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INTEGRATION FORMULAS INVOLVING FOURIER-FEYNMAN TRANSFORMS VIA A FUBINI THEOREM

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ABSTRACT. In this paper we use a general Fubini theorem established in [13] to obtain several Feynman integration formulas involving analytic Fourier-Feynman transforms. Included in these formulas is a general Parseval's relation.

1. Introduction and preliminaries

Let $C_0[0,T]$ denote one-parameter Wiener space, that is the space of $\mathbf{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)m(dx).$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable (s.i.m.) [8,15] provided $\rho E\in\mathcal{M}$ for all $\rho>0$, and a s.i.m. set N is said to be scale-invariant null provided $m(\rho N)=0$ for each $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F\approx G$. For a rather detailed discussion of s.i.m. and its relation with other topics see [15]. It was also pointed out in [15, p. 170] that the concept of s.i.m., rather than Borel measurability

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or Wiener measurability, is precisely correct for the analytic Fourier-Feynman transform theory and the analytic Feynman integration theory. Segal [19] gives an interesting discussion of the relationship between scale change in $C_0[0,T]$ and certain questions in quantum field theory.

Throughout this paper we will assume that each functional F (or G) we consider satisfies the conditions:

$$F : C_0[0,T] \to \mathbf{C} \text{ is defined } s-a.e. \text{ and is } s.i.m.. \tag{1.1}$$

$$\int_{C_0[0,T]} |F(\rho x)| m(dx) < \infty \text{ for each } \rho > 0. \tag{1.2}$$

Let $\mathbf{C}_+$ and $\tilde{\mathbf{C}}_+$ denote the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part respectively. Let F satisfy conditions (1.1) and (1.2) above, and for $\lambda > 0$, let

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x) m(dx).$$

If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for λ in $\mathbf{C}_+$ we write

$$\int_{C_0[0,T]}^{anw_\lambda} F(x) m(dx) = J^*(\lambda). \tag{1.3}$$

Let $q \neq 0$ be a real parameter and let F be a functional whose analytic Wiener integral exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{anf_q} F(x) m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{anw_\lambda} F(x) m(dx) \tag{1.4}$$

where $\lambda \to -iq$ through values in $\mathbf{C}_+$. Finally, for notational purposes, we let

(1.5)

$$\int_{C_0[0,T]}^{an_\lambda} F(x) m(dx) \equiv \begin{cases} \int_{C_0[0,T]}^{anw_\lambda} F(x) m(dx), & \lambda \in \mathbf{C}_+ \\ \int_{C_0[0,T]}^{anf_q} F(x) m(dx), & \lambda = -iq \in \tilde{\mathbf{C}}_+ - \mathbf{C}_+. \end{cases}$$

The following Fubini theorem established in [13], plays a major role in this paper.

THEOREM 1. *Assume that F satisfies conditions (1.1) and (1.2) above and is such that its analytic Feynman integral $\int_{C_0[0,T]}^{anf_q} F(x)m(dx)$ exists for all $q \in \mathbf{R} - \{0\}$. Then for all $a, b \in \mathbf{R}$ and all $(\lambda, \beta) \in \tilde{\mathbf{C}}_+ \times \tilde{\mathbf{C}}_+$ with $\lambda + \beta \neq 0$,*

$$\begin{aligned} &\int_{C_0[0,T]}^{an_\beta} \left(\int_{C_0[0,T]}^{an_\lambda} F(ay+bz)m(dy) \right) m(dz) \\ (1.6) \qquad &= \int_{C_0[0,T]}^{an_{(\lambda\beta)/(\lambda b^2+\beta a^2)}} F(x)dx \\ &= \int_{C_0[0,T]}^{an_\lambda} \left(\int_{C_0[0,T]}^{an_\beta} F(ay+bz)m(dz) \right) m(dy). \end{aligned}$$

In section 2 below we use Theorem 1 to help us establish several Feynman integration formulas involving Fourier-Feynman transforms. Finally, in section 3 we establish additional integration formulas including a general Parseval's relation.

2. Fourier-Feynman transforms

The concept of an L_1 analytic Fourier-Feynman transform (FFT) was introduced by Brue in [1]. In [5], Cameron and Storvick introduced an L_2 analytic FFT. In [14], Johnson and Skoug developed an L_p analytic FFT for $1 \leq p \leq 2$ which extended the results in [1,5] and gave various relationships between the L_1 and the L_2 theories. In [10], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and in [11,12] obtained various results involving the FFT and the convolution product. Also see [7,16 and 18] for further work on these topics.

