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TRAVEL TIME TOMOGRAPHY

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ABSTRACT. We survey recent results on the inverse kinematic problem arising in geophysics. The question is whether one can determine the sound speed (index of refraction) of a medium by measuring the travel times of the corresponding ray paths. We emphasize the anisotropic case.

1. The inverse kinematic problem

This inverse problem arose in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. An early success of this inverse method was the estimate by Herglotz [12] and Wiechert and Zoeppritz [26] of the diameter of the Earth and the location of the mantle, crust and core. The assumption used in those papers is that the index of refraction (speed of waves) depends only on the radius. A more realistic model is to assume that the it depends on position. The inverse kinematics problem can be formulated mathematically as determining, on a bounded domain (the Earth), a Riemannian metric given by $ds^2 = \frac{1}{c^2(x)}dx^2$ where c is a positive function from the length of geodesics (travel times) joining points in the boundary.

More recently it has been realized, by measuring the travel times of seismic waves, that the inner core of the Earth might exhibit anisotropic behavior, that is the speed of waves depends also on direction there with the fast direction parallel to the Earth's spin axis. In [1] it is given this explanation for the different time residuals of rays turning in the inner core. Given the complications presented by modeling the Earth as an anisotropic elastic medium we consider a simpler model of anisotropy,

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namely that the wave speed is given by a symmetric, positive definite matrix $g = (g_{ij})(x)$, that is, a Riemannian metric in mathematical terms. The problem is to determine the metric from the lengths of geodesics joining points in the boundary (the surface of the Earth in the motivating example).

This leads to the general question of whether given a compact Riemannian manifold with boundary one can determine the Riemannian metric in the interior knowing the lengths of geodesics joining points on the boundary. This is a problem that also appears naturally in rigidity questions in Riemannian geometry. We proceed to formulate in geometric terms in the next section the travel time tomography problem which is known in differential geometry as the boundary rigidity problem.

In section 2 we formulate the boundary rigidity problem for Riemannian manifolds with boundary as well as the related problems of conjugacy rigidity for the same manifolds, and the question of determining a closed manifolds from its length spectrum, i.e. the length of closed geodesics. In section 3 we consider the linearized problem which leads to the question of invertability of the ray transform, a problem in integral geometry. The question of spectral rigidity is also reviewed. In section 4 we describe a recent semiglobal result [13] on boundary rigidity for domains in Euclidean space.

2. The boundary rigidity problem

Ridigity problems in differential geometry can be briefly formulated as follows: to what extent is the local geometry of a Riemannian manifold determined if some global properties are known? In particular, are two manifolds isometric under the assumption that the corresponding global properties are the same? In the last case the manifold is said to be rigid with respect to the corresponding global property. The boundary rigidity problem can be stated as to what extent is a Riemannian metric on a compact manifold with boundary determined from the distances between boundary points. We give below a more precise mathematical formulation. Let (M, g) be a compact Riemannian manifold with boundary ∂M, and g' be another Riemannian metric on M. We say that g and g' have the same *boundary distance-function* if $d_g(x,y) = d_{g'}(x,y)$ for arbitrary boundary points $x, y \in \partial M$, where d_g (resp. $d_{g'}$) represents distance in M with respect to g (resp. g'). It is easy to give examples of pairs of metrics with the same boundary distance-function. Namely, if $\varphi : M \to M$ is an arbitrary diffeomorphism of M onto itself which is the identity on the boundary, then the metrics g and $g' = \varphi^* g$ have the same boundary distance-function. We say that a compact Riemannian manifold is **boundary rigid** if this is the only type of nonuniqueness. Many examples can be given of manifolds that are not boundary rigid. For instance, if an inner point $x_0 \in M \setminus \partial M$ is such that $\text{dist}\,(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x,y)$, then we can change the metric g in a neighborhood of x_0 without changing the boundary distance-function. These examples show that the boundary rigidity problem should be considered under some restrictions on the geometry of geodesics. The most usual of such restrictions is simplicity of the metric. A Riemannian manifold (M, g) (or the metric g) is called **simple** if the boundary ∂M is strictly convex and any two points $x, y \in M$ are joined by a unique geodesic. The natural conjecture is that every simple manifold is boundary rigid. The problem in this generality was proposed by Michel [14]. Simple Riemannian manifolds with boundary are boundary rigid under some rather restrictive assumptions on the curvature [11], [14], [2]. In [3], [19] is proved that negatively curved surfaces with boundary are boundary rigid.

