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PERMUTATIONS WITH PARTIALLY FORBIDDEN POSITIONS

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ABSTRACT. In this paper we consider the enumeration problem of permutations with partially forbidden positions, generalizing the notion of permutations with forbidden positions. As an alternative approach to this problem, we investigate the permanent maximization problem over some classes of $(0,1)$-matrices which have a given number of 1's some of which lie in prescribed positions.

1. Introduction

The problem of enumerating 'permutations with forbidden positions' retains substantial importance in the theory of combinatorics. A typical example is the well-known derangement problem. A derangement of $\{1,2,\cdots,n\}$ is a permutation σ of $\{1,2,\cdots,n\}$ with the property that $\sigma(i)\neq i$ for all $i=1,2,\cdots,n$. The problem is to find the number of derangements. This kind of problem can be converted into one of the evaluation of the permanent of certain $(0,1)$-matrix with 0's in the 'forbidden' positions and 1's elsewhere. For a matrix $A=[a_{ij}]$, the *permanent* of A, per$A$, is defined by

$$\mathrm{per}A=\sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)},$$

where S_n stands for the symmetric group on $\{1,2,\cdots,n\}$. The number d_n of derangements of $\{1,2,\cdots,n\}$, the n-th derangement number, is equal to the permanent of J_n-I_n, where J_n and I_n denote the all 1's matrix of order n and the identity matrix of order n respectively. It is

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well known that

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

The derangement problem can be generalized to a problem of enumerating permutations σ of $\{1, 2, \cdots, n\}$ satisfying $\sigma(i) \neq i$ for at least k of the i's in $\{1, 2, \cdots, n\}$, where k is an integer such that $1 \leq k \leq n$. The corresponding permutation matrices are 'partially' forbidden to have their 1's in the main diagonal in the sense that they are allowed to have upto $n - k$ 1's in the main diagonal. It is easy to see that the number we are looking for in this problem is equal to

$$\sum_{i=0}^{n-k} \binom{n}{i} d_{n-i}.$$

In this paper, we consider a problem of enumerating the permutations with partially forbidden positions for some other settings of interest.

Let n be a positive integer and $S = [s_{ij}]$ be a $(0,1)$-matrix of order n. Let d be an integer such that $0 \leq d \leq n^2 - \#(S)$, where and in the sequel for a $(0,1)$-matrix A, $\#(A)$ denotes the number of 1's in A. Let $\mathcal{R}(S, d)$ denote the class of all $(0,1)$-matrices A of order n such that $A \geq S$ and $\#(A - S) = d$, where and in the sequel, $A \geq S$ (resp. $A \leq S$) means that every entry of A is bigger (resp. less) than or equal to the corresponding entry of S. If $d = 0$, then $\mathcal{R}(S, d)$ consists of the matrix S only, and the number of permutations σ of $\{1, 2, \cdots, n\}$ such that $\sigma(i) \neq j$ whenever $s_{ij} = 0$ equals per$S$ as in the case of derangement problem. We are interested in the following problem:

(i) What is the maximum value of the permanent function over the class $\mathcal{R}(S, d)$?
(ii) At which matrices in $\mathcal{R}(S, d)$ is this value achieved?

This problem for $S = O$ was investigated by Brualdi, Goldwasser and Michael[1]. Specifically they determined an upper bound for the permanent of a matrix in $\mathcal{R}(O, d)$ for a given d, and they determined all matrices in $\mathcal{R}(O, d)$ with maximum permanent for d with $n \leq d \leq 2n$ and for d with $n^2 - 2n \leq d \leq n^2$.

An $n \times n$ $(0,1)$ matrix $A = [a_{ij}]$ with row sum vector $(r_1, r_2, \cdots, r_n)$ is called a Ferrers' matrix if $r_1 \leq r_2 \leq \cdots \leq r_n$ and $a_{i1} \geq a_{i2} \geq \cdots \geq a_{in}$ for every $i = 1, 2, \cdots, n$, and is denoted by $F(r_1, r_2, \cdots, r_n)$. It is well known that, if $r_i \geq i$ for all $i = 1, 2, \cdots, n$, then

$$\mathrm{per}F(r_1, r_2, \cdots, r_n) = \prod_{i=1}^{n}(r_i - i + 1) \tag{1}$$

[3]. For integers n, k with $0 \leq k \leq n-1$, let $F(n,k) = [f_{ij}]$ be the $n \times n$ $(0,1)$ matrix defined by $f_{ij} = 1$ if and only if $j < i + k$. For example $F(n,1) = \Delta_n$ where

$$\Delta_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

is the $n \times n$ lower triangular $(0,1)$-matrix with $(n^2+n)/2$ 1's in and under the main diagonal positions, and $F(n,2)$ is the $n \times n$ lower Hessenberg matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