In this paper, for simplicity, we restrict our discussion to the case $p = 1$; however most of our results hold for all $p \in [1, 2]$. Also, throughout this section, we will assume that the functionals $F : C_0[0,T] \to \mathbf{C}$ satisfy the hypotheses of our Fubini theorem, namely Theorem 1 above.

For $\lambda \in \mathbf{C}_+$ and $y \in C_0[0,T]$, let

$$(2.1) \qquad (T_\lambda(F))(y) = \int_{C_0[0,T]}^{anw_\lambda} F(y+x)m(dx).$$

Then for $q \in \mathbf{R} - \{0\}$ (see [10,p.663]), the L_1 analytic FFT, $T_q^{(1)}(F)$ of

F, is defined by the formula ($\lambda \in \mathbf{C}_+$)

$$(T_q^{(1)}(F))(y) = \lim_{\lambda \to -iq} (T_\lambda(F))(y) \tag{2.2}$$

for s-a.e. $y \in C_0[0,T]$ whenever this limit exists. That is to say,

$$(T_q^{(1)}(F))(y) = \int_{C_0[0,T]}^{anf_q} F(y+x)m(dx) \tag{2.3}$$

for s-a.e. $y \in C_0[0,T]$. We note that if $T_q^{(1)}(F)$ exists and if $F \approx G$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(F) \approx T_q^{(1)}(G)$.

In equations (2.4), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13) below, we establish various analytic Feynman integration formulas involving Fourier-Feynman transforms.

THEOREM 2. *Let F be as in Theorem 1 above and let $r > 0$ be given. Then for all q_1 and q_2 in $\mathbf{R} - \{0\}$ with $q_1 + q_2 \neq 0$,*

$$\begin{aligned} &\int_{C_0[0,T]}^{anf_{rq_2}} (T_{q_1}^{(1)}(F))(\sqrt{r}z)m(dz) \\ &\qquad = \int_{C_0[0,T]}^{anf_{(q_1q_2)/(q_1+q_2)}} F(x)m(dx) \\ &\qquad = \int_{C_0[0,T]}^{anf_{rq_1}} (T_{q_2}^{(1)}(F))(\sqrt{r}y)m(dy). \end{aligned} \tag{2.4}$$

Proof. Using equation (2.3) and the first equality in equation (1.6) with $a = 1$, $b = \sqrt{r}$, $\lambda = -iq_1$ and $\beta = -irq_2$, we obtain that

$$\begin{aligned} &\int_{C_0[0,T]}^{anf_{rq_2}} (T_{q_1}^{(1)}(F))(\sqrt{r}z)m(dz) \\ &\qquad = \int_{C_0[0,T]}^{anf_{rq_2}} \Big(\int_{C_0[0,T]}^{anf_{q_1}} F(\sqrt{r}z+y)m(dy)\Big)m(dz) \\ &\qquad = \int_{C_0[0,T]}^{anf_{(rq_2q_1)/(rq_1+rq_2)}} F(x)m(dx) \\ &\qquad = \int_{C_0[0,T]}^{anf_{(q_1q_2)/(q_1+q_2)}} F(x)m(dx). \end{aligned} \tag{2.5}$$

Also using equation (2.3) and the second equality in equation (1.6) with $a = \sqrt{r}$, $b = 1$, $\beta = -iq_2$ and $\lambda = -irq_1$, we obtain that

$$
\begin{aligned}
&\int_{C_0[0,T]}^{anf_{rq_1}} (T_{q_2}^{(1)}(F))(\sqrt{r}y)m(dy) \\
(2.6) \qquad &= \int_{C_0[0,T]}^{anf_{rq_1}} \Big(\int_{C_0[0,T]}^{anf_{q_2}} F(\sqrt{r}y + z)m(dz)\Big)m(dy) \\
&= \int_{C_0[0,T]}^{anf_{(rq_1q_2)/(rq_2+rq_1)}} F(x)m(dx) \\
&= \int_{C_0[0,T]}^{anf_{(q_1q_2)/(q_1+q_2)}} F(x)m(dx).
\end{aligned}
$$

Now equation (2.4) follows from equations (2.5) and (2.6). □

Our first corollary below says that the Feynman integral with parameter q_2 of the FFT with parameter q_1 equals the Feynman integral with parameter q_1 of the FFT with parameter q_2 provided $q_1 + q_2 \neq 0$.