Important progress has been made in the case that the metrics are conformal to the Euclidean case, that is the metric is of the form $ds^2 = \frac{1}{c^2} dx^2$ with c a positive function, the original problem arising in geophysics. The first general result is due to Mukhometov [16] for two dimensional geodesically convex domains with boundary who proved unique determination of the sound speed and stability estimates. This was generalized to higher dimensions in [18]. The method of [16] is very original, it depends on a form of an energy inequality. For geodesically convex domains in Euclidean space this was generalized in [17] for metrics in the same conformal class. A more general form of this result is due to Croke [2] who gives a beautiful geometric proof that two conformal metrics, on a strongly geodesic minimizing manifold (SGM), with the same boundary distance function are in fact the same. Croke proof doesn't give stability estimates as does the method of [16]. SGM means, roughly speaking, that every geodesic segment whose interior lies in the interior of the manifold is the unique minimizing path between its endpoints. It is a weaker condition than geodesically convex.

Recent progress has been made on the local problem, that is one assumes a-priori that two metrics with the same boundary distance function are close to each other, in some appropriate sense. The problem is whether the corresponding manifolds are isometric. In section 4 we review the result of [25] on this problem as well as a recent semiglobal result [24] where is assumed that one metric is close to the Euclidean metric and the other just satisfied some a-priori bounds on the sectional curvatures.

Boundary rigidity is closely related to **conjugacy rigidity** that is defined as follows. Two compact Riemannian manifolds M and N (possibly with boundary) have conjugate geodesic flows if there is a homeomorphism F between the unit tangent bundles S^1M and S^1N such that $F \circ g_M^t = g_N^t \circ F$, where g^t denotes the geodesic flow. M is called **conjugacy rigid** if any Riemannian manifold N with geodesic flow conjugate to the geodesic flow of M is isometric to M. Boundary rigidity implies conjugacy rigidity [2]. On the other hand, there are examples of manifolds that are not conjugacy rigid.

The boundary rigidity problem has the following analog for manifolds without boundary: to what extent is a Riemannian metric on a closed manifold determined by the **length spectrum**, i.e., by the set of lengths of closed geodesics? The problem makes sense only for metrics that have sufficiently many closed geodesics. Therefore the problem is usually considered for negatively curved metrics. In particular, if two metrics of negative curvature on a compact manifold have the same marked length spectrum (i.e., in each free homotopy class the two closed geodesics representing that class, one in each metric, have the same length) then they have C^0-conjugate geodesic flows. It is a conjecture that they must be isometric. The following result is proved in this direction [8]: If M and N are nonpositively curved manifolds of rank more than 1 and their geodesic flows are C^0-conjugate, then M and N are isometric.

3. The ray transform

We consider (M, g) a Riemannian manifold with boundary. In this section we study the linearization of the map

$$g \longrightarrow d_g \tag{3.1}$$

in the direction of a $C_0^\infty(\Omega)$-tensor field $f_{ij}, i, j = 1, ..., n$.

As we will point out also in section 4 it is useful to consider, instead of geodesics, bicharacteristic curves in the cotangent space of the manifold which project to geodesics. Let $h_g = \sum_{i,j=1}^n g^{ij}\xi_i\xi_j$ be the principal symbol of the Laplace-Beltrami operator where (g^{ij}) denotes the inverse of the metric g. We consider the Hamiltonian vector field H_{h_g} and we denote by $(x(t), \xi(t))$ the integral curves of H_{h_g} on the energy level

