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STOCHASTIC MEHLER KERNELS VIA OSCILLATORY PATH INTEGRALS

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ABSTRACT. The configuration space and phase space oscillatory path integrals are computed in the case of the stochastic Schrödinger equation for the harmonic oscillator with a stochastic term of the form $(K\psi_t)(x) \circ dW_t$, where K is either the position operator or the momentum operator, and W_t is the Wiener process. In this way formulae are derived for the stochastic analogues of the Mehler kernel.

1. Introduction

The stochastic partial differential equation of Schrödinger type

$$id\psi_t(x) = \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right)\psi_t(x)dt + (K\psi_t)(x) \circ dW_t$$

belongs to a class of equations relevant to non-linear filtering theory, cf. the Zakai equation [14], and to quantum filtering and measurement, see Belavkin's equation [6], [4]. In particular, for $-\frac{1}{2}\frac{d^2}{dx^2} + V$ and K being selfadjoint operators, the equation was considered by Hudson and Parthasarathy [8] and called the quantum filtering equation.

We shall employ oscillatory path integrals extending the approach of Albeverio and Høegh-Krohn [2] to find the Green's function of the above equation with the harmonic oscillator potential $V(x) = \frac{1}{2}a^2x^2$. In this context we are going to consider two cases: a) the position operator $(K\psi)(x) = x\psi(x)$, and b) the momentum operator $(K\psi)(x) = -i\frac{d}{dx}\psi(x)$. In both cases we shall find an elegant formula for the Green's function, see (16) and (25) below, generalizing the well-known Mehler kernel formula. This will be achieved by computing the appropriate path integral: a configuration space oscillatory path integral in the position

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operator case and a phase space oscillatory path integral in the momentum operator case. The former path integral is precisely that introduced by Albeverio and Høegh-Krohn in [2]. However, the latter uses a different set of paths, namely phase space valued ones. It was introduced in [12] by extending Albeverio and Høegh-Krohn's definition.

Finally, let us mention some other papers devoted to path integral representations of solutions to stochastic partial differential equations of Schrödinger type: Albeverio, Kolokol'tsov and Smolyanov [3], Belavkin and Smolyanov [5], Zastawniak [15], Truman and Zhao [13].

2. Configuration space oscillatory path integrals

We shall follow Albeverio and Høegh-Krohn's approach to Feynman path integrals [2], which was originally introduced in connection with the ordinary Schrödinger equation and has recently been extended by Zastawniak [15] to the case of stochastic PDEs of Schrödinger type. Albeverio and Høegh-Krohn's path integrals, also known as oscillatory or Fresnel type path integrals, lend themselves well to computation in concrete cases, as will be seen in the next section, in which we shall find a stochastic analogue of the Mehler kernel.

Let $\mathcal{H}$ be a separable real Hilbert space with scalar product $(\cdot, \cdot)$. We denote by $M(\mathcal{H})$ the Banach algebra of complex-valued Borel measures on $\mathcal{H}$ with the total variation $\|\mu\|$ of a measure $\mu \in M(\mathcal{H})$ serving as the norm and the convolution $\mu * \nu$ of measures $\mu, \nu \in M(\mathcal{H})$ playing the role of multiplication. The space of Fourier transforms

$$\hat{\mu}(X) = \int_{\mathcal{H}} e^{i(X,\bar{X})} \mu(d\bar{X})$$

of measures $\mu \in M(\mathcal{H})$ will be denoted by $F(\mathcal{H})$. The latter is also a Banach algebra with norm $\|\hat{\mu}\| := \|\mu\|$ and pointwise multiplication, the Fourier transform $\wedge : M(\mathcal{H}) \to F(\mathcal{H})$ being an isomorphism of Banach algebras.

Given a bounded linear selfadjoint operator $B : \mathcal{H} \to \mathcal{H}$ with bounded inverse, we shall write

$$\langle X, Y \rangle = (X, BY)$$

for any $X, Y \in \mathcal{H}$. We assume that $B = I + L$, where L is a trace class operator, and denote by $\det B$ the Fredholm determinant of B. We shall also write

$$\sqrt{\det B} = \exp\left(\frac{i\pi}{2} \operatorname{Ind} B\right) \sqrt{|\det B|},$$

where $\operatorname{Ind} B$ is the number of negative eigenvalues of B counted with multiplicity.