Let d be an integer such that $0 \leq d \leq n^2 - \#(F(n,k))$. In this paper, we maximize the permanent over $\mathcal{R}(F(n,k),d)$ for the case that n is sufficiently large compared with d, and for the case that $d \leq 2$. The problem concerning the maximization and minimization of spectral radii of matrices in the class $\mathcal{R}(\Delta_n, d)$ was investigated by Brualdi and Hwang [2]. By (1), it follows that

$$\mathrm{per}F(n,k) = k^{n-k}k!. \tag{2}$$

2. Maximum permanent over $\mathcal{R}(F(n,k),d)$

For a matrix A of order n and for $\alpha, \beta \subset \{1, 2, \cdots, n\}$, let $A(\alpha|\beta)$ denote the matrix obtained from A by deleting all rows in α and all columns in β, and let $A[\alpha|\beta] = A(\overline{\alpha}|\overline{\beta})$, where $\overline{\alpha}$ and $\overline{\beta}$ stand for the complements of α and β relative to $\{1, 2, \cdots, n\}$ respectively. The following lemma is well known. The proof we give here is a little bit simpler than that in [4, p.19]. For integers r, n with $0 \le r \le n$, let $Q_{r,n}$ denote the set of all r-subsets of $\{1, 2, \cdots, n\}$.

LEMMA 1 ([4, p. 18]). *If A, B are matrices of order n, then*

$$\text{per}(A+B) = \sum_{r=0}^{n} \sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha|\beta]\text{per}B[\overline{\alpha}|\overline{\beta}]. \tag{3}$$

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$, and

$$f(x) = \text{per}(Ax+B) = \sum_{r=0}^{n} c_r x^r. \tag{4}$$

In the expansion of

$$\sum_{\sigma \in S_n} (a_{1\sigma(1)}x + b_{1\sigma(1)})(a_{2\sigma(2)}x + b_{2\sigma(2)}) \cdots (a_{n\sigma(n)}x + b_{n\sigma(n)}),$$

the coefficient c_r of x_r clearly equals

$$\sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha|\beta]\text{per}B[\overline{\alpha}|\overline{\beta}],$$

and formular (3) follows by plugging $x = 1$ into (4). □

Let L_k denote the *back diagonal* permutation matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

of order k, and for an integer d with $0 \le d \le (n-k)/2$. Let Γ_d denote the $n \times n$ matrix

$$\begin{bmatrix} O & L_d \\ O & O \end{bmatrix}.$$

We make a rough guess that the maximum value of the permanent over the class $\mathcal{R}(F(n,k), d)$ is achieved at a matrix with the 'additional' d 1's being placed close to the back diagonal. From now on in the sequel, let E_{ij} denote the square matrix of suitable order all of whose entries are 0 except for the (i,j)-entry which is 1.

LEMMA 2. *Let $X = [x_{ij}]$ be an $n \times n$ Ferrers matrix with $\Delta_n \leq X$ whose row sum vector and column sum vector are $(r_1, r_2, \cdots, r_n)$ and $(c_1, c_2, \cdots, c_n)$ respectively, and let p, q be integers with $1 \leq p < q < n$ such that $x_{pq} = 0$. Let $A = X + E_{1q} + E_{pn}$, $B = X + E_{1n} + E_{pq}$. Then* $\text{per}A \leq \text{per}B$ *where the inequality is strict unless $r_1 = r_2 = \cdots = r_p$ and $c_q = c_{q+1} = \cdots = c_n$.*

Proof. By Lemma 1, we have

$$\text{per}A = \text{per}X + \text{per}X(1|q) + \text{per}X(p|n) + \text{per}X(1,p|q,n),$$
$$\text{per}B = \text{per}X + \text{per}X(1|n) + \text{per}X(p|q) + \text{per}X(1,p|q,n).$$