$h_g = 1$. We are going to use the following parameterization of those integral curves. Let us denote

$$ST^*\partial\Omega_- := \{(z,\omega) \in ST^*\partial\Omega;\ z \in \partial\Omega,\ \omega \in S^{n-1},\ g^{-1}\omega \cdot \nu(z) \le 0\}.$$

where $\nu(z)$ is the outer unit normal to $\partial\Omega$. Let us introduce the measure $d\mu(z,\omega) = g^{-1}\omega \cdot \nu(z) dS_z d\omega$ on $ST^*\partial\Omega_-$, where dS_z and $d\omega$ are the surface measures on $\partial\Omega$ and S^{n-1}, respectively. Then $(x(t),\xi(t)) = (x(t;z,\omega),\xi(t;z,\omega))$ is defined as the integral curve of H_{h_g} issued from $(z,\omega) \in ST^*\partial\Omega_-$.

Now we define the geodesic X-ray transform or **ray transform** by

$$I_g(f)(z,\theta) = \int_\gamma \sum_{i,j=1}^{n} f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt, \tag{3.2}$$

where $\gamma(t) = \gamma(t;z,\theta)$ is the geodesic issued from, $(z,\theta) \in ST\partial\Omega_-$ parameterized by its arc-length. Here $ST\partial\Omega_-$ consists of all unit (with respect to the metric) vectors on the boundary pointing inside Ω. Since the tangent vector to the geodesic is related to the covector or ξ by the formula $g\dot\gamma = \xi$, we get

$$I_g(f)(z,\theta) = \int_\gamma \sum_{i=1,j}^{n} m^{ij}(\gamma(t))\xi_i(t)\xi_j(t)\,dt \tag{3.3}$$

where $m = g^{-1}fg^{-1}$ or, in coordinates $m^{ij} = \sum_{i',j'} g^{ii'} f_{i'j'} g^{j'j}$. It is easy to see that if

$$d_{g+tf} = 0 \quad \forall t$$

then

$$I_g(f) = 0. \tag{3.4}$$

Of course the transform $f \to I_g(f)$ is not injective because the distance function is invariant under change of variables which are the identity at the boundary. For the linearized problem this corresponds to $I_g(dv) = 0$ for any tensor field v satisfying $v|_\Gamma = 0$. Here dv denotes the symmetric covariant derivative of v.

By Theorem 3.3.2 of [21] we can uniquely decompose the tensor f_{ij} into its solenoidal and potential parts, i.e.

$$f = f^s + dv, \quad v|_\Gamma = 0. \tag{3.5}$$

The natural conjecture is that

$$I_g(f^s) = 0 \Longrightarrow f^s = 0. \tag{3.6}$$

We denote by $(S^2\tau'_M)$ the space of tensor fields and by $Z(S^2\tau'_M)$ the kernel of the ray transform. Let $P(S^2\tau'_M) \subset C^\infty(S^2\tau'_M)$ be the space of all potential fields. For such a field, the integrand in 3.2 is the total derivative with respect to t. We thus have the inclusion

$$P(S^2\tau'_M) \subset Z(S^2\tau'_M). \tag{3.7}$$

The conjecture, which is a fundamental one in integral geometry, can be stated as follows: for which classes of Riemannian manifolds is the inclusion 3.7 equality?

In this case the manifold (M,g) is called a **deformation boundary rigid manifold**. As we have indicated the deformation boundary rigidity problem is a linearization of the boundary rigidity problem. See Chapter 1 of [21] for a detailed discussion of the relationship between these two problems. The right-hand side of 3.7 can be considered as the tangent space, at the point g, to the manifold of metrics on M with the same boundary distance function as g; while the left-hand side of (3.7) is the tangent space, at the point g, to the manifold of metrics that are isometric to g via an isometry which is the identity on the boundary. We recall some known results regarding the deformation boundary rigidity problem. Equality in (3.7) was first proved [20] for a compact Riemannian manifold of nonpositive sectional curvature with strictly convex boundary. Then this result was generalized [22] by replacing the condition of nonpositivity of the curvature with some weaker generic curvature condition. In particular, this condition is satisfied for every sufficiently small convex piece of an arbitrary Riemannian manifold. For a simple Riemannian manifold, it is known [23] that the inclusion 3.7 has a finite codimension. Before stating the recent result of [24] we recall a definition.