Every $f \in F(\mathcal{H})$ can be expressed as

$$f(X) = \int_{\mathcal{H}} e^{\frac{i}{2}\langle X, \bar{X}\rangle} \mu(d\bar{X}), \tag{1}$$

where $\mu \in M(\mathcal{H})$. Then the *oscillatory path integral of* f (*with respect to* B) is defined by

$$\int_{\mathcal{H}} e^{\frac{i}{2}\langle X, X\rangle} f(X)\, dX = \frac{1}{\sqrt{\det B}} \int_{\mathcal{H}} e^{-\frac{i}{2}\langle X, X\rangle} \mu(dX). \tag{2}$$

This is essentially Albeverio and Høegh-Krohn's definition [2], except for the factor $1/\sqrt{\det B}$.

We fix $t > 0$ and denote by $\mathcal{H}_t$, the Hilbert space of absolutely continuous functions $X : [0, t] \to \mathbb{R}$ with derivative $X' \in L^2[0, t]$ such that $X(t) = 0$. The scalar product in $\mathcal{H}_t$ is defined by

$$(X, Y)_t = \int_0^t X_s' Y_s'\, ds$$

for any $X, Y \in \mathcal{H}_t$.

For any fixed $a > 0$ we put

$$(BX)_s = X_s + a^2 \int_t^s \int_0^r X_u\, du\, dr, \quad X \in \mathcal{H}_t, \quad s \in [0, t]. \tag{3}$$

Then $B : \mathcal{H}_t \to \mathcal{H}_t$ is a bounded linear operator. Since $B = 1 + L$, where L is a trace class operator, the Fredholm determinant of B is well defined and can be shown to be

$$\det B = \cos(at), \tag{4}$$

see [11], Lemma 4.1 in [7], or Example 2.4 in [1].

The bounded inverse B^{-1} exists if $a^2 \notin \sigma(A)$, the spectrum of the Sturm-Liouville operator A on $L^2[0, t]$ defined by $Au = -u''$ on the domain $D(A)$ consisting of all $u \in L^2[0, t]$ such that $u'' \in L^2[0, t]$ and $u_0 = u_t' = 0$. The inverse B^{-1} is given by [16]

$$B^{-1} = I + a^2 K^{-1} R_{a^2} K,$$

where $R_{a^2} = \left(I - a^2 A\right)^{-1}$ is the resolvent of A and $(Ku)_s = u_{t-s}$ for any $u \in L^2[0, t]$ and $s \in [0, t]$. Another useful expression gives B^{-1} in terms of the Green's function G of $A - a^2 I$:

$$(B^{-1}X)_s = X_s + a^2 \int_0^t G_{t-s, t-r}\, X_r\, dr. \tag{5}$$

Here

$$G_{r,s} = G_{s,r} = \frac{\sin(ar)\cos(at-as)}{a\cos(at)}$$

for any $0 \le r \le s \le t$.

Let us introduce functions $\sigma, \eta : [0,t] \to \mathbb{R}$ such that

$$\sigma'' + a^2\sigma = 0, \quad \sigma_0 = 0, \quad \sigma_t = 1,$$
$$\eta'' + a^2\eta = 0, \quad \eta_0 = 1, \quad \eta_t = 0,$$

that is, for any $s \in [0,t]$

$$\sigma_s = \frac{\sin(as)}{\sin(at)}, \quad \eta_s = \frac{\sin(at-as)}{\sin(at)}. \tag{6}$$

We shall also need the following closed subspaces of $\mathcal{H}_t$:

$$[\eta] = \{x\eta : x \in \mathbb{R}\}, \quad \mathcal{H}_t^0 = \{X \in \mathcal{H}_t : X(0) = 0\}.$$

Clearly,

$$\mathcal{H}_t = [\eta] \oplus \mathcal{H}_t^0.$$

Consider the bilinear form

$$\langle X, Y \rangle_t = \int_0^t X_s' Y_s' ds - a^2 \int_0^t X_s Y_s ds \tag{7}$$

defined for all $X, Y : [0,t] \to \mathbb{R}$ such that $X', Y' \in L^2[0,t]$. In particular, for any $X, Y \in \mathcal{H}_t$

$$\langle X, Y \rangle_t = (X, BY)_t,$$

and for any $X, Y \in \mathcal{H}_t^0$

$$\langle X, \sigma \rangle_t = \langle X, \eta \rangle_t = 0.$$

If $V, \varphi \in F(\mathbb{R})$, then the solution to the Cauchy problem

$$i\frac{\partial}{\partial t}\psi_t(x) = \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{a^2x^2}{2} + V(x)\right)\psi_t(x), \qquad \psi_0(x) = \varphi(x) \tag{8}$$

can be represented as

$$\psi_t(x) = e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t} \int_{\mathcal{H}_t} e^{\frac{i}{2}\langle X,X\rangle_t} f(X)\, dX, \tag{9}$$

where the functional

$$f(X) = e^{ix\langle X,\sigma\rangle_t} e^{-i\int_0^t V(X_s + x\sigma_s)ds}\varphi(X_0), \quad X \in \mathcal{H}_t,$$

belongs to $F(\mathcal{H}_t)$ and the path integral over $\mathcal{H}_t$ (with respect to B) is understood as in (2). This well-known result was established in [2]. (Here we use a slightly different notation.)

