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EMPIRICAL REALITIES FOR A MINIMAL DESCRIPTION RISKY ASSET MODEL. THE NEED FOR FRACTAL FEATURES

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ABSTRACT. The classical Geometric Brownian motion (GBM) model for the price of a risky asset, from which the huge financial derivatives industry has developed, stipulates that the log returns are iid Gaussian. However, typical log returns data show a distribution with much higher peaks and heavier tails than the Gaussian as well as evidence of strong and persistent dependence. In this paper we describe a simple replacement for GBM, a fractal activity time Geometric Brownian motion (FATGBM) model based on fractal activity time which readily explains these observed features in the data. Consequences of the model are explained, and examples are given to illustrate how the self-similar scaling properties of the activity time check out in practice.

1. Introduction: mathematical models

Simple mathematical models of systems in science, engineering and business have been successful in providing much quantitative insight into their likely behaviour and performance. Nevertheless, the explanatory value of the simple models can be expected to have limitations, and predictions made therefrom to require qualification. It must be regarded as unusual, indeed remarkable, when a simple mathematical model has a key role in spawning a multi-billion dollar industry. But this is indeed what has happened with the geometric Brownian motion (GBM) model for the price of a risky asset. The option pricing industry was largely fueled by the success of Black and Scholes in obtaining an analytical pricing formula for the European option under the GBM model. The

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simple GBM model made this possible, and now thousands of traders and investors use the formula every day.

The ground breaking papers of Black and Scholes (1973) and Merton (1973) on option pricing were recognized by the award of the Nobel Prize for Economics to Scholes and Merton in 1997. Black, who died in 1995, would undoubtedly have shared in the award had he lived long enough.

The year 1973 also saw the introduction of exchange traded options and stock, exchange rate, commodity and futures options, and other financial derivatives, are now traded all over the world. The market for these has now grown to the extent that its worth is regularly reported as more than $US20 trillion.

When used in relation to financial instruments, options are generally understood as contracts between two parties which give one party the right, but not the obligation, to buy or sell an underlying asset at some time in the future. The concept is certainly not new. For example, the ancient Phonecians, Romans and Greeks traded options against outgoing cargos from seaports. What is new is the very widespread use of these instruments for the offsetting of risk.

The paradigm (GBM) model in mathematical finance which has facilitated all this activity is our natural point of reference. Under this model the price P_t at time t of a risky asset is

$$P_t = P_0 \exp[\mu t + \sigma W(t)]$$

where P_0 is the price at time 0, $\mu, \sigma^2 > 0$ are fixed constants and $W(t)$ is a standard Brownian motion. Then the corresponding log returns

$$X_t = \log P_t - \log P_{t-1} = \mu + \sigma(W(t) - W(t-1))$$

are iid Gaussian with mean μ and variance σ^2. Few models could be simpler.

In contrast, the typical log returns data shows:

- a pronounced leptokurtic distribution (much higher peaks and heavier tails than Gaussian),
- a time series with high volatility and (often) intermittency, quite unlike Gaussian white noise,
- evidence of strong dependence.

In Section 2 we outline the statistical evidence against the GBM model, treating each of the abovementioned features in turn. In Section 3 we explain the fractal activity time geometric Brownian motion (FAT-GBM) model introduced by Heyde (1999) and indicate how this resolves the statistical problems experienced by the GBM model. Empirical evidence of support for the fractal scaling of the model is also supplied. In Section 4 it is indicated how the FATBGM model can be used for option pricing, and a direct comparison with the corresponding results of Black and Scholes is given.

2. The basic evidence

2.1. Tails

It is now generally accepted that leptokurtosis with heavier tails than the normal is a necessary model feature for marginal distributions of returns. Arguments continue over how heavy they are and certainly hyperbolic distributions fit the central zone of the distribution well. But the extreme observations, say 1% of the data, which typically go out 6 or more standard deviations and are the features of most interest and concern, require tails which are asymptotically of Pareto type, namely of the form

$$P(|X_t| > x) \sim cx^{-\alpha}, \quad c > 0, \quad \alpha > 0. \tag{1}$$

as $x \to \infty$. Indeed, there is considerable support for the use of a t_ν-distribution for X_t, ν being the degrees of freedom, and typically $3 < \nu < 6$. Support for this form dates back at least to Praetz (1972) and recent comparisons using market index data have been given in Hurst, Platen and Rachev (1997) and Hurst and Platen (1997). If X_t has a t_ν-distribution, then (1) holds with $\alpha = \nu$ and X_t has an infinite k-th moment for $k \geq \nu$.