We say that a Riemannian manifold (M,g) *has no focal points* if, for every geodesic $\gamma : [a,b] \to M$ and every nonzero Jacobi field $Y(t)$ along γ satisfying the initial condition $Y(a) = 0$, the modulus $|Y(t)|$ is a strictly increasing function on $[a,b]$, i.e., $d|Y(t)|^2/dt > 0$ for $t \in [a,b]$.

In [24] it was solved the deformation boundary rigidity problem for Riemannian surfaces with no focal points and with strictly convex boundary. More precisely we have:

THEOREM 3.1. *A compact two-dimensional simply connected Riemannian manifold with strictly convex boundary and with no focal points is deformation boundary rigid.*

If a Riemannian manifold has no focal points, then it has no conjugate points. This implies that the manifolds satisfying the hypotheses of the Theorem are simple and, in particular, are diffeomorphic to the disk B^2.

Now we consider the corresponding question for closed Riemannian manifolds, i.e., compact Riemannian manifolds without boundary. For a symmetric tensor field $f \in C^\infty(S^2\tau'_M)$ on a closed Riemannian manifold (M, g), we can consider If over closed geodesics γ. As before let $Z(S^2\tau'_M)$ be the space of all fields f such that $If(\gamma) = 0$ for every closed geodesic γ. By $P(S^2\tau'_M)$ we denote the range of the corresponding ray transform for this case . The elements of $P(S^2\tau'_M)$ are also called potential fields. The inclusion holds for every closed Riemannian manifold. The natural question is: for what classes of closed Riemannian manifolds does (3.7) holds in this setting? For an Anosov manifold of nonpositive sectional curvature, the inclusion is equality. This fact is proved in [6] for negatively curved manifolds, but it is only the nonpositivity of the curvature and Anosov type of the geodesic flow that are used in the proof. Without constrains on the curvature, it is proved [9] that the inclusion has a finite codimension for an Anosov manifold.

We recall that a Riemannian manifold is an *Anosov manifold* if its geodesic flow is of Anosov type. For such a manifold, the set of closed geodesics is dense in the set of all geodesics. The class of Anosov manifolds contains the class of closed negatively curved manifolds.

The deformation rigidity problem is closely related to the **isospectral** problem. Let g^τ be a family of metrics depending on a parameter such that $g^0 = g$. Such a family is called an *isospectral deformation* if the spectrum of the Laplace — Beltrami operator Δ^τ of the metric g^τ is independent of τ. A deformation g^τ is called a *trivial deformation* if there exists a family φ^τ of diffeomorphisms of M such that $g^\tau = (\varphi^\tau)^* g$. A manifold (M, g) is called **spectrally rigid** if it does not admit a nontrivial isospectral deformation.

The following result is proved in [10]: an Anosov manifold is spectrally rigid if the inclusion (3.7) is equality. The statement is formulated in [10] for negatively curved manifolds. However, the same proof applies to Anosov manifolds. In view of this result, the left-hand side of can be considered as the space of trivial infinitesimal deformations, while the right-hand side is the space of infinitesimal isospectral deformations. In particular, under the hypotheses of Theorem 3.1, (M, g) is a spectrally rigid surface. More precisely we have

THEOREM 3.2. *Two-dimensional Anosov surfaces with no focal points are spectrally rigid.*

4. Semiglobal rigidity

The following result concerning the boundary rigidity problem is proven in [24].

THEOREM 4.1. *Let $D \subset \mathbf{R}^n$ be a closed bounded domain with smooth strictly convex (with respect to the Euclidean metric) boundary ∂D. Let g_1, g_2 be Riemannian metrics on D satisfying $d_{g_1} = d_{g_2}$. Then under appropriate bounds for the sectional curvatures of the metric g_1, and assuming that the metric g_2 is sufficiently close to the Euclidean metric we have that g_1 and g_2 are isometric via an isometry that is the identity on the boundary.*

This result generalizes the local result, proven in [25] where both metrics are assumed to be sufficiently close to the euclidean metric.