3. Oscillatory path integral computation of the stochastic Mehler kernel: position operator case

We shall consider the Cauchy problem for the stochastic Schrödinger equation

$$id\psi_t(x) = \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{a^2x^2}{2}\right)\psi_t(x)dt + x\psi_t(x)\circ dW_t. \tag{10}$$

The stochastic term $x\psi_t(x)\circ dW_t$ involves the position operator acting on the wave function.

Our goal is to compute the kernel $G_t(x,y)$, so that for any $\varphi \in F(\mathbb{R})$

$$\psi_t(x) = \int_{\mathbb{R}} G_t(x,y)\varphi(y)\,dy$$

is a solution to (10) with initial condition $\psi_0 = \varphi$.

To this end we shall show that f defined by

$$f(X) = e^{ix\langle X,\sigma\rangle_t} e^{-i\int_0^t (X_s + x\sigma_s)\,dW_s}\varphi(X_0), \quad X \in \mathcal{H}_t \tag{11}$$

(here the exponent involves an Itô stochastic integral) is an $F(\mathcal{H}_t)$-valued random variable. Then, following [15], we can represent the solution as a path integral

$$\psi_t(x) = e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t}\int_{\mathcal{H}_t} e^{\frac{i}{2}\langle X,X\rangle_t} f(X)\,dX \tag{12}$$

similar to (9). By computing this path integral we shall find an elegant explicit expression for $G_t(x,y)$.

Since the initial condition φ belongs to $F(\mathbb{R})$, so does the function $\vartheta : \mathbb{R} \to \mathbb{C}$ defined by

$$\vartheta(y) = e^{ixy\langle\eta,\sigma\rangle_t} e^{-i\int_0^t (x\sigma_s + y\eta_s)dW_s}\varphi(y),$$

that is,

$$\vartheta(y) = \int_{\mathbb{R}} e^{iyq}\nu(dq)$$

for some $\nu \in M(\mathbb{R})$. Then, for any $Y \in [\eta]$

$$\begin{aligned}\vartheta(Y_0) &= \int_{\mathbb{R}} e^{iY_0 q}\nu(dq) = \int_{\mathbb{R}} e^{i\langle Y,\Theta(q)\rangle_t}\nu(dq) \\ &= \int_{[\eta]} e^{i\langle Y,\bar{Y}\rangle_t}\left(\nu\circ\Theta^{-1}\right)(d\bar{Y}),\end{aligned}$$

where

$$\Theta : \mathbb{R} \ni q \mapsto \frac{q}{\langle\eta,\eta\rangle_t}\eta \in [\eta]\,.$$

We define $\alpha \in \mathcal{H}_t$ by

$$\alpha_s = -\int_s^t W_r dr, \quad s \in [0,t]$$

and take

$$\xi = B^{-1}\alpha - \left(B^{-1}\alpha\right)_0 \eta \in \mathcal{H}_t^0, \tag{13}$$

the projection of $B^{-1}\alpha \in \mathcal{H}_t$ onto $\mathcal{H}_t^0$. Then for all $Z \in \mathcal{H}_t^0$

$$e^{-i\int_0^t Z_s dW_s} = e^{i\langle Z,\alpha\rangle_t} = e^{i\langle Z,\xi\rangle_t} = \int_{\mathcal{H}_t^0} e^{i\langle Z,\bar{Z}\rangle_t}\delta_\xi(d\bar{Z}),$$

where δ_X denotes the Dirac delta measure concentrated at $X \in \mathcal{H}_t$.

