, which manifests itself in the deviation of nearby geodesics, corresponds to the Newtonian tidal force which causes relative motion of nearby test masses. The spacetime manifold $(\mathcal{M},g)$ is subject to *Einstein's equations*:

$$\mathrm{Ric}-\frac{1}{2}Rg=2T$$

which constitute the laws of the theory. Here Ric is the Ricci curvature, R the scalar curvature of $(\mathcal{M},g)$, and T is the energy-momentum-stress tensor of matter. In this lecture I shall for the most part confine attention to the vacuum Einstein equations:

$$\mathrm{Ric}=0$$

In general relativity the setup of the initial value problem requires some caution. For, it is meaningless to say that we specify conditions at $t = 0$, or on a hypersurface in spacetime, when the spacetime itself is the unknown in the problem. We therefore proceed as follows. We define an *initial data set* in general relativity to be a triplet $(\mathcal{H}, \overline{g}, k)$, where $(\mathcal{H}, \overline{g})$ is a 3-dimensional complete Riemannian manifold and k a 2-covariant symmetric tensorfield on $\mathcal{H}$, satisfying the *constraint equations*:

$$\overline{\nabla} \cdot k - d\mathrm{tr}k = 0$$

$$\overline{R} - |k|^2 + (\mathrm{tr}k)^2 = 0$$

where $\overline{\nabla}$ is the covariant derivative and $\overline{R}$ the scalar curvature of $(\mathcal{H}, \overline{g})$. We then define a *future development* of this initial data set to be a Lorentzian 4-manifold $(\mathcal{M}, g)$, with boundary, satisfying the Einstein equations, such that with $\mathcal{H}$ identified with the boundary of $\mathcal{M}$, $\overline{g}$ and k coincide respectively with the first and second fundamental form of $\mathcal{H}$ induced from $(\mathcal{M}, g)$ The constraint equations then correspond to the contracted Codazzi and twice contracted Gauss equations of the embedding of $\mathcal{H}$ in $\mathcal{M}$. Moreover, it is required that each past directed causal curve issuing from any given point of $\mathcal{M}$ intersects $\mathcal{H}$ at one and only one point. This requirement, ensures that the future development is actually determined by the initial data, and constitutes part of the definition of the notion *future development*. By a *trivial* initial data set we mean a complete spacelike hypersurface in Minkowski spacetime, with its induced metric and second fundamental form.

The first question posed in the initial value problem is the question of local existence and uniqueness of solutions. In fluid mechanics this question was answered in the affirmative by bringing the compressible Euler equations in the form of a symmetric hyperbolic first order system, once the local theory of such systems had been developed. In general relativity the theorems answering this question in the affirmative where first established in the work of Choquet-Bruhat [5] by making use of "wave coordinates", namely local coordinates such that each of the coordinate functions is a local solution of the wave equation on $(\mathcal{M}, g)$. In these coordinates the Einstein equations reduce to a system of nonlinear wave equations.

The local uniqueness theorem for Einstein's equations, Euler's equations of compressible fluid flow, as indeed for any system of partial differential equations of hyperbolic type, takes the form of a domain of dependence theorem. The nature of the issue of uniqueness is such that the local theorem leads in a straightforward manner, as in the case of a system of ordinary differential equations, to a global theorem. The formulation of the theorem in the case of general relativity requires some caution, because of the fact that the spacetime manifold itself is the unknown. The theorem, proved by Choquet-Bruhat and Geroch [4], asserts the existence, for each given initial data set, of a unique *maximal future development.*

The next fundamental question posed in the initial value problem is the question of the existence of a global solution. Now, in fluid mechanics, as in all cases where there is a given fixed background spacetime, by a "global solution" we simply mean a solution which exists for all time. This would be meaningless in the context of general relativity where the notion of "time" derives from the notion of "spacetime", the unknown in the problem. Thus in general relativity we speak instead of the completeness of the maximal future development. *Completeness* is taken to mean timelike and null geodesic completeness toward the future. The question of completeness, for arbitrary initial data, of the maximal development was answered in the negative by the theorem of Penrose [8], which asserts that if the initial data set $(\mathcal{H}, \overline{g}, k)$ has $\mathcal{H}$ non-compact and containing a closed *trapped surface* $\mathcal{S}$, the boundary of a domain of compact closure in $\mathcal{H}$, then the corresponding maximal future development is incomplete. A trapped surface is a surface $\mathcal{S}$ such that an infinitesimal displacement of $\mathcal{S}$ in $\mathcal{M}$ along the congruence of future-directed outgoing null normals to $\mathcal{S}$ would pointwise decrease the area element.