Let $Y = X[p+1, \cdots, q|p, \cdots, q-1]$ and let

$$a = r_2(r_3 - 1)\cdots(r_p - p + 2),$$
$$a' = r_1(r_2 - 1)\cdots(r_{p-1} - p + 2),$$
$$b = c_{n-1}(c_{n-2} - 1)\cdots(c_q - (n-q-1)),$$
$$b' = c_n(c_{n-1} - 1)\cdots(c_{q+1} - (n-q-1)).$$

Then $a \geq a'$ with equality if and only if $r_1 = r_2 = \cdots = r_p$, and $b \geq b'$ with equality if and only if $c_q = c_{q+1} = \cdots = c_n$. Now by the formula (1),

$$\begin{aligned}
&\text{per}X(1|n) \\
&= \text{per}X[2, \cdots, n|1, \cdots, n-1] \\
&= r_2\text{per}X[3, \cdots, n|2, \cdots, n-1] \\
&= r_2(r_3 - 1)\text{per}X[4, \cdots, n|3, \cdots, n-1] \\
&\quad\vdots \\
&= r_2(r_3 - 1)\cdots(r_p - p + 2)\text{per}X[p+1, \cdots, n|p, \cdots, n-1] \\
&= a\ \text{per}X[p+1, \cdots, n|p, \cdots, n-1].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&= \text{per}X[p+1, \cdots, n|p, \cdots, n-1] \\
&= c_{n-1}\text{per}X[p+1, \cdots, n-1|p, \cdots, n-2] \\
&= c_{n-1}(c_{n-2} - 1)\text{per}X[p+1, \cdots, n-2|p, \cdots, n-3]
\end{aligned}$$

$$\begin{aligned} &\vdots \\ &= c_{n-1}(c_{n-2}-1)\cdots(c_q-(n-q-1)) \\ &\quad \times \operatorname{per}X[p+1,\cdots,q|p,\cdots,q-1] \\ &= b \operatorname{per}Y. \end{aligned}$$

Thus $\operatorname{per}X(1|n) = ab \operatorname{per}Y$. Similarly we can show that $\operatorname{per}X(1|q) = ab' \operatorname{per}Y$, $\operatorname{per}X(p|n) = a'b \operatorname{per}Y$ and $\operatorname{per}X(p|q) = a'b' \operatorname{per}Y$. Hence

$$\begin{aligned} \operatorname{per}B - \operatorname{per}A &= (ab + a'b' - ab' - a'b)\operatorname{per}Y \\ &= (a-a')(b-b')\operatorname{per}Y \\ &\geq 0, \end{aligned}$$

where equality holds if and only if either $a = a'$ or $b = b'$. Thus the inequality is strict unless $r_1 = r_2 = \cdots = r_p$ and $c_q = c_{q+1} = \cdots = c_n$. □

Let $\mathbf{Max}(S,d)$ denote the set of all matrices $A \in \mathcal{R}(S,d)$ such that $\operatorname{per}A \geq \operatorname{per}X$ for all $X \in \mathcal{R}(S,d)$.

THEOREM 3. *Let c, d and n be positive integers such that $n \geq c+2d$. If d equals 1 or 2, then $F(n,c) + \Gamma_d$ is the unique matrix in $\mathbf{Max}(F(n,c),d)$.*

Proof. Let $G = F(n,c)$. By Lemma 2, it is straightforward that $\mathbf{Max}(F(n,c),1)$ consists of the single matrix $G + \Gamma_1$. To prove the theorem for the case $d = 2$, let $A \in \mathbf{Max}(F(n,c),2)$ and let $A_1 = G + E_{1,n} + E_{2,n}$, $A_2 = G + \Gamma_2$. Then by Lemma 2 again and by taking flip along the back diagonal, if necessary, we may assume that $A = A_1$ or $A = A_2$. By Lemma1,

$$\begin{aligned} \operatorname{per}A_1 &= \operatorname{per}G + \operatorname{per}G(1|n) + \operatorname{per}G(2|n), \\ \operatorname{per}A_2 &= \operatorname{per}G + \operatorname{per}G(1|n) + \operatorname{per}G(2|n-1) + \operatorname{per}G(1,2|n-1,n) \end{aligned}$$

so that

$$\operatorname{per}A_2 - \operatorname{per}A_1 = \operatorname{per}G(2|n-1) + \operatorname{per}G(1,2|n-1,n) - \operatorname{per}G(2|n).$$