COROLLARY 1 TO THEOREM 2. *Let F be as in Theorem 2. Then for all q_1 and q_2 in $\mathbf{R} = \{0\}$ with $q_1 + q_2 \neq 0$,*

$$
(2.7) \qquad \int_{C_0[0,T]}^{anf_{q_2}} (T_{q_1}^{(1)}(F))(z)m(dz) = \int_{C_0[0,T]}^{anf_{q_1}} (T_{q_2}^{(1)}(F))(y)m(dy).
$$

COROLLARY 2 TO THEOREM 2. *Let F be as in Theorem 2. Then for all q in $\mathbf{R} - \{0\}$,*

$$
\begin{aligned}
(2.8) \qquad \int_{C_0[0,T]}^{anf_q} (T_q^{(1)}(F))(y)m(dy) &= \int_{C_0[0,T]}^{anf_{q/2}} F(x)m(dx) \\
&= \int_{C_0[0,T]}^{anf_q} F(\sqrt{2}x)m(dx).
\end{aligned}
$$

Proof. The first equality in equation (2.8) follows by letting $r = 1$ and $q_1 = q_2 = q$ in equation (2.4). The second equality follows from the formula

$$
\int_{C_0[0,T]}^{anf_{kq}} F(x)m(dx) = \int_{C_0[0,T]}^{anf_q} F(x/\sqrt{k})m(dx)
$$

established in [13] for $k > 0$. □

COROLLARY 3 TO THEOREM 2. *Let F be as in Theorem 2. Then for all q in $\mathbf{R} - \{0\}$,*

$$\begin{aligned}\int_{C_0[0,T]}^{anf_{-q}} (T_{q/2}^{(1)}(F))(z)m(dz) &= \int_{C_0[0,T]}^{anf_q} F(x)m(dx) \\ &= \int_{C_0[0,T]}^{anf_{q/2}} (T_{-q}^{(1)}(F))(y)m(dy).\end{aligned} \tag{2.9}$$

Proof. Simply choose $q_1 = q/2$, $q_2 = -q$ and $r = 1$ in equation (2.4). □

THEOREM 3. *Let F be as in Theorem 2 and let $q_1, q_2, \ldots, q_n$ be elements of $\mathbf{R} - \{0\}$ with*

$$\sum_{j=1}^{k} \frac{q_1 q_2 \cdots q_k}{q_j} \neq 0 \quad \text{for } k = 2, \ldots, n.$$

Then, for s-a.e. $z \in C_0[0,T]$,

$$\begin{aligned}&(T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\ldots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\ldots)))(z) \\ &= \int_{C_0[0,T]}^{anf_{\alpha_n}} F(z+x)m(dx) \\ &= (T_{\alpha_n}^{(1)}(F))(z)\end{aligned} \tag{2.10}$$

where

$$\alpha_n = \frac{q_1 q_2 \cdots q_n}{\sum_{j=1}^{n} \frac{q_1 q_2 \cdots q_n}{q_j}} = \frac{1}{\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n}}.$$

Proof. Using equation (2.3), and then equation (1.6) repeatedly, we obtain that

$$\begin{aligned}&(T_{q_n}^{(1)}(T_{q_{n-1}}^{(1)}(\ldots(T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))\ldots)))(z) \\ &= \int_{C_0[0,T]}^{anf_{q_n}}(\int_{C_0[0,T]}^{anf_{q_{n-1}}}(\ldots(\int_{C_0[0,T]}^{anf_{q_2}}(\int_{C_0[0,T]}^{anf_{q_1}} F(z+y_1+y_2+\cdots+y_n)m(dy_1)) \\ &\qquad \cdot m(dy_2))\ldots)m(dy_{n-1}))m(dy_n) \\ &= \int_{C_0[0,T]}^{anf_{\alpha_n}} F(z+x)m(dx) \\ &= (T_{\alpha_n}^{(1)}(F))(z)\end{aligned}$$

for s-a.e. $z \in C_0[0,T]$. □

Choosing $q_j = q$ for $j = 1, 2, \ldots, n$, we obtain the following corollary to Theorem 3.