Figure 1 contains histograms for the density of X_t for representative examples of index (Standard & Poors 500), foreign exchange ($US/ Deutchmark) and common stock (Chevron) data. Here S&P contains 8813 daily observations (Nov. 64 to Nov. 99), USD/DEM contains 5764 daily observations (Jan. 71 to Dec. 93) and Chevron contains 7561 daily observations (Jan. 70 to Dec. 99). Estimates of ν for the data in Figure 1 are all between 3 and 4.

For a general discussion of the issue of tail size see Chapter IV.2 of Shiryaev (1999).

2.2. Dependence

The next issue for modelling is that of strong dependence in the data. Log returns time series are very different from white noise. They exhibit more volatility and sometimes (especially for index data) a clustering of

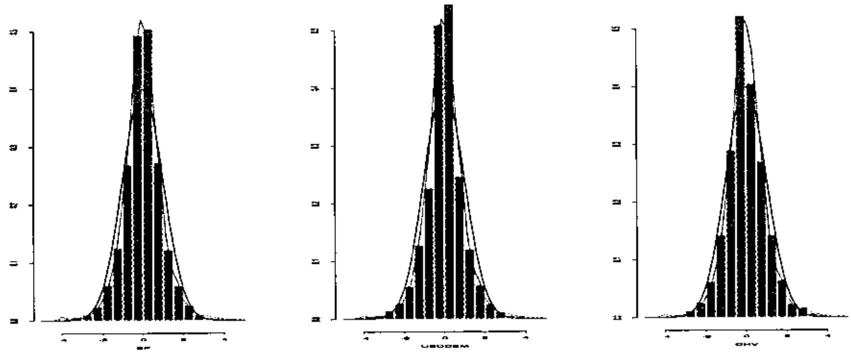

FIGURE 1. Histograms for S&P, USD/DEM and Chevron (left to right), with corresponding $N(0,1)$ (lower curve) and t_4 density (scaled to have variance 1) (upper curve) superimposed

extremes (intermittency). Sample autocorrelations of the log returns always die away quickly but this is not the case for their absolute values or squares which have non-negligible values for very large lags. This phenomenon has been noted by various authors. See for example Greene and Fielitz (1979), Ding and Granger (1996), Granger and Ding (1996). From these persistent correlations we must conclude that the absolute values and squares of the return times exhibit long range dependence (LRD), classically defined (for stationary finite variance processes) as holding if $\sum_{k=0}^{\infty} \gamma_k$ diverges, γ_k being the autocorrelation at lag k. For a discussion of the definition and its extensions see Heyde and Yang (1997). For a comprehensive overview of the subject of LRD see Beran (1994).

Figures 2-4 contain plots of the autocorrelations, over 200 day periods, of the log returns, their squares, and their absolute values, again for the representative examples S&P, USD/DEM and Chevron.

3. Modelling to retain the essential features of the GBM model

The objective is a minimal description model, the simplest practicable extension of the GBM model which has the necessary features outlined above in Section 2 and is also sufficiently tractable to enable it to be used for such things as the pricing of options. This can all be achieved

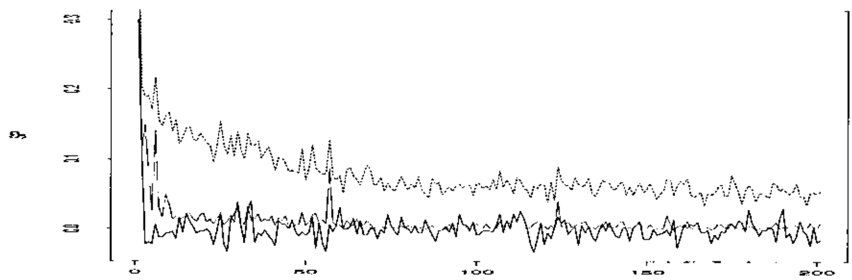

FIGURE 2. Autocorrelation function of the returns, their squares and their absolute values (low to high) for S&P.