Now the functional defined by (11) can be written as

$$\begin{aligned} f(X) &= \vartheta(Y_0)e^{-i\int_0^t Z_s dW_s} \\ &= \int_{[\eta]} e^{i\langle Y,\bar{Y}\rangle_t}\left(\nu\circ\Theta^{-1}\right)(d\bar{Y}) \int_{\mathcal{H}_t^0} e^{i\langle Z,\bar{Z}\rangle_t}\delta_\xi(d\bar{Z}) \\ &= \int_{\mathcal{H}_t} e^{\frac{i}{2}\langle X,\bar{X}\rangle_t}\mu(d\bar{X}), \end{aligned}$$

where $X = Y + Z$ is the unique decomposition of $X \in \mathcal{H}_t$ into vectors $Y \in [\eta]$ and $Z \in \mathcal{H}_t^0$, and where

$$\mu = \left(\nu\circ\Theta^{-1}\right)\otimes\delta_\xi.$$

It follows that $f \in F(\mathcal{H}_t)$ and we are ready to compute the path integral (12). First we find that

$$\int_{[\eta]} e^{-\frac{i}{2}\langle Y,Y\rangle_t}\left(\nu\circ\Theta^{-1}\right)(dY) = \sqrt{\frac{\langle\eta,\eta\rangle_t}{2\pi i}}\int_{\mathbb{R}} e^{\frac{i}{2}y^2\langle\eta,\eta\rangle_t}\vartheta(y)\,dy,$$

$$\int_{\mathcal{H}_t^0} e^{-\frac{i}{2}\langle Z,Z\rangle_t}\delta_\xi(dZ) = e^{-\frac{i}{2}\langle\xi,\xi\rangle_t}.$$

This gives

$$\begin{aligned} \psi_t(x) &= e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t}\int_{\mathcal{H}_t} e^{\frac{i}{2}\langle X,X\rangle_t} f(X)\,dX \\ &= \frac{1}{\sqrt{\det B}}e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t}\int_{\mathcal{H}_t} e^{-\frac{i}{2}\langle X,X\rangle_t}\mu(dX) \\ &= \sqrt{\frac{\langle\eta,\eta\rangle_t}{2\pi i\det B}}e^{-\frac{i}{2}\langle\xi,\xi\rangle_t}\int_{\mathbb{R}} e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t}e^{\frac{i}{2}y^2\langle\eta,\eta\rangle_t}\vartheta(y)\,dy \\ &= \int_{\mathbb{R}} G_t(x,y)\varphi(y)\,dy, \end{aligned}$$

where

$$G_t(x,y) = \sqrt{\frac{\langle\eta,\eta\rangle_t}{2\pi i \det B}}\, e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t} e^{\frac{i}{2}y^2\langle\eta,\eta\rangle_t} e^{ixy\langle\eta,\sigma\rangle_t} \times e^{-i\int_0^t (x\sigma_s + y\eta_s)dW_s} e^{-\frac{i}{2}\langle\xi,\xi\rangle_t}. \tag{14}$$

It remains to find $\langle\sigma,\sigma\rangle_t$, $\langle\eta,\eta\rangle_t$, $\langle\eta,\sigma\rangle_t$, and $\langle\xi,\xi\rangle_t$. The first three expressions are easily computed by substituting formulae (6) for σ and η into (7):

$$\langle\sigma,\sigma\rangle_t = \langle\eta,\eta\rangle_t = \frac{a}{\tan(at)}, \quad \langle\eta,\sigma\rangle_t = -\frac{a}{\sin(at)}. \tag{15}$$

The expression $\langle\xi,\xi\rangle_t$, which leads to double stochastic integrals, requires some careful treatment to ensure that the integrand of the outer stochastic integral is non-anticipating. Namely,

$$\begin{aligned}\langle\xi,\xi\rangle_t &= \int_0^t W_s^2 ds - 2a\int_0^t \left[\int_0^r \frac{\cos(as)\cos(at-ar)}{\sin(at)} W_s W_r ds\right] dr \\ &= \frac{2}{a}\int_0^t \left[\int_0^r \frac{\sin(as)\sin(at-ar)}{\sin(at)} \circ dW_s\right] \circ dW_r.\end{aligned}$$

The first equality is obtained simply by substituting the definition (13) of ξ into (7) and applying formula (5) for B^{-1}. The second equality follows by applying the Itô formula twice. The symbol $\circ$ designates Stratonovich stochastic integrals. Inserting the above expressions into (14) and using (4) for $\det B$, we obtain the following result:

THEOREM 1. *The Green's function of the stochastic Schrödinger equation* (10) *is given by*

$$\begin{aligned}G_t(x,y) = {}&\sqrt{\frac{a}{2\pi i \sin(at)}} \exp\left(ia\frac{(x^2+y^2)\cos(at) - 2xy}{2\sin(at)}\right) \\ &\times \exp\left(-i\int_0^t \frac{x\sin(as) + y\sin(at-as)}{\sin(at)} \circ dW_s\right) \\ &\times \exp\left(-\frac{i}{a}\int_0^t \left[\int_0^r \frac{\sin(as)\sin(at-ar)}{\sin(at)} \circ dW_s\right] \circ dW_r\right).\end{aligned} \tag{16}$$

Observe that the expression in the first line on the right-hand side is the well-known Mehler kernel for the harmonic oscillator. In particular, the choice of the branch of the square root is the same as in the Mehler kernel. We shall call the above formula the *stochastic Mehler kernel* (*position operator case*).

4. Phase space oscillatory path integrals

The stochastic Schrödinger equation (10) involves the position operator acting on the wave function in the stochastic term $x\psi_t(x) \circ dW_t$. In what follows we shall also consider an equation with the stochastic term $-i\frac{d}{dx}\psi_t(x) \circ dW_t$, in which the momentum operator takes place of the position operator. To this end we need to extend the definition of oscillatory path integrals to the case of phase space valued paths.

In the case of the configuration space oscillatory path integrals considered in Section 2 the paths belong to $\mathcal{H}_t$ and their derivatives to $\mathcal{L}_t = L^2[0,t]$. It is therefore natural to take $\mathcal{L}_t$ to be the set of momentum space paths and $\mathcal{H}_t \times \mathcal{L}_t$ to be the set of phase space paths. $\mathcal{H}_t \times \mathcal{L}_t$ is a Hilbert space equipped with the scalar product

$$(X,P;Y,Q) = \int_0^t X_s' Y_s' \, ds + \int_0^t P_s Q_s \, ds$$

defined for any $X, Y \in \mathcal{H}_t$ and $P, Q \in \mathcal{L}_t$. We also introduce the *kinetic energy bilinear form*

$$[X,P;Y,Q] = \int_0^t X_s' Q_s \, ds + \int_0^t Y_s' P_s \, ds - \int_0^t P_s Q_s \, ds$$

on $\mathcal{H}_t \times \mathcal{L}_t$. Clearly, $[X,P;Y,Q] = (X,P;S(Y,Q))$, where

$$S(Y,Q) = (\int_t^{(\cdot)} Q_s \, ds, Y' - Q)$$

is a bounded linear operator in $\mathcal{H}_t \times \mathcal{L}_t$.

Given a bounded linear selfadjoint operator C on $\mathcal{H}_t \times \mathcal{L}_t$ with bounded inverse C^{-1}, we shall consider the continuous bilinear form

$$\langle X,P;Y,Q\rangle = [X,P;C(Y,Q)] = (X,P;SA(Y,Q))$$

on $\mathcal{H}_t \times \mathcal{L}_t$. We also assume that $C = I + T$, where T is a trace class operator.

DEFINITION 1. The phase space oscillatory path integral (with respect to C) of a functional

$$f(X,P) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{i\langle X,P;Y,Q\rangle} \mu(dY,dQ),$$

where $\mu \in M(\mathcal{H}_t \times \mathcal{L}_t)$, is defined by

$$\int_{\mathcal{H}_t \times \mathcal{L}_t} e^{\frac{i}{2}\langle X,P;X,P\rangle} f(X,P) \, dX \, dP = \frac{1}{\sqrt{\det C}} \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{-\frac{i}{2}\langle Y,Q;Y,Q\rangle} \mu(dY,dQ).$$

This definition follows closely the pattern of Albeverio and Høegh-Krohn's configuration space oscillatory path integral in that integration is performed over a Hilbert space of paths. It proves sufficient to tackle the deterministic Schrödinger equation. However, to deal with the stochastic Schrödinger equation we need to extend the set of momentum space paths from $\mathcal{H}_t \times \mathcal{L}_t$ to $\mathcal{C}_t \times \mathcal{L}_t$, where $\mathcal{C}_t$ is the set of continuous functions from $[0,t]$ to $\mathbb{R}$ vanishing at t. This can readily be done if the bilinear from $\langle X, P; Y, Q\rangle$ can be extended from $\mathcal{H}_t \times \mathcal{L}_t$ to a continuous bilinear form on $\mathcal{C}_t \times \mathcal{L}_t$. In this case we adopt the following definition.