COROLLARY 1 TO THEOREM 3. *Let F be as in Theorem 3 and let q be an element of $\mathbf{R} - \{0\}$. Then for s-a.e. $z \in C_0[0,T]$,*

$$(2.11)\quad (T_q^{(1)}(T_q^{(1)}(F)))(z) = (T_{q/2}^{(1)}(F))(z) = \int_{C_0[0,T]}^{anf_q} F(z+\sqrt{2}x)m(dx),$$

$$(2.12)\quad (T_q^{(1)}(T_q^{(1)}(T_q^{(1)}(F))))(z) = (T_{q/3}^{(1)}(F))(z) = \int_{C_0[0,T]}^{anf_q} F(z+\sqrt{3}x)m(dx),$$

and in general,

$$(2.13)\quad \begin{aligned} (T_q^{(1)}(T_q^{(1)}(\ldots(T_q^{(1)}(F))\ldots)))(z) &= (T_{q/n}^{(1)}(F))(z) \\ &= \int_{C_0[0,T]}^{anf_q} F(z+\sqrt{n}x)m(dx). \end{aligned}$$

COROLLARY 2 TO THEOREM 3. *Let F be as in Theorem 3 and let q_1 and q_2 be elements of $\mathbf{R} - \{0\}$ with $q_1 + q_2 \neq 0$. Then for s-a.e. $z \in C_0[0,T]$,*

$$(2.14)\quad (T_{q_2}^{(1)}(T_{q_1}^{(1)}(F)))(z) = (T_{\frac{q_1 q_2}{q_1+q_2}}^{(1)}(F))(z) = (T_{q_1}^{(1)}(T_{q_2}^{(1)}(F)))(z).$$

REMARK 1. We note that the hypotheses (and hence the conclusions) of Theorems 1-3 and their corollaries above are indeed satisfied by many large classes of functionals. These classes of functionals include:

(a) The Banach algebra S defined by Cameron and Storvick in [6]; also see [9,12,18].

(b) Various spaces of functionals of the form

$$F(x) = \exp\{\int_0^T f(t, x(t))dt\}$$

for appropriate $f : [0,T] \times \mathbf{R} \to \mathbf{C}$; see for example [5,11 and 14].

(c) Various spaces of functionals of the form

$$F(x) = f(\int_0^T \alpha_1(t)dx(t), \ldots, \int_0^T \alpha_n(t)dx(t))$$

for appropriate f as discussed in [10,16].

(d) Various spaces of functionals of the form

$$F(x) = \exp\{\int_0^T \int_0^T f(s,t,x(s),x(t))dsdt\}$$

for appropriate $f : [0,T]^2 \times \mathbf{R}^2 \to \mathbf{C}$; see for example [12].

REMARK 2. In a unifying paper [17], Lee obtains some similar results for several different integral transforms including the Fourier-Feynman transform. However, the results in this paper hold for much more general functionals F. For example, in our notation, Lee requires the functional $F(x + \lambda y)$ to be an entire function of λ over $\mathbf{C}$ for each x and y in $C_0[0,T]$ whereas we don't even require F to be a continuous function. The classes of functionals studied by Yeh in [20] and Yoo in [21] for the Fourier-Wiener transform are similar to those used by Lee in [17].

3. Further applications

First we state the definition of the convolution product of two functionals F and G on $C_0[0,T]$ as given by Huffman, Park and Skoug in [10, p. 663]. This definition is different than the definition given by Yeh in [20] and used by Yoo in [21]. In [20] and [21], Yeh and Yoo study the relationship between their convolution product and Fourier-Wiener transforms. For $\lambda \in \tilde{\mathbf{C}}_+$, the convolution product (if it exists) of F and G is defined by the formula (see equation (1.5) above)

$$(F * G)_\lambda(y) = \int_{C_0[0,T]}^{an_\lambda} F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})m(dx) \tag{3.1}$$

for s-a.e. $y \in C_0[0,T]$. When $\lambda = -iq$, we usually denote $(F * G)_\lambda$ by $(F * G)_q$.

In our first theorem of this section, we show that the Fourier-Feynman transform of the convolution product is a product of their transforms.