Let us confine attention in the following to the case that the initial data are asymptotically flat.

The theorem of Penrose (and its subsequent extensions by Hawking and Penrose [6]) led naturally to the following question: is there any non-trivial asymptotically flat initial data whose maximal development is complete? This question was answered by my joint work with Klainerman [1] on the stability of Minkowski spacetime. This work shows, in fact, that any asymptotically flat initial data which is sufficiently close to the trivial data $(\mathcal{H}_0, \overline{g}_0, k_0)$, where $\mathcal{H}_0 = \Re^3$, $\overline{g}_0$ is the Euclidean metric and $k_0 = 0$, has a complete maximal development $(\mathcal{M}, g)$. More precisely, *sufficiently close to* $(\mathcal{H}_0, \overline{g}_0, k_0)$ can be taken to mean the following:

$$\inf_{p\in\mathcal{H}_0, a>0} a^{-1} \int_{\mathcal{H}_0} \left\{ \sum_{n=0}^{1} (d_p^2 + a^2)^{n+1} |\overline{\nabla}^n \overline{\mathrm{Ric}}|^2 + \sum_{n=0}^{2} (d_p^2 + a^2)^n |\overline{\nabla}^n k|^2 \right\} d\mu_{\overline{g}} \; < \; \varepsilon$$

for a suitable $\varepsilon > 0$. Here $\overline{\nabla}$ denotes the covariant derivative operator, $\overline{\mathrm{Ric}}$ the Ricci curvature, and d_p the distance function from the point p, of the Riemannian manifold $(\mathcal{H}_0, \overline{g})$. Moreover, the spacetime $(\mathcal{M}, g)$ becomes flat along any geodesic as the affine parameter tends to infinity. In fact, our work contains detailed results on the asymptotic behavior of $(\mathcal{M}, g)$. The proof was based on the derivation of growth estimates for certain quantities, norms controlling the spacetime curvature, built from vectorfields which generate the action of a group on the spacetime manifold, an action defined in such a way that its a long-time asymptote is of that of a group of conformal motions. A novel kind of continuity argument was employed, involving an elaborate bootstrap, in which the construction of the group action starts from the final hypersurface possessing certain maximal properties and proceeds toward the past to the initial hypersurface. This work is reviewed in [2].

Returning to the compressible Euler equations, it had long been known that special solutions of these equations arising from smooth initial data develop singularities, as the first spatial derivatives of (v, ρ, s) blow up within a finite time interval, signaling shock formation. However, the first work to demonstrate this in general was the work of Sideris [11].

The trivial solution in the context of the theory of compressible perfect fluids, the analogue of the Minkowski spacetime, is not a single solution but any one of the set of *constant states* (v_0, ρ_0, s_0), where $v_0 = 0$ and ρ_0, s_0 are constants. Sideris considers initial data at $t = 0$ which coincide with those of a constant state outside a ball of radius R. The hypothesis of his theorem is that there exists a $R' \in (0, R)$ such that for each $r \in (R', R)$ the integral:

$$\int_{B_R \setminus B_r} |x|^{-1} (|x| - r)^2 (\rho - \rho_0) d^3x$$

is positive while the integral

$$\int_{B_R \setminus B_r} |x|^{-3}(|x|^2 - r^2)\rho < x, v > d^3x$$

is non-negative. Here B_r is the ball of radius r with center at the origin in Euclidean space. Moreover it is assumed that $s \geq s_0$, and that the equation of state is such that p is a convex function of ρ at constant s and an increasing function of s at constant ρ. The conclusion of the theorem is that the maximal time interval of existence of a C^1 solution is finite. Note that according to this theorem even arbitrarily small perturbations of a constant state, having support in a ball of fixed radius, lead to breakdown in a finite time. Thus the analogue of the stability of Minkowski spacetime is false.