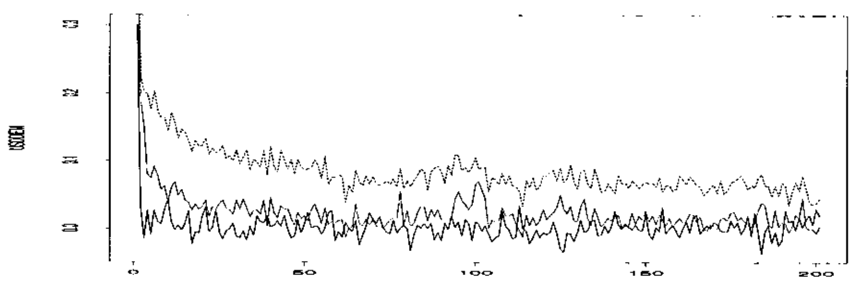

FIGURE 3. Autocorrelation function of the returns, their squares and their absolute values (low to high) for USD/DEM.

successfully with the fractal activity time geometric Brownian motion (FATGBM) model introduced by Heyde (1999), as we will now indicate.

3.1. The FATGBM subordinator model

We retain the GBM assumption that the $\{\log P_t\}$ process should have stationary differences, but now with heavy tails and the absolute values and squares of the differences should exhibit LRD. To achieve this, we suppose that

$$P_t = \exp[\mu t + \sigma W(T_t)]$$

where the (market) activity time $\{T_t\}$ is a positive increasing random process with stationary differences which is independent of the Brownian motion $\{W(t)\}$ and the differences of the $\{T_t\}$ process are LRD and have

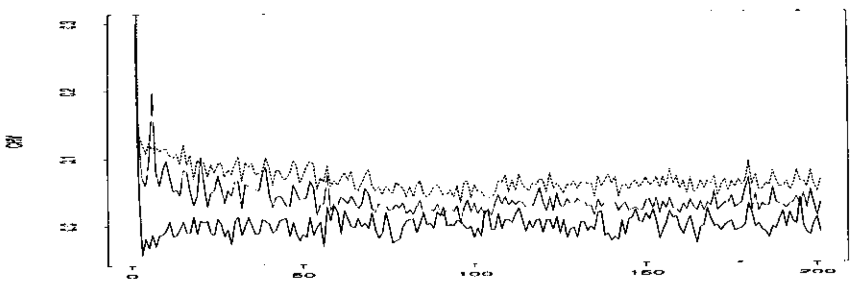

FIGURE 4. Autocorrelation function of the returns, their squares and their absolute values (low to high) for Chevron.

heavy tails. Clock time just does not correspond to activity time (also called trading time or intrinsic market time).

The activity time process $\{T_t\}$ is assumed to have a finite mean. Then, by the ergodic theorem, and in view of its stationary differences, $T_t \sim ct$ almost surely as $t \to \infty$, for some $c > 0$, and without loss of generality we may choose $c = 1$, the scaling being absorbed into the σ.

One of the important things about the model is that $\{W(T_t), \mathcal{F}(W(s), s \leq T_t)\}$ is a martingale. So the efficient market hypothesis is being retained and there are no arbitrage opportunities.

We now show that the FATGBM model has all the features suggested by the data. We have

$$\begin{aligned} X_t &= \log P_t - \log P_{t-1} \\ &= \mu + \sigma(W(T_t) - W(T_{t-1})) \\ &\stackrel{\mathrm{d}}{=} \mu + \sigma(T_t - T_{t-1})^{1/2} W(1), \end{aligned}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution and the heavy tails of the X_t come from those of $\tau_t = T_t - T_{t-1}$. Also, for $k = 1, 2, \ldots$ and centered variables,

$$\begin{aligned} \mathrm{cov}(X_t, X_{t+k}) &= 0 \\ \mathrm{cov}(|X_t|, |X_{t+k}|) &= (2/\pi)\sigma^2 \mathrm{cov}({\tau_t}^{1/2}, {\tau_{t+k}}^{1/2}) \\ \mathrm{cov}({X_t}^2, {X_{t+k}}^2) &= \sigma^4 \mathrm{cov}(\tau_t, \tau_{t+k}), \end{aligned}$$

the last if $E{\tau_t}^2 < \infty$. Thus, LRD of $\{|X_t|\}$ and $\{{X_t}^2\}$ follows from that of $\{{\tau_t}^{1/2}\}$ and $\{\tau_t\}$ respectively.