The proof of 4.1 relies on a new identity for the difference of the metrics which was derived in [25]. The linearized version of the identity at the euclidean metric gives, roughly speaking, that the integrals along the geodesics (lines in the linear case) of the difference of the two metrics is zero. Then one concludes that the metrics are the same in those coordinates by inverting the X-ray transform. In section 3 of [25] the local result was proven by using the identity and a perturbation argument that leads to the inversion of a Fourier integral operator.

The main identity

We give complete details of the identity proved in [25].

It is known that if we have two metrics g_1, g_2 with the same boundary function then the Taylor series of the metrics, in appropriate coordinates, coincides at the boundary [13], [15]. Thus, without loss of generality we assume that g_1 and g_2 can be extended outside Ω as e and the so extended metrics belong to $C^\infty(\mathbf{R}^n)$. From now on we will denote by g_1 and g_2 the extended metrics.

Let $x^{(0)} \in \Gamma, \xi^{(0)} \in S^2$ such that $\nu(x^{(0)}) \cdot g^{-1}\xi^{(0)} < 0$. The integral curves of $H_{h_{g_j}}, j = 1, 2$ tangent to the energy $h_{g_j} = 1$ are denoted by

$(x_{g_j}, \xi_{g_j}), j = 1, 2$. They solve the Hamiltonian system

(4.1)

$$\begin{cases} \frac{d}{ds}x_m &= \sum_{j=1}^{n} g^{mj}\xi_j, \quad \frac{d}{ds}\xi_m &= -\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial g^{ij}}{\partial x_m}\xi_i\xi_j, \quad m = 1, ..., n, \\ x|_{s=0} &= x^{(0)}, \quad \xi|_{s=0} &= \xi^{(0)}. \end{cases}$$

Here g is either g_1 or g_2, while the initial conditions are the same for both metrics. We remark that if $\xi^{(0)} \cdot g^{-1}\xi^{(0)} = 1$, then s is the arc-length in (4.1). The assumption that the boundary distance function for the two metrics is the same implies the following property.

LEMMA 4.1. *Let g_1, g_2 be two Riemannian metrics in $\bar{\Omega}$ with $\bar{\Omega}$ strictly convex with respect to anyone of them and assume $g_1|_\Gamma = g_2|_\Gamma$. Assume also $d_{g_1} = d_{g_2}$. Let x_{g_m}, ξ_{g_m}, $m = 1, ..., n$, be the solution of (4.1) with the same initial conditions*

$$x_{g_1}(0) = x_{g_2}(0) = x^{(0)}, \quad \xi_{g_1}(0) = \xi_{g_2}(0) = \xi^{(0)}.$$

Then

$$x_{g_1}(t) = x_{g_2}(t) \in \Gamma, \quad \xi_{g_1}(t) = \xi_{g_2}(t), \tag{4.2}$$

where t is the common length of the corresponding geodesics joining $x^{(0)}$ and $x_{g_1}(t) = x_{g_2}(t)$ provided that $\xi^{(0)} \cdot g^{-1}\xi^{(0)} = 1$.

Proof. We give a proof due to Michel [14]. Let x_{g_1} be the geodesics related to g_1 defined above. Denote by $s \mapsto y_{g_2}(s)$ the geodesics associated to g_2 joining $x_{g_1}(0)$ and $x_{g_1}(t) \in \Gamma$, where t is the length of $x_{g_1}x_{g_2}$. In other words, $y_{g_2}(0) = x_{g_1}(0)$, $y_{g_2}(t) = x_{g_1}(t)$. Note, that t is also the length of that geodesic. By [Corollary 2.3] [14], the geodesics x_{g_1} and y_{g_2} are tangent at the common endpoints. Since y_{g_2} solves (4.1) with $g = g_2$ and initial data $y_{g_2} = x^{(0)}$, $\xi(0) = \eta^{(0)}$ with some $\eta^{(0)}$, we get that $\eta^{(0)} = \xi^{(0)}$, because the two metrics coincide on the boundary. Therefore, y_{g_2} solves (4.1) with $g = g_2$ and by the uniqueness of that solution we get that $y_{g_2} = x_{g_2}$. This proves the lemma.