THEOREM 4. *Let F be as in Theorem 1 and assume that $G : C_0[0,T] \to \mathbf{C}$ satisfies the same conditions as F; i.e., G is s.i.m., $\int_{C_0[0,T]} |G(\rho x)|\, m(dx) < \infty$ for all $\rho > 0$ and the analytic Feynman integral of G exists for all $q \in \mathbf{R} - \{0\}$. Furthermore assume that $T_q^{(1)}((F * G)_q)$ exists for all $q \in \mathbf{R} - \{0\}$. Then for all $q \in \mathbf{R} - \{0\}$,*

$$(T_q^{(1)}((F * G)_q))(z) = (T_q^{(1)}(F))(z/\sqrt{2})(T_q^{(1)}(G))(z/\sqrt{2}) \tag{3.2}$$

for s-a.e. z in $C_0[0,T]$.

Proof. Because of the assumptions on F and G, all three of the transforms in equation (3.2) exist; thus we only need to establish the equality. For $\lambda > 0$, using (2.1) and (3.1), we see that

$$(T_\lambda((F * G)_\lambda))(z) = \int_{C_0[0,T]} (F * G)_\lambda(z + \lambda^{-1/2}y) m(dy)$$

$$= \int_{C_0[0,T]}\int_{C_0[0,T]} F(\frac{z}{\sqrt{2}} + \lambda^{-1/2}(\frac{y+x}{\sqrt{2}}))G(\frac{z}{\sqrt{2}} + \lambda^{-1/2}(\frac{y-x}{\sqrt{2}}))m(dx)m(dy)$$

for s-a.e. $z \in C_0[0,T]$. But $w_1 = (y+x)/\sqrt{2}$ and $w_2 = (y-x)/\sqrt{2}$ are independent standard Wiener processes, and hence

$$\begin{aligned}(T_\lambda((F * G)_\lambda))(z) &= \int_{C_0[0,T]} (\int_{C_0[0,T]} F(z/\sqrt{2} + w_1/\sqrt{\lambda})G(z/\sqrt{2} + w_2/\sqrt{\lambda}) \\ &\quad m(dw_2))m(dw_1) \\ &= \int_{C_0[0,T]} F(z/\sqrt{2} + w_1/\sqrt{\lambda})m(dw_1) \\ &\quad \int_{C_0[0,T]} G(z/\sqrt{2} + w_2/\sqrt{\lambda})m(dw_2) \\ &= (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2})\end{aligned}$$

for s-a.e. $z \in C_0[0,T]$. Now by analytic extensions through $\mathbf{C}_+$ we obtain that

$$(T_\lambda((F * G)_\lambda))(z) = (T_\lambda(F))(z/\sqrt{\lambda})(T_\lambda(G))(z/\sqrt{2}) \tag{3.3}$$

holds throughout $\mathbf{C}_+$. Finally, equation (3.2) follows from equation (3.3) by letting $\lambda \to -iq$, since all three of the transforms in (3.2) exist. □

COROLLARY 1 TO THEOREM 4. *Let F be as in Theorem 4. Then for all $q \in \mathbf{R} - \{0\}$,*

$$(T_q^{(1)}((F*1)_q))(z) = (T_q^{(1)}(F))(z/\sqrt{2}).$$

Furthermore, if $T_q^{(1)}((F*F)_q)$ exists, then

$$(T_q^{(1)}((F*F)_q))(z) = [(T_q^{(1)}(F))(z/\sqrt{2})]^2.$$

In our next theorem, we establish a general Parseval's relation; for related results in the Fourier-Wiener theory see Cameron [2], Cameron and Martin [3,4], and Il Yoo [21].

THEOREM 5. *Let F and G be as in Theorem 4. Furthermore, assume that F and G are continuous on $C_0[0,T]$. Then for all $q \in \mathbf{R} - \{0\}$,*

$$\begin{aligned}\int_{C_0[0,T]}^{anf_{-q}} (T_q^{(1)}(F))(z/\sqrt{2})(T_q^{(1)}(G))(z/\sqrt{2})m(dz)\\ = \int_{C_0[0,T]}^{anf_q} F(x/\sqrt{2})G(-x/\sqrt{2})m(dx).\end{aligned} \tag{3.4}$$