DEFINITION 2. The phase space oscillatory path integral (with respect to C) of a functional

$$f(X,P) = \int_{\mathcal{C}_t \times \mathcal{L}_t} e^{i\langle X,P;Y,Q\rangle} \mu(dY, dQ),$$

where $\mu \in M(\mathcal{C}_t \times \mathcal{L}_t)$, is defined by

$$\int_{\mathcal{C}_t \times \mathcal{L}_t} e^{\frac{i}{2}\langle X,P;X,P\rangle} f(X,P)\, dX\, dP = \frac{1}{\sqrt{\det C}} \int_{\mathcal{C}_t \times \mathcal{L}_t} e^{-\frac{i}{2}\langle Y,Q;Y,Q\rangle} \mu(dY, dQ).$$

We shall consider the bilinear from

$$\langle X, P; Y, Q\rangle_t = \int_0^t Q_s\, dX_s + \int_0^t P_s\, dY_s - \int_0^t P_s Q_s\, ds - a^2 \int_0^t X_s Y_s\, ds$$

defined for all square integrable functions $P, Q : [0,t] \to \mathbb{R}$ and almost all (with respect to the Wiener measure) continuous functions $X, Y : [0,t] \to \mathbb{R}$, where $\int_0^t Q_s\, dX_s$ and $\int_0^t P_s\, dY_s$ are understood as Paley-Wiener-Zygmund integrals [9] (or simply stochastic Itô integrals in modern terminology).

Next, we introduce an operator C such that

$$C(X,P) = (BX, P + (BX - X)')$$

for any $X \in \mathcal{H}_t$ and $P \in \mathcal{L}_t$. Clearly,

$$\langle X, P; Y, Q\rangle_t = [X, P; C(Y,Q)] \tag{17}$$

for any $X, Y \in \mathcal{H}_t$ and $P, Q \in \mathcal{L}_t$. The operator is of the form $C = 1 + T$, where T is trace class. It turns out that

$$\det C = \det B, \quad \operatorname{Ind} C = \operatorname{Ind} B, \quad \sqrt{\det C} = \sqrt{\det B}. \tag{18}$$

The inverse of C is given by

$$C^{-1}(X,P) = (B^{-1}X, P + (B^{-1}X - X)'). \tag{19}$$

5. Oscillatory path integral computation of the stochastic Mehler kernel: momentum operator case

In this section we shall consider the Cauchy problem for the stochastic Schrödinger equation

$$id\psi_t(x) = \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{a^2x^2}{2}\right)\psi_t(x)dt - i\frac{d}{dx}\psi_t(x)\circ dW_t \tag{20}$$

with the momentum operator acting on the wave function in the stochastic term.

Our goal is to compute the kernel $G_t(x,y)$, so that for any $\varphi \in F(\mathbb{R})$

$$\psi_t(x) = \int_{\mathbb{R}} G_t(x,y)\varphi(y)\,dy$$

is a solution to (20) with initial condition $\psi_0 = \varphi$.

Following [12], the Cauchy problem for (20) can be solved by computing the path integral

$$\psi_t(x) = e^{\frac{i}{2}x^2\langle\sigma,\sigma';\sigma,\sigma'\rangle_t} \int_{\mathcal{C}_t\times\mathcal{L}_t} e^{\frac{i}{2}\langle X,P;X,P\rangle_t} f(X,P)\,dX\,dP, \tag{21}$$

where

$$f(X,P) = e^{ix\langle X,P;\sigma,\sigma'\rangle_t} e^{-i\int_0^t (P_s + x\sigma'_s)\,dW_s}\varphi(X_0). \tag{22}$$

To this end we need to find a measure $\mu \in M(\mathcal{C}_t \times \mathcal{L}_t)$ such that

$$f(X,P) = \int_{\mathcal{C}_t\times\mathcal{L}_t} e^{i\langle X,P;\bar{X},\bar{P}\rangle_t}\,\mu(d\bar{X},d\bar{P}).$$

First of all observe that

$$\langle X,P;\sigma,\sigma'\rangle_t = \langle X,P\rangle_t\,,$$
$$\langle \sigma,\sigma';\sigma,\sigma'\rangle_t = \langle \sigma,\sigma\rangle_t\,,$$

where the bilinear form on the right-hand side is that defined by (7). This simplifies slightly the expressions in (21) and (22).