Other features of the model are conditional heteroscedasity,

$$\mathrm{var}(X_t|\mathcal{F}_{t-1}) = \sigma^2 E(\tau_t|\mathcal{F}_{t-1}),$$

and leptokurtosis

$$\mathrm{Kurtosis}(X_t) = 3(1 + \mathrm{var}\tau_t) > 3.$$

The model provides a coherent formulation for all time scales in contrast to the discrete time ARCH, GARCH type models which may not rescale well.

3.2. Fractal scaling of the activity time process

We now consider the activity time process in more detail, noting that it should exhibit LRD.

Although the activity time process $\{T_t\}$ is not observed directly it can be empirically constructed. Note that the use of Ito's formula gives

$$d \log P_t - \frac{dP_t}{P_t} = -\frac{1}{2}\sigma^2 dT_t$$

from which the increments of the $\{T_t\}$ process may be obtained in a discretized approximation. This allows the process to be checked for scaling properties and a remarkable fact emerges. To a good degree of first approximation, the process $\{T_t - t\}$ is self-similar with index $H, \frac{1}{2} < H < 1$. That is, for positive c,

$$T_{ct} - ct \stackrel{\mathrm{d}}{=} c^H(T_t - t),$$

where here $\stackrel{\mathrm{d}}{=}$ denotes equality of finite dimensional distributions. The desired LRD properties then follow as a simple consequence.

The approximate self-similarity can easily be checked via estimation of H over a wide range of time scales. Suppose that the price P_t is observed over a fixed time interval $[0, \mathcal{T}]$ and let

$$S_q(\delta) = \sum_{i=1}^{N} |T_{i\delta} - T_{(i-1)\delta} - \delta|^q / N$$

where $\mathcal{T} = N\delta$ and $0 < q \leq p$ with $ET_1{}^p < \infty$. If $\{T_t - t\}$ is self-similar, then

$$\log ES_q(\delta) = Hq \log \delta + \log E|T_1 - 1|^q$$

for $0 < q \leq p$. Now the ergodic theorem ensures that $S_q(\delta) \stackrel{\text{a.s.}}{\to} ES_q(\delta)$ as $N \to \infty$, and one can estimate H separately at each of a broad range of δ values. A high degree of consistency of these estimates is observed in practice.

Figures 5-7 contain plots of estimated values of $(\log ES_q(\delta)-\log E|T_1-1|^q)/\log\delta$ against q for values $\delta = 5, 10, ..., 1280$ and for q in the allowable range where T_t has a finite qth moment for examples of index, foreign exchange and common stock data. The estimates of H are very stable over a period from 5 days to 5 years. The values are all $H = 0.78$ for S&P (Figure 5), $H = 0.75$ for USD/DEM (Figure 6) and $H = 0.76$ for Chevron (Figure 7).

Additional support for the claim of approximate self-similarity can be obtained from an analysis of the scaling properties of the process $\{\sum_{j=1}^t {X_j}^2\}$. This is so because $\{T_t\}$ is the quadratic variation process associated with the $\{\log P_t\}$ process, and hence if

$$X_{\delta t} = \log P_{\delta t} - \log P_{\delta(t-1)},$$

then

$$T_t = \lim_{\delta\to 0+} \sum_{j=1}^{[t/\delta]} X_{\delta j}^2 \text{ a.s.}$$

Thus, $\{\sum_{j=1}^t {X_j}^2\}$ approximates $\{T_t\}$ and its scaling can be conveniently checked using wavelets. In Figure 8 one sample plot is given indicating a straight line plot over 8 octaves as is expected for a self-similar model. The estimated value of 2.2 for the scaling parameter α clearly indicates underlying long range dependence. Here we have used software based mainly on LDestimate.m written in Matlab for the logscale diagram, a wavelet analysis framework for time series. For an introduction and uses, see e.g. Abry et al. (2000) and Veitch and Abry (1999). More details concerning scaling will be provided in a forthcoming paper of Heyde and Liu (2001).