Since

$$G(2|n-1) = \left[\begin{array}{c|c} (L_{n-2}\mathbf{h}_c)^T & 0 \\ \hline F(n-2,c+2) & \mathbf{h}_c \end{array}\right],$$

$$G(1,2|n-1,n) = F(n-2,c+2)$$

and

$$G(2|n) = \left[\begin{array}{c|c} (L_{n-2}\mathbf{h}_c)^T & 0 \\ \hline F(n-2, c+2) & \mathbf{h}_{c+1} \end{array}\right],$$

where $\mathbf{h}_k$ stands for the sum of the last k columns of the identity matrix of order $n-2$ for $k = c, c+1$, we get, by formula (2), that

$$\begin{aligned} \mathrm{per}G(2|n-1) &= c^2 \ \mathrm{per}F(n-3, c+1) = c^2(c+1)^{n-c-4}(c+1)!, \\ \mathrm{per}G(1,2|n-1,n) &= (c+2)^{n-c-4}(c+2)! = (c+2)^{n-c-3}(c+1)!, \\ \mathrm{per}G(2|n) &= c \ \mathrm{per}F(n-2, c+1) = c(c+1)^{n-c-3}(c+1)!, \end{aligned}$$

whence

$$\begin{aligned} &\mathrm{per}A_2 - \mathrm{per}A_1 \\ &= (c+1)![c^2(c+1)^{n-c-4} + (c+2)^{n-c-3} - c(c+1)^{n-c-3}] \\ &\geq (c+1)!(c+1)^{n-c-4}[c^2 + (c+2) - c(c+1)] \\ &= 2(c+1)!(c+1)^{n-c-4} > 0. \end{aligned}$$

Thus it follows that A_2 is the matrix at which the permanent function attains at its maximum, and the proof is complete. □

LEMMA 4. *Let c and d be fixed positive integers, and let $A \in$* **Max** *$(F(n,c), d)$. If n is sufficiently large, then the $d \times d$ submatrix $A[1, \cdots, d|n-d+1, \cdots, n]$ in the upper right corner of A is a permutation matrix.*

Proof. Let $G = F(n,c)$ and $U = A - G$. Let p be the largest integer less than or equal to n such that the row p of U is not a zero vector, and let q be the smallest integer less than or equal to n such that the column q of U is not a zero vector. Let $U_0 = U[1, \cdots, p|q, \cdots, n]$. Then by Lemma2, the matrix U_0 can not have a zero row or zero column. Note that U_0 is a permutation matrix if and only if $p = n - q + 1 = d$. We may assume that $p \leq n - q + 1$ by taking flip along the back diagonal if necessary. Suppose that U_0 is not a permutation matrix. Then $p < d$. There is no $k \times k$ permutation submatrix of U if $k > p$. Let k be an integer such that $1 \leq k \leq p$. Let $\alpha_k = \{1, \cdots, k\}$, $\beta_k = \{n-k+1, \cdots, n\}$. Then for every k-subset α of $\{1, \cdots, p\}$ and every k-subset β of $\{n-q+1, \cdots, n\}$, $\mathrm{per}G(\alpha|\beta) \leq \mathrm{per}G(\alpha_k|\beta_k)$ because

$G(\alpha|\beta) \leq G(\alpha_k|\beta_k)$. Since $G(\alpha_k|\beta_k) = F(n-k, k+c)$, we have, by formula (2), that

$$\mathrm{per}G(\alpha|\beta) \leq (k+c)^{n-2k-c}(k+c)! = \frac{(k+c)!}{(k+c)^{2k+c}}(k+c)^n.$$

Since the number of $k \times k$ permutation submatrices of U does not exceed $\binom{d}{k}$, and since $\mathrm{per}G = c^{n-c}c!$, we have from Lemma 1 that

$$\mathrm{per}A \leq \sum_{k=0}^{p} = \frac{(k+c)!}{(k+c)^{2k+c}}(k+c)^n \leq d! \sum_{k=0}^{p}(k+c)^n.$$