Proof. Because of our assumptions on F and G, the analytic Feynman integrals on both sides of equation (3.4) certainly exist. Also recall that in the definition of the analytic Feynman integral (1.4), we assumed that λ could approach $-iq$ in an arbitrary fashion through values in $\mathbf{C}_+$. Thus, using (3.2), Theorem 1, (3.1), and the continuity of F and

G, we obtain that

$$\begin{aligned}
&\int_{C_0[0,T]}^{anf_{-q}} (T_q^{(1)}(F))(z/\sqrt{2})(T_q^{(1)}(G))(z/\sqrt{2})m(dz) \\
&\quad = \int_{C_0[0,T]}^{anf_{-q}} (T_q^{(1)}((F*G)_q))(z)m(dz) \\
&\quad = \lim_{p\to 0^+} \int_{C_0[0,T]}^{anw_{p+qi}} (T_{p-qi}((F*G)_{p-qi}(z)m(dz) \\
&\quad = \lim_{p\to 0^+} \int_{C_0[0,T]}^{anw_{p+qi}} \left(\int_{C_0[0,T]}^{anw_{p-qi}} (F*G)_{p-qi}(z+y)m(dy) \right) m(dz) \\
&\quad = \lim_{p\to 0^+} \int_{C_0[0,T]}^{anw_{(p^2+q^2)/2p}} (F*G)_{p-qi}(w)m(dw) \\
&\quad = \lim_{p\to 0^+} \int_{C_0[0,T]} (F*G)_{p-qi}(\sqrt{\frac{2p}{p^2+q^2}}w)m(dw) \\
&\quad = \lim_{p\to 0^+} \int_{C_0[0,T]} \left(\int_{C_0[0,T]}^{anw_{p-qi}} F(\sqrt{\frac{p}{p^2+q^2}}w + \frac{x}{\sqrt{2}}) \right. \\
&\qquad \left. G(\sqrt{\frac{p}{p^2+q^2}}w - \frac{x}{\sqrt{2}})m(dx) \right) m(dw) \\
&\quad = \int_{C_0[0,T]} \left(\int_{C_0[0,T]}^{anf_q} F(x/\sqrt{2})G(-x/\sqrt{2})m(dx) \right) m(dw) \\
&\quad = \int_{C_0[0,T]}^{anf_q} F(x/\sqrt{2})G(-x/\sqrt{2})m(dx)
\end{aligned}$$

as desired. □

Our first corollary gives an alternative form of Parseval's relation.

COROLLARY 1 TO THEOREM 5. *Let F and G be as in Theorem 5. Then for all q in $\mathbf{R}-\{0\}$,*

$$\int_{C_0[0,T]}^{anf_{-q}} (T_{q/2}^{(1)}(F))(z)(T_{q/2}^{(1)}(G))(z)m(dz) = \int_{C_0[0,T]}^{anf_q} F(x)G(-x)m(dx). \tag{3.5}$$

Corollary 2 to Theorem 5. *Let F be as in Theorem 5 and assume that $T_q^{(1)}((F*F)q)$ exists for all $q \in \mathbf{R}-\{0\}$. Then*
(3.6)
$$\int_{C_0[0,T]}^{anf_{-q}} [(T_q^{(1)}(F))(z/\sqrt{2})]^2 m(dz) = \int_{C_0[0,T]}^{anf_q} F(x/\sqrt{2})F(-x/\sqrt{2})m(dx).$$

Theorem 6. *Let F be as in Theorem 1. Furthermore, assume that F is continuous on $C_0[0,T]$. Then for all q in $\mathbf{R}-\{0\}$,*

$$(T_{-q}^{(1)}(T_q^{(1)}(F)))(y) = F(y) \tag{3.7}$$

for s-a.e. $y \in C_0[0,T]$.