The FATGBM model is distinctively different from earlier subordinator models (which date back to Mandelbrot and Taylor (1967); see Heyde (1999) for more details) in its recognition of self-similar scaling and the associated long range dependence.

Subordinator models are also closely related to stochastic volatility models in which a stochastic differential equation is hypothesized for the volatility process $\{\sigma(t)\}$ which would be related to the activity time process $\{T_t\}$ herein by $\sigma^2 T_t = \int_0^t \sigma^2(u)du$. The FATGBM model circumvents such assumptions by using the observed self-similarity.

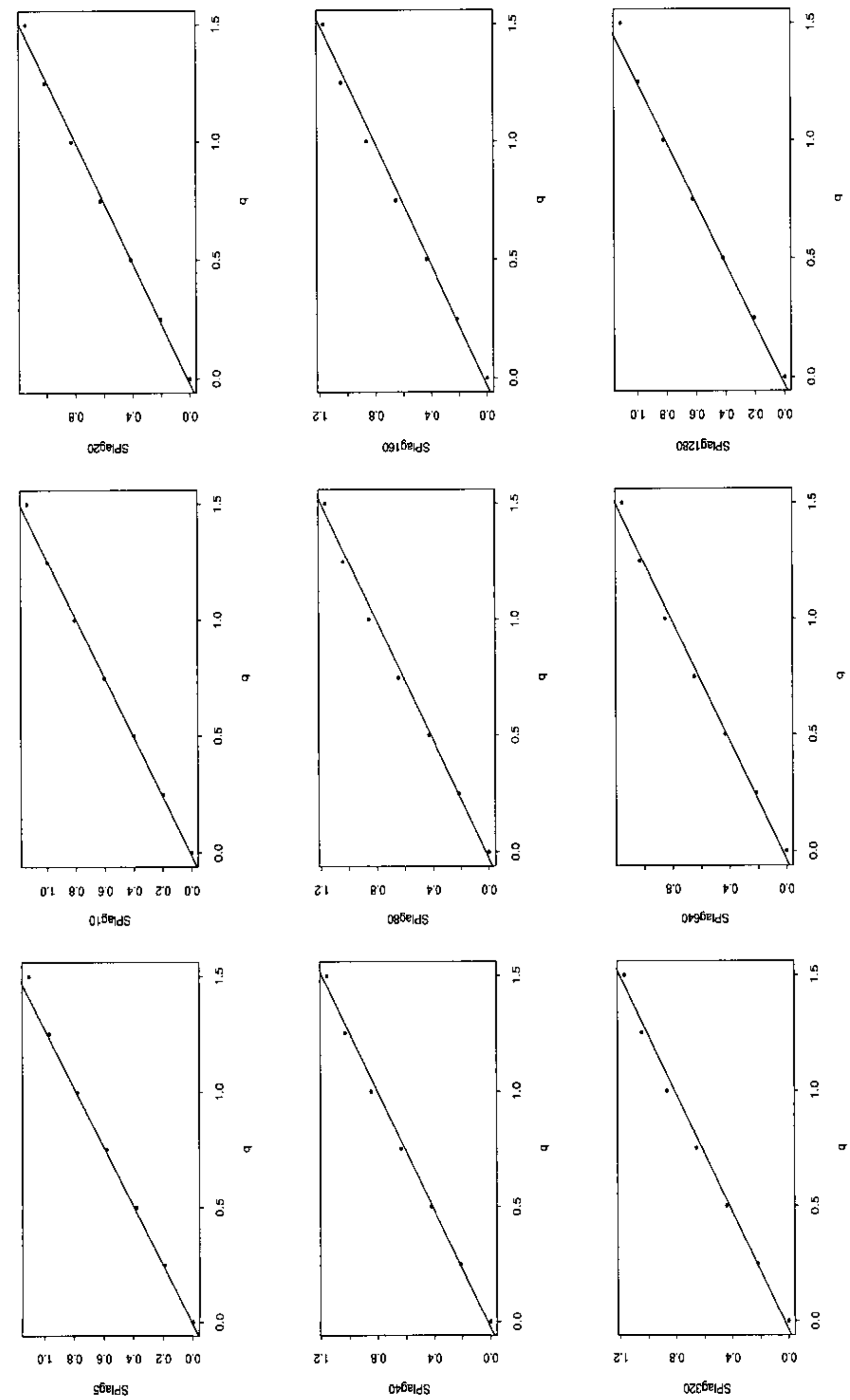

FIGURE 5. Scaling plots for S&P. Periods from 5 to 1280 days. Slope estimates for all nine plots $H = 0.78$.