On the other hand, letting $\alpha_0 = \{1, \cdots, d\}$, $\beta_0 = \{n-d+1, \cdots, n\}$, we have

$$\begin{aligned}\mathrm{per}(G + \Gamma_d) &\geq \mathrm{per}G(\alpha_0|\beta_0)\ \mathrm{per}\Gamma_d(\alpha_0|\beta_0) \\ &= \mathrm{per}G(\alpha_0|\beta_0) \\ &= (d+c)^{n-2d-c}(d+c)! = \frac{(d+c)!}{(d+c)^{2d+c}}(d+c)^n,\end{aligned}$$

since $G(\alpha_0|\beta_0) = F(n-d, d+c)$. Thus

$$\begin{aligned}\frac{\mathrm{per}A}{\mathrm{per}(G+\Gamma_d)} &\leq \frac{d!(d+c)^{2d+c}}{(d+c)!}\sum_{k=0}^{p}\left(\frac{k+c}{d+c}\right)^n \\ &\leq \frac{d!(d+c)^{2d+c}}{(d+c)!}(p+1)\left(\frac{p+c}{d+c}\right)^n.\end{aligned}$$

Since $p < d$, the above inequality tells us that $\mathrm{per}A < \mathrm{per}(G + \Gamma_d)$ for every sufficiently large n, which is impossible since $G + \Gamma_d \in \mathcal{R}(G, d)$ and $A \in \mathbf{Max}(G, d)$. Therefore it has to be that U_0 is a permutation matrix. □

THEOREM 5. *Let c and d be fixed positive integers. If n is sufficiently large, then the maximum permanent over $\mathcal{R}(F(n,c), d)$ is achieved uniquely at the matrix $F(n,c) + \Gamma_d$.*

Proof. We prove the theorem by induction on d. The case $d \leq 2$ is treated in Theorem 3, and the induction starts. As in Lemma 4, let $G = F(n,c)$, $A \in \mathbf{Max}(G,d)$, and $U = A - G = [u_{ij}]$. Then by Lemma 4, $U[1, \cdots, d|n-d+1 \cdots, n]$ is a permutation matrix of order d. Note that any square submatrix of U is a permutation matrix unless it has a zero row or zero column. We call a subset $\delta = \{(i_1, j_1), \cdots, (i_k, j_k)\}$ of $\{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$ a k-diagonal of U if $U = [i_1, \cdots, i_k\, |j_1, \cdots, j_k]$ is a permutation matrix of order k. For a k-diagonal $\delta = \{(i_1, j_1),$

$\cdots, (i_k, j_k)\}$ of U, let $G(\delta) = G(i_1, \cdots, i_k | j_1, \cdots, j_k)$. Let $\Delta(k, U)$ denote the set of all k-diagonals of U. By Lemma 1, we see that

$$\text{per}A = \text{per}G + \sum_{k=1}^{d} \sum_{\delta \in \Delta(k,U)} \text{per}G(\delta).$$

We first show that $u_{1n} = 1$. Suppose, on the contrary, that $u_{1n} = 0$. Then there exist integers p, q with $1 < p \le d$ and $n - d + 1 \le q < n$ such that $u_{1q} = u_{pn} = 1$. Then, clearly, $u_{pq} = u_{1n} = 0$. Let $B = A - E_{1q} + E_{1n} + E_{pq} - E_{pn}$, and $V = B - G$. Then $B \in \mathcal{R}(G, d)$, $V = U - E_{1q} + E_{1n} + E_{pq} - E_{pn}$, and

$$\text{per}B = \text{per}G + \sum_{k=1}^{d} \sum_{\delta \in \Delta(k,V)} \text{per}G(\delta).$$

A $\delta \in \Delta(k, U)$ belongs to exactly one of the following four cases ;

$$\begin{aligned}
&\textit{case}\ \ (\text{i}) : (1,q) \notin \delta \text{ and } (p,n) \notin \delta, \\
&\textit{case}\ \ (\text{ii}) : (1,q) \in \delta \text{ and } (p,n) \notin \delta, \\
&\textit{case}\ \ (\text{iii}) : (1,q) \notin \delta \text{ and } (p,n) \in \delta, \\
&\textit{case}\ \ (\text{iv}) : (1,q) \in \delta \text{ and } (p,n) \in \delta.
\end{aligned}$$

Let $\delta_0 = \delta - \{(1,q), (p,n)\}$, and let $\delta' = \delta$ if δ belongs to case (i), $= \delta_0 \cup \{(1,n)\}$ if δ belongs to case (ii), $= \delta_0 \cup \{(p,q)\}$ if δ belongs to case (iii), $= \delta_0 \cup \{(1,n), (p,q)\}$ if δ belongs to case (iv). Then

$$\text{per}B - \text{per}A = \sum_{k=1}^{d} \sum_{\delta \in \Delta(k,U)} (\text{per}G(\delta') - \text{per}G(\delta)).$$