The chief drawback of the theorems of both Penrose and Sideris is that they tell us nothing about the nature of the breakdown. In the case of the theorem of Penrose, we do not even know whether the maximal development $(\mathcal{M}, g)$ is locally inextendible as a solution in a C^2 fashion (in reference to g). In fact, it could well be that under the hypotheses of the theorem the maximal development does extend as a solution in a C^2 fashion. This larger spacetime $(\mathcal{M}', g')$ would no longer be a development of the initial data, so for points $p' \in \mathcal{M}' \setminus \mathcal{M}$ there would exist past directed causal curves issuing from p' which do not terminate on $\mathcal{H}$. The boundary of $\mathcal{M}$ in $\mathcal{M}'$ would then be a *Cauchy horizon* and the extension beyond this boundary would not be determined by the initial data on $\mathcal{H}$.

Another limitation of the theorem of Penrose is in relation to the hypothesis that a closed trapped surface is present in the initial data. The methods of the theorem and of its extensions due to Hawking and Penrose are not capable of demonstrating that the concept of a closed trapped surface is *evolutionary*. By this I mean that a closed trapped surface can in fact form in the evolution, starting from initial data in which no such surfaces are present. To see more clearly what is at issue here, let us introduce the concept of an *anti-trapped surface*. An anti-trapped surface is a surface S such that an infinitesimal displacement of S in $\mathcal{M}$ along the congruence of future-directed incoming null normals to S would pointwise increase the area element. Using the methods of Penrose's theorem we can easily show that closed anti-trapped surfaces are *non-evolutionary*. That is, if no closed anti-trapped surfaces are present in the initial data $(\mathcal{H}, \overline{g}, k)$, then no such surfaces are contained in the maximal development of $(\mathcal{H}, \overline{g}, k)$. I believe this is an interesting point, for it relates to the way that entropy increase and irreversibility arise from the way in which initial conditions are set up and a direction of time is chosen toward which to evolve. It is reminiscent of what Landau has shown in the framework of compressible perfect fluid flow ([7]), namely that if we start from smooth initial conditions, then only those shock discontinuities can develop which have the property that along each flow line in space-time the entropy per particle jumps upward as the hypersurface of discontinuity is traversed (from the past to the future).

The above summarize what we actually know at present about the initial value problem for the general compressible Euler equations and for the general vacuum Einstein equations with asymptotically flat initial data.

In the case of the compressible Euler equations we do not possess a positive result even for small but otherwise arbitrary compactly supported perturbations of a constant state. Since by the result of Sideris a smooth solution does not exist for all time, to obtain a global result we must allow for solutions which contain shock discontinuities. The difficulty however is in finding an appropriate class of solutions wide enough, to permit global existence but narrow enough so that uniqueness can be proved. For only such a result can be considered satisfactory from the physical point of view. I should remark at this point that no such result is known at present *even when plane symmetry is imposed*, so that the problem reduces to one in a single space dimension. A detailed understanding of the singularities that actually form starting from arbitrary smooth initial data would constitute great progress toward the desired goal.

In the case of the Einstein equations the methods which we currently possess are incapable of attacking the general initial value problem *in the large*, that is when the initial data are no longer confined to a suitable neighborhood of trivial data and *no symmetry* is imposed. In relation to the initial value problem in the large we can at present only formulate conjectures, not prove theorems. Chief among the conjectures that have been formulated is the conjecture that has been termed "weak cosmic censorship". This conjecture was conceived by Penrose [9], not long after his singularity theorem. It is usually stated as follows:

Generic asymptotically flat initial data have a maximal future development possessing a *complete future null infinity.*

In heuristic terms this means that, if we disregard exceptional initial conditions, no singularities are observed from infinity, even though observations from infinity are allowed to continue indefinitely.

Almost equal in importance to the weak cosmic censorship conjecture, is the conjecture which has been termed "strong cosmic censorship". It was also conceived by Penrose [10], who was motivated by the consideration that the weak cosmic censorship conjecture does not suffice in so far as it allows for singularities which are locally visible although not accessible to observation from infinity. Such singularities may affect the unique predictability of the outcomes of observations by local observers. The strong cosmic censorship conjecture was thus designed to forbid the formation of singularities of this type. A concrete mathematical formulation is the following.