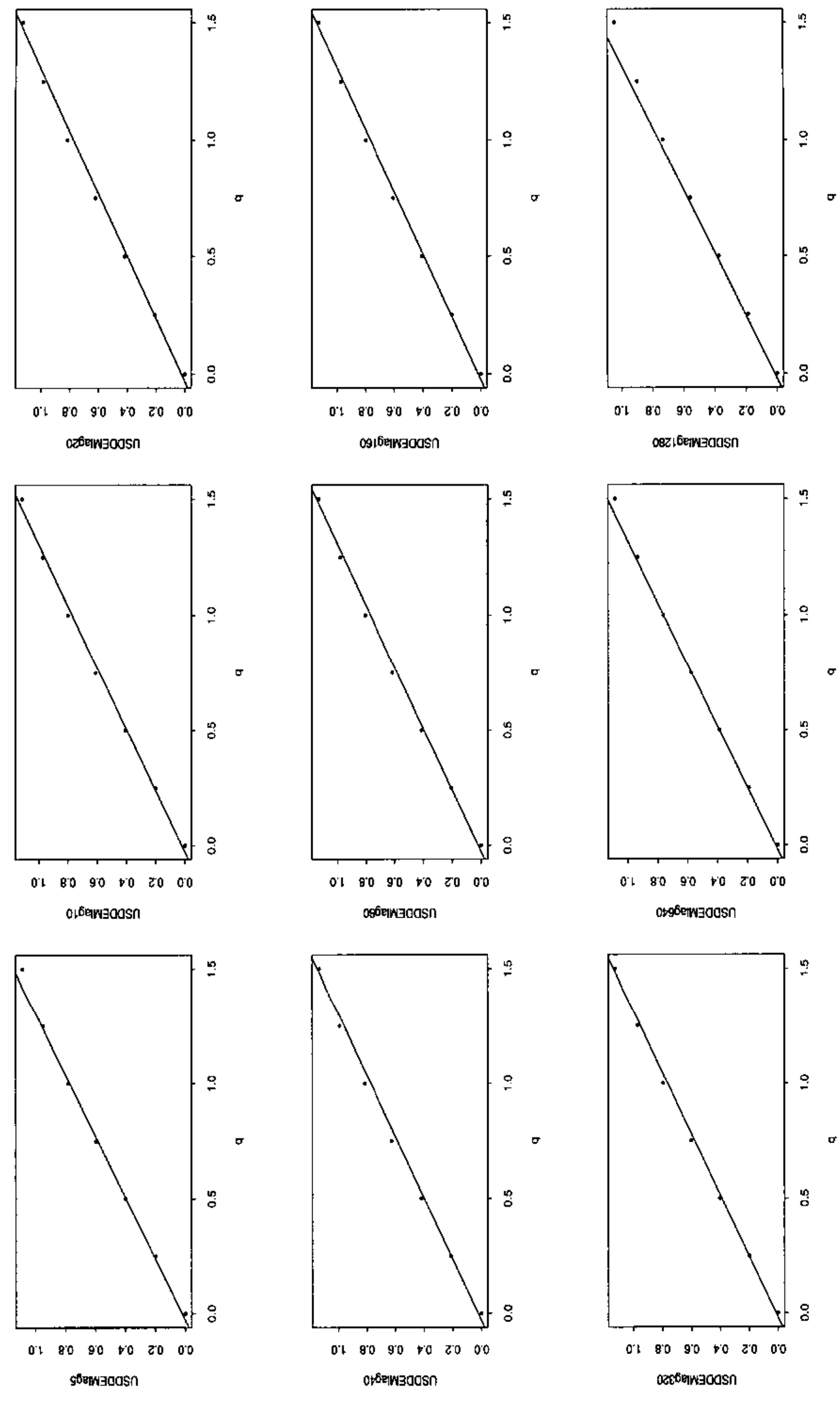

FIGURE 6. Scaling plots for USD/DEM. Periods from 5 to 1280 days. Slope estimates for all nine plots $H = 0.75$.