Proof. Proceeding as in the proof of Theorem 5 and using equations (2.2), (2.1), (1.6) and the continuity of F, we obtain that for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}(T_{-q}^{(1)}(T_q^{(1)}(F)))(y) &= \lim_{p\to 0^+}(T_{p+qi}(T_{p-qi}(F)))(y)\\ &= \lim_{p\to 0^+}\int_{C_0[0,T]}^{anw_{p+qi}}\left(\int_{C_0[0,T]}^{anw_{p-qi}} F(y+z+x)m(dx)\right)m(dz)\\ &= \lim_{p\to 0^+}\int_{C_0[0,T]}^{anw_{(p^2+q^2)/(2p)}} F(y+w)m(dw)\\ &= \lim_{p\to 0^+}\int_{C_0[0,T]} F\left(y+\sqrt{\frac{2p}{p^2+q^2}}w\right)m(dw)\\ &= \int_{C_0[0,T]} F(y)m(dw)\\ &= F(y).\end{aligned}$$

□

Corollary 1 to Theorem 6. *Let E be the class of all continuous functionals $F : C_0[0,T] \to \mathbf{C}$ satisfying the hypotheses of Theorem 1. Let $T_0^{(1)}$ denote the identity map; i.e., $T_0^{(1)}(F) \approx F$. Then*

$$\left\{T_q^{(1)}\right\}_{q\in\mathbf{R}}$$

forms an abelian group acting on E with $(T_q^{(1)})^{-1} = T_{-q}^{(1)}$.

REMARK 3. Looking at equation (3.2) above, together with its proof, it is quite tempting to conjecture that for s-a.e. z in $C_0[0,T]$,

$$(3.8) \quad (T_{q_1}^{(1)}((F*G)_{q_2}))(z) = (T_{\frac{2q_1q_2}{q_1+q_2}}^{(1)}(F))(z/\sqrt{2})(T_{\frac{2q_1q_2}{q_1+q_2}}^{(1)}(G))(z/\sqrt{2})$$

provided that $q_1 + q_2 \neq 0$. But in general, equation (3.8) holds if and only if $q_1 = q_2 = q$, in which case equation (3.8) reduces to equation (3.2). The proof given above to establish equation (3.2) fails to work for equation (3.8) since for $\lambda > 0$ and $\beta > 0$,

$$\begin{aligned}
&(T_\lambda((F*G)_\beta))(z) \\
&= \int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F\Big(\frac{z}{\sqrt{2}} + \frac{y}{\sqrt{2\lambda}} + \frac{x}{\sqrt{2\beta}}\Big) G\Big(\frac{z}{\sqrt{2}} + \frac{y}{\sqrt{2\lambda}} - \frac{x}{\sqrt{2\beta}}\Big) \\
&\quad m(dx)\Big) m(dy),
\end{aligned}$$

while $w_1 = \frac{y}{\sqrt{2\lambda}} + \frac{x}{\sqrt{2\beta}}$ and $w_2 = \frac{y}{\sqrt{2\lambda}} - \frac{x}{\sqrt{2\beta}}$ are independent processes if and only if $\lambda = \beta$.

In particular (see section 3 of [12] for the appropriate definitions), for F and G in the Banach algebra S with corresponding finite Borel measures f and g in $M(L_2[0,T])$ and using equations (3.2) and (3.5) of [12], it is easy to see that

$$\begin{aligned}
&(3.9) \\
&(T_{\frac{2q_1q_2}{q_1+q_2}}^{(1)}(F))(z/\sqrt{2}) \\
&\qquad = \int_{L_2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}} \int_0^T v(t)dz(t) - \frac{i(q_1+q_2)}{4q_1q_2} \int_0^T v^2(t)dt\Big\} df(v),
\end{aligned}$$

$$\begin{aligned}
&(3.10) \\
&(T_{\frac{2q_1q_2}{q_1+q_2}}^{(1)}(G))(z/\sqrt{2}) \\
&\qquad = \int_{L_2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}} \int_0^T w(t)dz(t) - \frac{i(q_1+q_2)}{4q_1q_2} \int_0^T w^2(t)dt\Big\} dg(w),
\end{aligned}$$

and that

(3.11)

$$(T_{q_1}^{(1)}((F*G)_{q_2}))(z) = \int_{L_2^2[0,T]} \exp\{\frac{i}{\sqrt{2}}\int_0^T [v(t)+w(t)]dz(t)\} \cdot \exp\{-\frac{i(q_1+q_2)}{4q_1q_2}\int_0^T [v^2(t)+w^2(t)]dt\} \cdot \exp\{\frac{i(q_1-q_2)}{2q_1q_2}\int_0^T v(t)w(t)dt\} df(v)dg(w).$$

Now a careful examination of equations (3.9), (3.10) and (3.11) shows that equation (3.8) holds if and only if $q_1 = q_2$.

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