If δ belongs to case (i) or (iv), then $\text{per}G(\delta') - \text{per}G(\delta) = 0$. So, by letting $\Delta^*(k-1, U)$ denote the set of all $(k-1)$-diagonals which do not contain one of $(1,q), (p,n)$, we have

$$\begin{aligned}
&\sum_{k=1}^{d} \sum_{\delta \in \Delta(k,U)} (\text{per}G(\delta') - \text{per}G(\delta)) \\
&= \sum_{k=1}^{d} \sum_{\delta \in \Delta^*(k-1,U)} (\text{per}G(\delta_{1,n}) + \text{per}G(\delta_{p,q}) - \text{per}G(\delta_{1,q}) - \text{per}G(\delta_{p,n})),
\end{aligned}$$

where $\delta_{i,j} = \delta_0 \cup \{(i,j)\}$, for $(i,j) \in \{(1,n), (p,q), (1,q), (p,n)\}$. But

$$\begin{aligned}
&(\text{per}G(\delta_{1,n}) + \text{per}G(\delta_{p,q}) - \text{per}G(\delta_{1,q}) - \text{per}G(\delta_{p,n})) \\
&= \text{per}(G(\delta_0) + E_{1n} + E_{pq}) - \text{per}(G(\delta_0) + E_{1q} + E_{pn}),
\end{aligned}$$

which is positive by Lemma 2 because $G(\delta_0)$ is a Ferrers matrix, contradicting the maximality of A. Thus it is proved that $u_{1n} = 1$. Now expanding per$A$ along the first row, we have

$$\text{per}A = c^2\ \text{per}(F(n-2,c) + U(1|n)) + \text{per}(F(n-1,c+1) + U(1,n)).$$

By induction,

$$\text{per}(F(n-2,c) + U(1|n)) \le \text{per}(F(n-2,c) + \Gamma_{d-1})$$
$$\text{per}(F(n-1,c+1) + U(1|n)) \le \text{per}(F(n-1,c+1) + \Gamma_{d-1}),$$

where any of the equalities holds if and only if $U(1|n) = \Gamma_{d-1}$. Thus

$$\begin{aligned}\text{per}A &\le c^2\ \text{per}(F(n-2,c) + \Gamma_{d-1}) + \text{per}(F(n-1,c+1) + \Gamma_{d-1})\\ &= \text{per}(F(n,c) + \Gamma_d),\end{aligned}$$

with equality if and only if $U(1|n) = \Gamma_{d-1}$. Since $A \in \mathbf{Max}(G,d)$, it must be that $U(1|n) = \Gamma_{d-1}$ and hence that $U(1|n) = \Gamma_d$, and the proof is complete. □

3. Matrices in $\mathbf{Max}(\Delta_n, d)$

If $c = 1$, then $F(n,c) = \Delta_n$. The permanent maximization problem over the class $\mathcal{R}(\Delta_n, d)$ can be interpreted in terms of graph theoretic terminologies as follows. Let D_n be the directed graph with n nodes $1, 2, \cdots, n$ and $(n^2+2)/2$ arcs (i,j), for all i, j with $i \ge j$. Then Δ_n is the adjacency matrix of D_n. To D_n, we would like to introduce some new arcs (i,j) with $i < j$. We call such an arc (i,j) a down-going arc. A spanning subgraph H of a directed graph D is called a 1-factor of D if the in-degree and the out-degree in H of each vertex equal 1. It is well known that the number of 1-factors of a directed graph D is equal to the permanent of the adjacency matrix of D. The problem here is to determine the set of d down-going arcs to be added to D_n in order to maximize the number of 1-factors of the resulting directed graph. We conjecture that the maximum permanent over $\mathcal{R}(\Delta_n, d)$ is achieved at the matrix $\Delta_n + \Gamma_d$ if $d \le n/2$. In what follows we evaluate the permanent of $\Delta_n + \Gamma_d$ in terms of some numbers defined by a recurrence relation similar to that of binomial coefficients, and show that every matrix in $\mathbf{Max}(\Delta_n, d)$ is fully indecomposable.

Let p be a fixed nonnegative integer. For nonnegative integers n, k, let $\binom{n}{k}_p$ be a number defined by

(i) $\binom{n}{1}_p = 1$, $\binom{n}{n}_p = 1$ for all $n = 1, 2, \cdots$,

(ii) $\binom{n}{k}_p = \binom{n-1}{k-1}_p + k^p\binom{n-1}{k}_p = 1$ for $k \le n-1$,

(iii) $\binom{n}{k}_p = 0$ if $k > n$.