Generic asymptotically flat initial data have a maximal future development which is locally inextendible as a Lorentzian manifold in a continuous manner.

Here the continuity requirement on the extension refers to the metric. According to this formulation, along a future inextendible timelike geodesic of bounded arc length (proper time), not only must the spacetime curvature components in a parallel orthonormal frame field along the geodesic blow up as the limiting value of the arc length is approached, but the corresponding deformation matrix, which expresses the components of a Jacobi field along the geodesic must itself not tend to a finite limit. This is necessary to ensure that no extension in a weaker sense is physically possible.

The choice of adjectives "strong" and "weak" in reference to the two cosmic censorship conjectures is not a happy one, for it seems to suggest that the "strong" conjecture implies the "weak" one, which is not the case. For, one can easily imagine a maximal development $(\mathcal{M}, g)$ terminating at a singular future light cone, which can be attached as a boundary to $\mathcal{M}$, thus obtaining a topological manifold $\overline{\mathcal{M}} = \mathcal{M} \bigcup \partial\mathcal{M}$, but such that the metric does not extend continuously, even locally, to $\partial\mathcal{M}$, nevertheless the area of cross sections of $\partial\mathcal{M}$ is well defined and has no upper bound, resulting in the incompleteness of future null infinity. In fact, there is no obvious relation between the two cosmic censorship conjectures. This is not surprising, for the "weak" conjecture has a global aspect not present in the "strong" conjecture.

In the effort to establish the truth of conjectures whose statement includes the caveat "generic" an understanding of the obstructions to the general validity of the conjectures is indispensable. It is only after we have gained an understanding of the reasons why a possible counterexample to general validity cannot belong to an open set of such examples, that we have any chance of proving the conjectures. In the case of the weak cosmic censorship conjecture, we must investigate the possibility of singularity formation which is not associated to closed trapped surfaces. (Singularities arising from closed trapped surfaces do not affect the general validity of the conjecture.) At present there are no mathematical results in this direction for the Einstein vacuum equations.

An analogous problem in the theory compressible perfect fluid flow would be to search for singularities which are not shock discontinuities. The problem admits a simple statement in the incompressible limit, for no shocks can develop in an incompressible fluid. In terms of the fluid velocity v and vorticity ω, vectorfields in Euclidean space depending on time, the Euler equations reduce in the incompressible limit to:

$$\frac{\partial \omega}{\partial t} + [v, \omega] = 0$$
$$\nabla \cdot v = 0 \quad \nabla \times v = \omega$$

Here $[v, \omega]$ denotes the commutator of the vectorfields v, ω. The initial condition is the specification of ω_0, the vectorfield ω at $t = 0$. It is assumed that ω_0 is a smooth vectorfield of compact support. If $[0, t_*)$ is the maximal time interval of existence, then the support of ω in space at each $t \in [0, t_*)$ remains compact. The problem is to show that there exist initial data for which t_* is finite. This problem is at present unsolved. It is a fundamental problem in hydrodynamics, because its solution would constitute a first step toward understanding the dynamical onset of turbulence.

In concluding, I would like to mention my work on the Einstein equations in the presence of spherical symmetry with a massless scalar field as the material model. The Einstein equations assume in this case the form:

$$\mathrm{Ric} = 2d\phi \otimes d\phi$$

As a consequence of these equations the function ϕ, the scalar field, satisfies the wave equation on $(\mathcal{M}, g)$. However, it is not appropriate for me to give here a complete summary of this work at this time. I shall refer to the review article [3] and the references cited therein. I shall only mention that the issues I have been discussing have been settled, in that limited context, in favor of both cosmic censorship conjectures, the caveat "generic" having been demonstrated to be necessary. The proof relies on a detailed understanding of the nature of the initial conditions which eventually lead to trapped surface formation.

However, it is wise to keep in mind that whereas some of the spherical concepts and methods are capable of generalization, others are not, and I believe that many new ideas and surprises will come before the fundamental issues are settled in the general case.

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