Consider the Hamiltonian system (4.1) with the following initial conditions

(4.3)

$$\begin{cases} \frac{d}{ds}x_m &= \sum_{j=1}^{n} g^{mj}\xi_j, \quad \frac{d}{ds}\xi_m &= -\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial g^{ij}}{\partial x_m}\xi_i\xi_j, \quad m = 1, ..., n, \\ x|_{s=-\rho} &= (-\rho, z), \quad \xi|_{s=-\rho} &= (1, 0, 0). \end{cases}$$

Here $z \in \mathbf{R}^2$, $\rho > 0$ is such that $g = e$ for $|x| > \rho$ and the solution $x = x(s, z)$, $\xi = \xi(s, z)$ depends on the parameter z. If $g = e$, then $x = (s, z) = (s, z_1, z_2)$.

Denote the solution of (4.1) by $x = x(s, x^{(0)}, \xi^{(0)})$, $\xi = \xi(s, x^{(0)}, \xi^{(0)})$. Let us introduce new notation

$$X := (x, \xi).$$

The solution to (4.1) related to g_1 and g_2, respectively, can therefore be written down as $X_{g_j} = X_{g_j}(s, X^{(0)}) = X_{g_j}(s, x^{(0)}, \xi^{(0)})$.

Set $F(s) := X_{g_2}(t-s, X_{g_1}(s, X^{(0)}))$. Here $t = t(X^{(0)})$ is the length of the geodesics issued from $X^{(0)}$ with endpoint on Γ and t is independent of $g = g_1$ or $g = g_2$. Notice that the x-component of $F(s)$ may not be in Ω but belongs to a neighborhood of Γ small with ε. By (4.2), $F(0) = X_{g_2}(t, X^{(0)}) = X_{g_1}(t, X^{(0)}) = F(t)$. Thus

$$\int_0^t F'(s)\, ds = 0. \tag{4.4}$$

Denote $V_{g_j} := (\partial H_{g_j}/\partial \xi, -\partial H_{g_j}/\partial x)$, $j = 1, 2$. Then

$$\begin{aligned} F'(s) = &- V_{g_2}(X_{g_2}(t-s, X_{g_1}(s, X^{(0)}))) \\ &+ \frac{\partial X_{g_2}}{\partial X^{(0)}}(t-s, X_{g_1}(s, X^{(0)}))V_{g_1}(X_{g_1}(s, X^{(0)})). \end{aligned} \tag{4.5}$$

We claim that

$$\begin{aligned} &V_{g_2}\big(X_{g_2}(t-s, X_{g_1}(s, X^{(0)}))\big) \\ &= \frac{\partial X_{g_2}}{\partial X^{(0)}}(t-s, X_{g_1}(s, X^{(0)}))V_{g_2}(X_{g_1}(s, X^{(0)})). \end{aligned} \tag{4.6}$$

Indeed, (4.6) follows from

$$\begin{aligned} 0 &= \frac{d}{ds}\bigg|_{s=0} X(T-s, X(s, X^{(0)})) \\ &= -V(X(T, X^{(0)})) + \frac{\partial X}{\partial X^{(0)}}(T, X^{(0)})V(X^{(0)}) \quad \forall T \end{aligned}$$

after setting $T = t - s$. Therefore, (4.4), (4.5) and (4.6) combined together imply

$$\int_0^t \frac{\partial X_{g_2}}{\partial X^{(0)}}(t-s, X_{g_1}(s, X^{(0)}))\,(V_{g_1} - V_{g_2})\,(X_{g_1}(s, X^{(0)}))\, ds = 0. \tag{4.7}$$

Formula (4.7) is the main result used in [25] to prove the local result mentioned before. This identity is a non-linear integral equation on the difference of the metrics g_1 and g_2.

The main idea of [13] is to rewrite the identity (4.7) in a way that it is possible to derive a ray transform, along the geodesics of the metric g_1, for the difference of the two metrics with a weight depending on the metric of g_2 and is therefore close to the standard weight. Now, with the assumptions on the sectional curvatures of g_1 one can apply similar arguments to [21] for the linearized problem to conclude that the metrics are the same in appropriate coordinates.

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