Note that $\binom{n}{k}_0 = \binom{n}{k}$ and $\binom{n}{k}_1 = S(n,k)$, the Stirling number of the second kind. In the next theorem we give a formula for the permanent of the matrix $\Delta_n + \Gamma_d$ in terms of the numbers $\binom{n}{k}_2$.

THEOREM 6. *Let d, n be integers such that $0 \le 2d \le n$.*

(a) *If $2d \le n-1$, then* $\mathrm{per}(\Delta_n + \Gamma_d) = \sum_{k=1}^{d+1} \binom{d+1}{k}_2 k^{n-2d}(k-1)!$.

(b) *If $2d = n$, then* $\mathrm{per}(\Delta_n + \Gamma_d) = \sum_{k=1}^{d} \binom{d}{k}_2 (k+1)!$.

Proof. For nonnegative integers n, k, r with $k + r \le n$, let $f_r(n,k) = \mathrm{per}(F(n,k) + \Gamma_r)$. We first prove the recurrence relation

$$f_r(n,k) = k^2 f_{r-1}(n-2,k) + f_{r-1}(n-1,k+1) \tag{5}$$

for $f_r(n,k)$. Let $A = F(n,k) + \Gamma_r$. Then $\mathrm{per}A = \mathrm{per}(A - E_{1n}) + \mathrm{per}A(1|n)$. Clearly $\mathrm{per}(A - E_{1n}) = k^2\mathrm{per}(F(n-2,k) + \Gamma_{r-1}) = k^2 f_{r-1}(n-2,k)$. Since $A(1|n) = F(n-1,k+1) + \Gamma_{r-1}$, we have $\mathrm{per}A(1|n) = f_{r-1}(n-1,k+1)$. Thus (5) follows. We see that $\mathrm{per}(\Delta_n + \Gamma_d) = f_d(n,1)$. We now show, for each integer r with $0 \le r \le \min\{d, (n-1)/2\}$, that

$$f_d(n,1) = \sum_{k=1}^{r+1} \binom{r+1}{k}_2 f_{d-r}(n-2r+k-1,k). \tag{6}$$

Clearly the equality (6) holds for $r = 0$. Suppose that (6) holds for $r-1$. Then by induction and by the recurrence relation (5), we get

$$f_d(n,1) = \sum_{k=1}^{r} \binom{r}{k}_2 f_{d-r+1}(n-2r+k+1,k)$$

$$
\begin{aligned}
&= \sum_{k=1}^{r} \binom{r}{k}_2 [k^2 f_{d-r}(n-2r+k-1,k) \\
&\quad + f_{d-r}(n-2r+k,k+1)] \\
&= \binom{r}{1}_2 f_{d-r}(n-2r,1) + k^2 \sum_{k=2}^{r} \binom{r}{k}_2 f_{d-r}(n-2r+k-1,k) \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k}_2 f_{d-r}(n-2r+k,k+1) + \binom{r}{r}_2 f_{d-r}(n-r,r+1) \\
&= \binom{r}{1}_2 f_{d-r}(n-2r,1) \\
&\quad + \sum_{k=2}^{r} \left(k^2 \binom{r}{k}_2 + \binom{r}{k-1}_2 \right) f_{d-r}(n-2r+k-1,k) \\
&\quad + \binom{r}{r}_2 f_{d-r}(n-r,r+1) \\
&= f_{d-r}(n-2r,1) \\
&\quad + \sum_{k=2}^{r} \binom{r+1}{k}_2 f_{d-r}(n-2r+k-1,k) + f_{d-r}(n-r,r+1) \\
&= \sum_{k=1}^{r+1} \binom{r+1}{k}_2 f_{d-r}(n-2r+k-1,k),
\end{aligned}
$$

completing the proof of (6). In case that $2d \leq n-1$, plugging $r = d$ into (6), we get

$$
\begin{aligned}
f_d(n,1) &= \sum_{k=1}^{d+1} \binom{d+1}{k}_2 f_0(n-2d+k-1,k) \\
&= \sum_{k=1}^{d+1} \binom{d+1}{k}_2 k^{n-2d}(k-1)!
\end{aligned}
$$

with the aid of formula (2). If $2d = n$, then plugging $r = d-1$ into (6), we get