ONE-POINTED GRAVITATIONAL GROMOV-WITTEN INVARIANTS FOR GRASSMANNIANS

BUMSIG KIM

ABSTRACT. We write down explicitly a recursive formula of one-pointed gravitational Gromov-Witten invariants and reduce the computation of them to a combinatoric problem which is not solved yet. The one-pointed invariants were played important role in Givental's program in mirror symmetry. In section 3, we describe the combinatoric problem which can be read independently.

1. Gromov-Witten theory

Let X be a projective complex manifold, that is, a compact complex manifold which has an embedding to a projective space.

Given $d \in H_2(X, \mathbb{Z})$, let

$$M_{g,n}(X, d) = \{[(f, \Sigma_g, x_1, \cdots, x_n)] \mid \Sigma_g \text{ - a Riemann surface of genus } g$$
$$\text{with distinct marked points } x_i,$$
$$f : \Sigma_g \to X \text{ holomorphic,}$$
$$f_*[\Sigma_g] = d\}.$$

Here we identify $(f, \Sigma_g, x_1, ..., x_n) \sim (f', \Sigma'_g, x'_1, ..., x'_n)$ if there is a bi-holomorphic map h from Σ_g to Σ'_g such that $f = f' \circ h$ and $h(x_i) = x'_i$ for all i.

Kontsevich [7] introduced a compactification of the moduli space of holomorphic maps from genus g Riemann surfaces Σ_g to X. It is denoted by $\overline{M}_{g,n}(X, d)$. It has a natural scheme structure as the coarse moduli space of the corresponding moduli problem [3].

Here we describe it when $g = 0$.

$$\overline{M}_{0,n}(X,d) = \{[(f, C, x_1, \cdots, x_n)] \,|\, C \text{ - a tree of } \mathbb{C}P^1 \text{with distinct}$$
$$\text{marked smooth points } x_i, f : C \to X$$
$$\text{holomorphic, } f_*[C] = d, \text{plus a}$$
$$\text{certain stability condition}\}.$$

A tree of $\mathbb{C}P^1$ is a compact connected curve with arithmetic genus 0 with at worst nodal singularities. Nodal points and marked points are called special points. The stability condition is as follows. Whenever an irreducible component D of C maps to a point in X under f, D has at least three special points. The condition is equivalent to the ampleness of $\omega_C(x_1 + x_2 + ... + x_n)$, where ω_C is the dualizing sheaf of C. If $C \subset \mathbb{C}P^n$, then $\omega_C \cong \Omega^n_{\mathbb{C}P^n} \otimes \wedge^{n-1}\mathcal{N}_{C/\mathbb{C}P^n}$. The element of the moduli space is called a stable map. So there is only trivial infinitesimal deformation of automorphisms of a stable map. There are natural evaluation morphism ev_i from the moduli space to X, defined by $ev_i([f, C, x_1, ..., x_n]) = f(x_i)$.

According to the deformation of stable maps, one defines $[\overline{M}_{g,n}(X,d)]^{\mathrm{virt}}$ as a rational homology class of $\overline{M}_{g,n}(X,d)$ [1]. We will neither enter this highly nontrivial part, nor need it in this article.

Anyway, we mention that intersection theory on the virtual class together with pull-backed cohomology classes of X under evaluation maps is called Gromov-Witten theory.

Let $\{T_i\}$ be a basis for $H^*(X)$ and $\{T_i\}$ be a dual basis, that is, $\int_X T_i \cup T^j = \delta_{i,j}$. Let c be the first Chern class of the universal cotangent line bundle over $\overline{M}_{0,1}(X,d)$, which is defined as follows. There is a forgetful morphism π from $\overline{M}_{g,n+1}(X,d)$ to $\overline{M}_{g,n}(X,d)$, defined by $\pi([(f, C, x_1, ..., x_{n+1})]) = [(f, \mathrm{Proj} \bigoplus_{i \geq 0} \Gamma(C, (\omega_C(x_1 + ... + x_n) \otimes (f^*H)^{\otimes 3})^i)]$, where H is a very ample line bundle over X. Denote by s_i the section associated to i-th marked points. Then let $L_i = s_i^* \omega_{\overline{M}_{g,n+1}(X,d)/\overline{M}_{g,n}(X,d)}$ and $U = L_1$. The fiber of the universal line bundle U over $[(f, C, x)]$ is, by definition, T_x^*C. For simplicity assume that $H_2(X, \mathbb{Z}) = \mathbb{Z}$ by a positive generator. Let q be a formal variable. Introduce following Givental [5]

$$S^X(q) = 1 + \sum_{d=1}^{\infty} \sum_i q^d T_i \int_{[\overline{M}_{0,1}(X,d)]^{\mathrm{virt}}} \frac{1}{1-c} \wedge ev_1^*(T^i)$$

where $\frac{1}{1-c} = 1 + c + c^2 + ...$

Here the integration denotes the paring of the virtual class with the cohomology classes. By definition, each pairing is zero if the degree of homology class are not equal to the degree of the cohomology class.