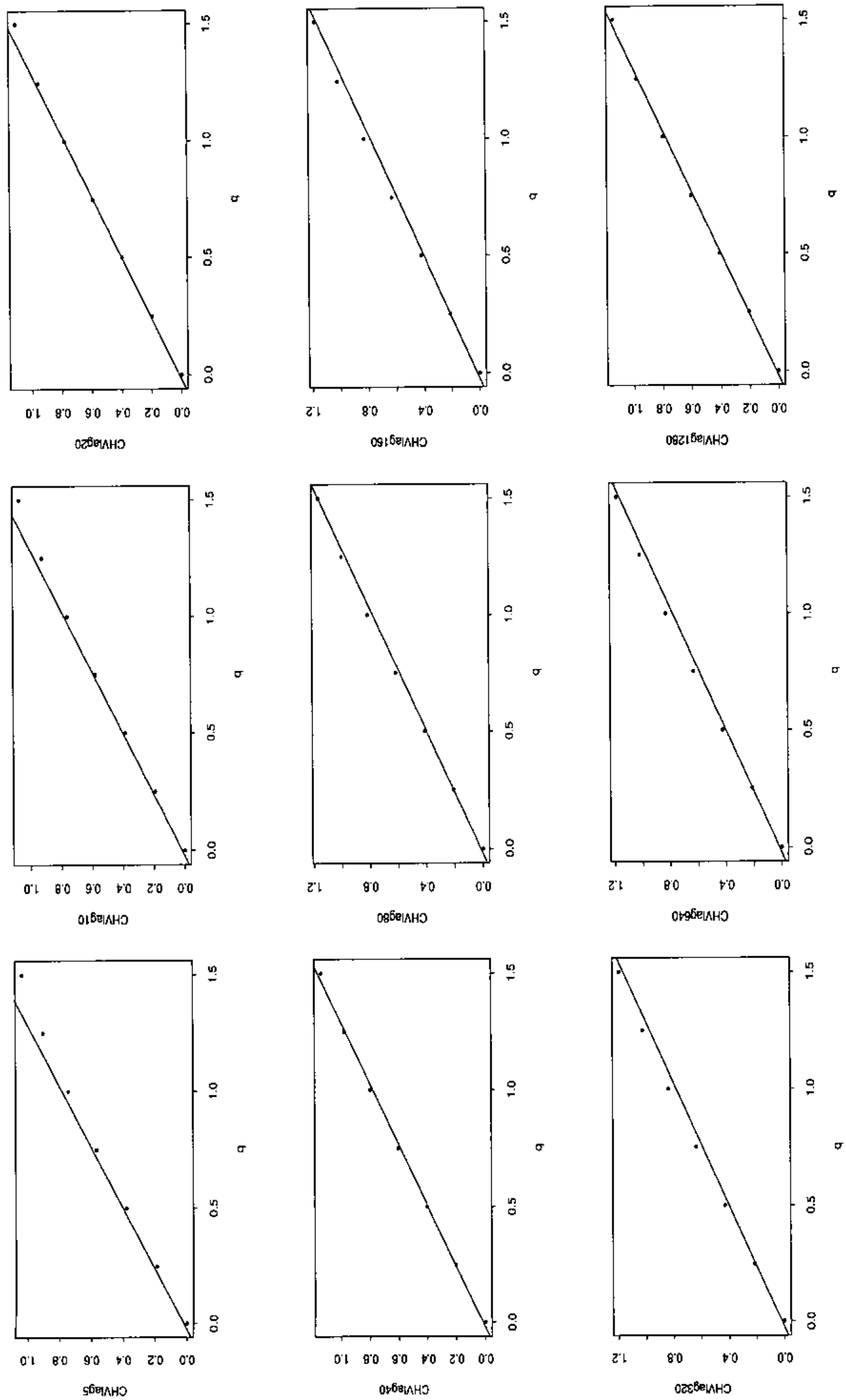

FIGURE 7. Scaling plots fro Chevron. Periods from 5 to 1280days. Slope estimates for all nine plots $H = 0.76$.