The space of paths $\mathcal{C}_t \times \mathcal{L}_t$ can be represented as a direct sum of the closed subspaces

$$[\eta,\eta'] = \left\{(x\eta,x\eta') : x\in\mathbb{R}\right\}, \quad (\mathcal{C}_t\times\mathcal{L}_t)^0 = \left\{(X,P)\in\mathcal{C}_t\times\mathcal{L}_t : X_0 = 0\right\},$$

that is,

$$\mathcal{C}_t\times\mathcal{L}_t = [\eta,\eta']\oplus(\mathcal{C}_t\times\mathcal{L}_t)^0.$$

For any $(X,P)\in(\mathcal{C}_t\times\mathcal{L}_t)^0$

$$\langle X,P;\eta,\eta'\rangle_t = \langle X,P;\sigma,\sigma'\rangle_t = 0.$$

Now, $(X,P) \in \mathcal{C}_t \times \mathcal{L}_t$ can be written uniquely as $(X,P) = (Y,Q) + (Z,R)$, where $(Y,Q) \in [\eta,\eta']$ and $(Z,R) \in (\mathcal{C}_t \times \mathcal{L}_t)^0$. Hence, $f(X,P)$ can be written as

$$f(X,P) = \theta(Y_0) e^{-i\int_0^t R_s\,dW_s},$$

where

$$\theta(y) = e^{ixy\langle\eta,\sigma\rangle_t} e^{-i\int_0^t (x\sigma_s' + y\eta_s')\,dW_s} \varphi(y).$$

Because φ belongs to $F(\mathbb{R})$, so does θ, that is,

$$\theta(y) = \int_{\mathbb{R}} e^{iyq} \nu(dq).$$

for some $\nu \in M(\mathbb{R})$. Consider the mapping

$$\Lambda : \mathbb{R} \ni q \mapsto \frac{q}{\langle\eta,\eta\rangle_t}(\eta,\eta') \in [\eta,\eta'].$$

Then

$$\theta(Y_0) = \int_{\mathbb{R}} e^{iY_0 q}\nu(dq) = \int_{\mathbb{R}} e^{i\langle Y,Q,\Lambda(q)\rangle_t}\nu(dq)$$
$$= \int_{[\eta,\eta']} e^{i\langle Y,Q;\bar{Y},\bar{Q}\rangle_t}(\nu\circ\Lambda^{-1})(d\bar{Y},d\bar{Q}).$$

We put $V_s = W_s - W_t$ and define (ξ,ζ) to be the projection of $C^{-1}(-V,0)$ onto $(\mathcal{C}_t \times \mathcal{L}_t)^0$, that is,

(23)

$$(\xi,\zeta) = (-B^{-1}V + (B^{-1}V)_0\eta, -(B^{-1}V - V)' + (B^{-1}V)_0\eta') \in (\mathcal{C}_t \times \mathcal{L}_t)^0.$$

Then

$$\langle Z,R;\xi,\zeta\rangle_t = [Z,R;-V,0] = -\int_0^t R_s\,dV_s$$

and

$$e^{-i\int_0^t R_s\,dV_s} = e^{i\langle Z,R;\xi,\zeta\rangle_t} = \int_{(\mathcal{C}_t\times\mathcal{L}_t)^0} e^{i\langle Z,R;\bar{Z},\bar{R}\rangle_t}\,\delta_{(\xi,\zeta)}(d\bar{Z},d\bar{R}).$$

It follows that

$$f(X,P) = \int_{\mathcal{C}_t\times\mathcal{L}_t} e^{i\langle X,P;\bar{X},\bar{P}\rangle_t}\mu(d\bar{X},d\bar{P}),$$

where

$$\mu = (\nu\circ\Lambda^{-1}) \otimes \delta_{(\xi,\zeta)}.$$

We are now in a position to compute the path integral (21). First we find that

$$\int_{[\eta,\eta']} e^{-\frac{i}{2}\langle Y,Q;Y,Q\rangle_t}(\nu\circ\Lambda^{-1})(dY,dQ) = \sqrt{\frac{\langle\eta,\eta\rangle_t}{2\pi i}}\int_{\mathbb{R}} e^{\frac{i}{2}y^2\langle\eta,\eta\rangle_t}\theta(y)\,dy,$$

$$\int_{(\mathcal{C}_t\times\mathcal{L}_t)^0} e^{-\frac{i}{2}\langle Z,R;Z,R\rangle_t}\,\delta_{(\xi,\zeta)}(dZ,dR) = e^{-\frac{i}{2}\langle\xi,\zeta;\xi,\zeta\rangle_t}.$$