It played an important role in a proof of mirror theorem [5]. In his proof, he computed it based on the localization method when X is a projective space. We apply the same method to get a recursive formula when X is a Grassmannian. We, however, failed to solve the recursive formula explicitly.

2. Grassmannians

Let $Gr(r,n)$ be the space of all r-subspaces of $\mathbb{C}^n$. It is a Grassmannian. It has a torus action induced by the standard torus $T = (\mathbb{C}^\times)^n$ action on $\mathbb{C}^n$. This induces the action on $\overline{M}_{0,k}(Gr(r,n),d)$, the space of stable maps to $Gr(r,n)$ with genus 0 and degree $d \in H_2(Gr(r,n),\mathbb{Z})$. That is, if $t \in T$ acts as $[(t \cdot f, C, x_1, ..., x_n)]$ for $[(f, C, x_1, ..., x_n)]$. The dimension of the moduli space is $\dim Gr(r,n) + nd + k - 3$. In this case, it turns out that the virtual class is exactly the fundamental class of the moduli space. And the moduli space is a compact connected complex analytic orbifold. To apply the localization theorem, we will work over the equivariant cohomology group $H_T^*(Gr(r,n), \mathbb{Q})$ of $Gr(r,n)$ under the torus action. Choose a homogeneous basis $T_0, ..., T_N$ of a free $H_T^*(pt)$-module $H_T^*(Gr(r,n), \mathbb{Q})$ and let $T^0, ..., T^N$ be the dual basis, that is $\int_{Gr(r,n)} T_i \cup T^j = \delta_{i,j}$. Identify $H_2(Gr(r,n)) = \mathbb{Z}$ by a generator represented by cycle

$$\{W \in Gr(r,n) \mid \mathbb{C}^{r-1} \times 0 \times ... \times 0 \subset W \subset \mathbb{C}^{r+1} \times 0 \times ... \times 0\}.$$

The set of all T-fixed points F are the collection of all subspaces spanned by r many vectors in the standard basis of $\mathbb{C}^n$.

Introduce a formal indeterminant h. Let

$$
\begin{aligned}
S(q,h) &= 1 + \sum_{d=1}^{\infty} q^d \sum_{i=1}^{N} T^i \int_{\overline{M}_{0,1}(Gr(r,n),d)} \frac{ev_1^*(T_i)}{h(h-c)} \\
&= 1 + \sum_{d=1}^{\infty} q^d ev_{1,*}\left(\frac{1}{h(h-c)} \right)
\end{aligned}
$$

as in the previous section, but here we use integration (=Gysin map) in the equivariant sense.

For $v \in F$, denotes by i_v the inclusion of $\{v\}$ to $Gr(r,n)$, Set

$$
\begin{aligned}
S_v(q,h) \quad &:= \quad i_v^* S(q,h) \\
&= \quad 1 + \sum_{d=1}^{\infty} q^d \int_{\overline{M}_{0,1}(Gr(r,n),d)} \frac{ev_1^*(\phi_v)}{h(h-c)} \\
&= \quad 1 + \sum_{d=1}^{\infty} q^d \frac{ev_1^*(\phi_v)c^{d-1}}{h^d(h-c)},
\end{aligned}
$$

where the latter equality is obtained by dimension counting of the moduli spaces.

Let $O(v)$ is the set of all T-fixed points of X which can be connected with v by an complex-one-dimensional orbit. If a stable map $(f, C, x_1, ..., x_n)$ represents a fixed point of the moduli space under the induced action, then the image variety of each irreducible component of C under f is just a fixed point of $Gr(r,n)$, or an one-dimensional invariant subvariety o. Since o has at least two fixed points, say v, w, in it, $T_v o$ is a one-dimensional invariant subspace of $T_v Gr(r,n)$ Since T action around v can be linearized, we easily conclude that $v \cap w$ (resp. $v \cup w$)is spanned by $r - 1$ (resp. $r + 1$) many vectors in the standard basis of $\mathbb{C}^n$ and $o = \{W | v \cap w \subset W \subset v \cup w\}$. Now apply the localization theorem and obtain the following as in [6].