This gives

$$\begin{aligned}
\psi_t(x) &= e^{\frac{i}{2}x^2\langle\sigma,\sigma';\sigma,\sigma'\rangle_t}\int_{\mathcal{C}_t\times\mathcal{L}_t} e^{\frac{i}{2}\langle X,P;X,P\rangle_t} f(X,P)\,dX\,dP \\
&= \frac{1}{\sqrt{\det C}}e^{\frac{i}{2}x^2\langle\sigma;\sigma\rangle_t}\int_{\mathcal{C}_t\times\mathcal{L}_t} e^{-\frac{i}{2}\langle X,P;X,P\rangle_t}\mu(dX,dP) \\
&= \sqrt{\frac{\langle\eta,\eta\rangle_t}{2\pi i\det C}}e^{\frac{i}{2}x^2\langle\sigma;\sigma\rangle_t}e^{-\frac{i}{2}\langle\xi,\zeta;\xi,\zeta\rangle_t}\int_{\mathbb{R}} e^{\frac{i}{2}y^2\langle\eta,\eta\rangle_t}\theta(y)\,dy \\
&= \int_{\mathbb{R}} G_t(x,y)\varphi(y)\,dy,
\end{aligned}$$

where

$$\begin{aligned}
G_t(x,y) = \sqrt{\frac{\langle\eta,\eta\rangle_t}{2\pi i\det C}}&e^{\frac{i}{2}x^2\langle\sigma,\sigma\rangle_t}e^{\frac{i}{2}y^2\langle\eta,\eta\rangle_t}e^{ixy\langle\eta,\sigma\rangle_t} \\
&\times e^{-i\int_0^t(x\sigma_s'+y\eta_s')\,dW_s}e^{-\frac{i}{2}\langle\xi,\zeta;\xi,\zeta\rangle_t}.
\end{aligned}\tag{24}$$

The expressions for $\langle\sigma,\sigma\rangle_t$, $\langle\eta,\eta\rangle_t$ and $\langle\eta,\sigma\rangle_t$ were found in Section 3, see (15). It remains to compute $\langle\xi,\zeta;\xi,\zeta\rangle_t$. This leads to double stochastic integrals and requires some careful treatment to ensure that the integrand of the outer stochastic integral is non-anticipating. Namely,

$$\begin{aligned}
\langle\xi,\zeta;\xi,\zeta\rangle_t &= -aV_0^2\frac{\cos(at)}{\sin(at)} + 2a^2V_0\int_0^t V_s\frac{\sin(at-as)}{\sin(at)}\,ds \\
&\quad + a^2\int_0^t V_s^2\,ds + 2a^3\int_0^t\left[\int_0^r\frac{\sin(as)\sin(at-ar)}{\sin(at)}V_rV_s\,ds\right]dr \\
&= -2a\int_0^t\left[\int_0^r\frac{\cos(as)\cos(at-ar)}{\sin(at)}\circ dW_s\right]\circ dW_r
\end{aligned}$$

The first equality is obtained by substituting (23) into (17) and using formulae (19) and (5) for C^{-1} and B^{-1}. The second equality follows by applying the Itô formula twice. Inserting these expressions into (24) and using formula (18) for $\det C$, we finally obtain the following result.

THEOREM 2. *The Green's function for the stochastic Schrödinger equation* (20) *is given by*

$$G_t(x,y) = \sqrt{\frac{a}{2\pi i \sin(at)}} \exp\left(ia \frac{(x^2+y^2)\cos(at) - 2xy}{2\sin(at)} \right) \times \exp\left(-ia \int_0^t \frac{x\cos(as) - y\cos(at-as)}{\sin(at)} \circ dW_s \right) \times \exp\left(ia \int_0^t \left[\int_0^r \frac{\cos(as)\cos(at-ar)}{\sin(at)} \circ dW_s \right] \circ dW_r \right). \tag{25}$$

Observe that, once again, the expression in the first line on the right-hand side is the well-known Mehler kernel for the harmonic oscillator. Here the choice of the branch of the square root is also the same as in the Mehler kernel. We shall call the above formula the *stochastic Mehler kernel* (*momentum operator case*).

6. Conclusions

Formulae (16) and (25) for the stochastic Mehler kernels, once computed by means of path integrals, can of course be verified directly by inserting into the stochastic partial differential equations (10) and (20), respectively. Nevertheless, such direct verification, even if straightforward, is much more tedious than the path integral derivation.

Using the above results as a starting point in his recent PhD thesis [10], L. Rincón computed the corresponding path integrals by discretisation (polygonal path approximation), once again confirming formulae (16) and (25) for the stochastic Mehler kernels. He also obtained analogous formulae for the Green's functions in the case of stochastic heat equations with similar stochastic terms.

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