$$
\begin{aligned}
Z_v(q,h) \quad &:= \quad S_v(qh,h) \\
&= \quad 1 + \sum_{d=1}^{\infty} q^d \frac{ev_1^*(\phi_v)c^{d-1}}{h-c} \\
&= \quad 1 + \sum_{d=1}^{\infty} \sum_{w \in O(v)} q^d \frac{ev_1^*(\phi_v)(-\alpha_{v,w}/d)^d}{(\alpha_{v,w} + dh)\mathrm{Euler}(N_{v,w,d})} Z_w(q, -\alpha_{v,w}/d),
\end{aligned}
$$

where $\alpha_{v,w}$ is the character of the tangent line of the one-dimensional orbit $o(v,w)$ at v, where the orbit connects an isolated fixed point v to another fixed point w and $N_{v,w,d}$ is the T-representation space $[H^0(\mathbb{P}^1, f^* TX)] - [0]$ and here f is the totally ramified d-fold map from $\mathbb{P}^1$ onto $o(v,w)$ over v and w.

3. Explicit description of the recursive formula

Consider formal variables (i.e., indeterminants) e_i, $i = 1, ..., n$. The vector space V with a basis $\{e_1, ..., e_n\}$ over $\mathbb{Q}$ is an inner product space with $\langle e_i, e_j \rangle = \delta_{i,j}$.

(These e_i denotes the character of the action $T = (\mathbb{C}^\times)^n$ on $\mathbb{C}$, $(t_1, ..., t_n) \cdot x = t_i x$ for $(t_1, ..., t_n) \in T$ and $x \in \mathbb{C}$.)

Fix an integer r such that $n - 1 \geq r \geq 1$. Consider the set F of all subsets of $\{e_1, ..., e_n\}$ which consist of r many elements. Note

$$|F| = \binom{n}{r}.$$

(F exactly corresponds the set of fixed points of $Gr(r, n)$, $v \in F$ corresponding the subspace spanned by vectors in v.)

For $v \in F$ and $\alpha \in V$, we denote $\alpha >_v 0$ if $\alpha = e_j - e_i$ and $e_i \in v$ and e_j is not in v. In this case, let $v(\alpha) = (v - e_i) \cup e_j$, which is another element in F.

We may call an element of F a vertex and α an edge if $\alpha >_v 0$ for some vertex v, where α connects v and $v(\alpha)$.

(The $\alpha = e_j - e_i$ is the character of $T_v o(v, v(\alpha))$. $\alpha >_v 0$ means that α occurs as one of the characters of $T_v Gr(r, n)$.)

Let q and h be other formal variables (as introduced in previous sections).

We would like to solve the following a system of *recurrence relations*.

For all $v \in F$, let

$$Z_v(q, h) = \sum_{d=0}^{\infty} A_{v,d}(h) q^d$$

where

$$A_{v,d}(h) \in \mathbb{Q}(e_1, ..., e_n)[[h^{-1}]]$$

and $A_{v,0}(h) = 1$.

Here $\mathbb{Q}(e_1, ..., e_n)$ is the quotient field of the polynomial ring over $\mathbb{Q}$ in $e_1, ..., e_n$ and $R[[h^{-1}]]$ is the ring of formal power series in h^{-1} over a ring R.

They satisfy the following recurrence relation.

For all $v \in F$,

$$Z_v(q, h) = 1 + \sum_{\infty > d \geq 1, \alpha >_v 0} q^d \frac{(-\alpha/d)^d}{(\alpha + dh) \prod_{\delta >_v 0} \prod_{k=1}^{\prime d < \delta, \alpha >} (\delta - k\frac{\alpha}{d})} Z_{v(\alpha)}(q, -\frac{\alpha}{d})$$

Here $\prod'$ means the product of the factors except the zero factor.

Open problem: Solve the recurrence system, i.e., find $A_{v,d}(h)$ explicitly.

We know the answer for the case $r = 1$ [5]. We explain it here. Let $r = 1$. Then

$$Z_{e_i}(q, h) = 1 + \sum_{d=1}^{\infty} q^d \frac{h^d}{\prod_{j=1}^n \prod_{k=1}^d (e_j - e_i + kh)}.$$

The proof can be seen if one considers the residue of the coefficient

$$A_{v,d}(h) = \frac{h^d}{\prod_{j=1}^n \prod_{k=1}^d (e_j - e_i + kh)}$$

at $(e_j - e_i)/d'$, $j \neq i$. First note that $A_{v,d}$ as a function in h, has simple poles. So,

$$A_{v,d}(h) = \sum_{d \geq d' \geq 1} \frac{1}{(e_j - e_i + d'h)} A_{v,d}(-(e_j - e_i)/d').$$

(Here we use the generalization of an equality, for example, if $a_1 \neq a_2$,

$$\frac{1}{(z - a_1)(z - a_2)} = \frac{1}{(z - a_1)(a_1 - a_1)} + \frac{1}{(z - a_2)(a_2 - a_1)}).$$

and

$$A_{v,d}(-(e_j - e_i)/d')$$
$$= \frac{1}{\left(\prod_{l \neq i,j} \prod_{k=1}^{d'} (e_l - e_i - k(e_j - e_i)/d')\right)}$$
$$\times \frac{(-(e_j - e_i)/d')^{d'}}{(-(e_j - e_i)/d')^{2d'-1} - (d' - 1)...(-1)d'!}$$
$$\times A_{v(\alpha),d-d'}(-(e_j - e_i)/d')$$
$$= \frac{(-(e_j - e_i)/d')^{d'}}{\left(\prod_{l \neq i,j} \prod_{k=1}^{d'} (e_1 - e_i - k(e_j - e_i)/d')\right)}$$
$$\times \frac{1}{\prod_{k=1}^{\prime 2d'} (e_j - e_i - k(e_j - e_i)/d')}$$
$$\times A_{v(\alpha),d-d'}(-(e_j - e_i)/d')$$

which is exactly the recursion relation for $r = 1$. Here $\alpha = e_j - e_i$.

4. Quantum hyperplane section principle

Let $X \subset Y \subset \mathbb{C}P^N$ and X be a zero locus of a line bundle L over X, where L is generated by holomorphic global sections. For simplicity, assume that $H_2(Y) = \mathbb{Z}$ by a positive generator and $H_2(X) \cong H_2(Y)$ by i_*, $i : X \subset Y$. We would like to compare $S^Y = \sum_d q^d S_d^Y$ and S^X, defined in section 1. To do so, introduce a correcting Euler class E_d for L. Define

$$c_1(L)_d := \prod_{i=0}^{\langle c_1(L), d \rangle} (c_1(L) + i) \in H^*(Y)$$

$$\tilde{S}^Y(q) = \sum q^d S_d^Y E_d$$

and

$$\tilde{S}^X = i_*(S^X(q)) \in H^*(Y)[[q]].$$

THEOREM. *Let* $Y = Gr(r, n)$ *be a Grassmannian, the space of* r *dimensional subspace in* $\mathbb{C}^n$, L *be a line bundle over* Y, *and let* $c_1(L) = lp$, *where* p/n *is the anticanonical class and* $l > 0$. *Then*

a) *if* $n - l > 1$, *then*

$$\tilde{S}^X(q) = \tilde{S}^Y(q),$$

b) *if* $n - l = 1$, *then*

$$\tilde{S}^X(q) = e^{-l!\binom{n-2}{r-1}q} \tilde{S}^Y(q),$$

c) *if* $n - l = 0$, *that is* X *is a Calabi-Yau manifold, then*

$$\tilde{S}^X(q) = e^{g(q)p} \frac{\tilde{S}^Y}{\phi_0}(qe^{g(q)})$$

where $\tilde{S}^Y = c_1(L)(\phi_0(q) + \phi_1(q)p +$ *higher degree terms in* $H^*(X))$ $(\phi_i(q) \in \mathbb{Q}[[q]])$ *and* $g(q)$ *is the unique formal power series of* q *satisfying* $\frac{\phi_1}{\phi_0}(qe^{g(q)}) = -g(q)$.

The theorem is proven in [6] except the term $-l!\binom{n-2}{r-1}$ in b). It can be computed directly since the term comes from degree 1 maps from $\mathbb{P}^1$ to $Gr(r, n)$.

REMARK. 1. Similar statement is true for a line bundle L such that $n - l \geq 0$ and $H^0(f^*(L)) = 0$ for all $f : \mathbb{C}P^1 \to X$. This give a proof for some local mirror symmetry phenomena.

2. Similar statement is true for a decomposable vector bundles $L_1 \oplus L_2 \oplus \ldots \oplus L_k$.

3. For recent developments in the generalization of the theorem, see
[2, 8, 4]

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Department of Mathematics
Pohang University of Science and Technology
Pohang 790-784, Korea
E-mail: bumsig@postech.edu