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PATH INTEGRALS ASSOCIATED WITH STURM-LIOUVILLE OPERATORS

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Abstract. We consider path integrals, more precisely projective systems of Fresnel distributions, associated with Sturm-Liouville boundary value problems.

Introduction

Among the rigorous approaches to the notion of path integral there is the notion of projective system of finite-dimensional summable distributions, closely related to the notion of prodistribution as set forth in the work of C. M. DeWitt [6]. In the case of the Schrödinger operator $-(\frac{d}{dt})^2$ the finite-dimensional marginal distributions, corresponding to subdivisions $\sigma : 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, are well known, i.e. the corresponding finite-dimensional action functionals S_σ are well known. They have the property that the corresponding finite dimensional distributions form a projective or compatible system.

The main object of this paper is to calculate the discrete actions S_σ also in the case of general Sturm-Liouville operators on an interval $[0, T]$

$$D = -\frac{d}{dt}p(t)\frac{d}{dt} + q(t), \quad p(t) > 0, \quad q(t) \in \mathbb{R}$$

accompanied by appropriate boundary conditions (BC), in such a way that one obtains a compatible system of finite dimensional distributions $C_\sigma \exp iS_\sigma$, so that at least for cylindrical test functionals Φ there can be no doubt as to the meaning of the path integral $\int e^{iS(x)}\Phi(x)\mathcal{D}(dx)$

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analogous to the Feynman integral for the simple Schrödinger operator $-(\frac{d}{dt})^2$.

We assume the equation $Dx = 0$, $x \in BC$ has the solution $x = 0$ only, so that there exists a Green function $K_t(s) = K(t,s)$ satisfying the conditions

$$DK_t = \delta_t, \quad K_t \in BC$$

where δ_t is the unit point mass at t. It is unfortunate, but unavoidable, that this notion of Green function is entirely different from the notion 'Green function' normally associated with path integrals, namely the kernel such that the corresponding integral operator gives the solution of the Schrödinger equation, which to distinguish it from the present Green function, we prefer to call the propagator (cf. [12]).

In studying this kind of path integral we distinguish the case where the operator D with boundary conditions (BC), is positive. In that case one can associate with it a Gaussian measure on the space of continuous functions $C[0,T]$, analogous to Wiener measure, and the path integral could in principle be obtained by analytic continuation from a probability measure. Also interesting from the point of view of path integrals is the case where $D,(BC)$ is not positive. In that case we describe the projective system of summable distributions only if the mesh $|\sigma| = \max_{i=1\ldots n}(t_i - t_{i-1})$ of the subdivision is sufficiently small. There is then no analogue of Wiener measure available, and we are in a way forced to develop, which is only very partially done here, a theory of Fresnel path integrals independent of analytic continuation from a Gaussian measure.

Consideration of general second order differential operators, even non symmetric, is not new of course, going back to Feynman himself ([10]). But we have not seen in the literature of path integrals the kind of exact projective systems obtained in this paper. The reason for this is doubtless that they are more of theoretical than of practical interest.

Of course the case of the harmonic oscillator is well known. In the last section of the paper we show that our result in this case conforms to the result of a precise calculation in L. S. Schulman's work [12].

Part of the paper is devoted to a brief exposition of L. Schwartz's notion of summable distribution. Summable distributions naturally occur as the finite-dimensional marginal distributions for path integrals. In particular, finite-dimensional Fresnel distributions $C_\sigma e^{iS_\sigma}$ are summable.

While writing this paper my attention was drawn[1] to the work of the physicists L. Chetouani, L. Dekar and T. F. Hamman, [4], [8] who consider just such Sturm-Liouville Schrödinger operators in order to accommodate position-dependent mass.

1. Variation and Sturm-Liouville Green functions

Consider the Sturm-Liouville Problem on $[0, T]$:

$$\begin{aligned} &Dx = f \\ &x \in BC \end{aligned} \tag{1}$$

where $Dx = -(px')' + qx$ and BC are separated boundary conditions:

$$L_0(x) = \alpha x(0) + \beta x'(0) = 0, \quad L_T(x) = \gamma x(T) + \delta x'(T) = 0 \tag{2}$$

or if $p(0) = p(T)$, periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{2'}$$

We assume p is continuously differentiable and $p(t) > 0$ for all $t \in [0, T]$. We also assume for simplicity[2] that the boundary conditions are such that $p(T)x(T)x'(T) = p(0)x(0)x'(0)$. This is the case for the Dirichlet boundary conditions: $x(0) = x(T) = 0$, that occur most often in connection with Feynman integral, and for the Wiener boundary conditions $x(0) = 0$, $x'(T) = 0$ that occur in connection with Wiener integral.

We assume that $Dx = 0$, $x \in BC$ implies $x = 0$. Then it is well known that there exists a unique Green function[3] K such that if $K_t(s) = K(s, t)$ we have, δ_t denoting the unit mass at t,

$$\begin{aligned} &DK_t = \delta_t \\ &K_t \in BC \end{aligned} \tag{3}$$

[1] by Imme van den Berg, whom it is my pleasure to thank hereby.

[2] We have omitted boundary conditions such that $p(T)x(T)x'(T) - p(0)x(0)x'(0) \geq 0$. cf. [5] p. 291.

[3] In the theory of path integrals the term 'Green function' usually denotes an entirely different object, namely the propagator of the one-parameter unitary group which has D as generator. To distinguish it from the Sturm-Liouville Green function this kernel is sometimes called the propagator cf. e.g. [12]

Let

$$S(x) = \frac{1}{2}(Dx, x) = \frac{1}{2}\int_0^1 p\dot{x}^2 + qx^2 dt \tag{4}$$

Let $\sigma = \{t_1, \dots, t_n\}$ where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, and let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$

Assume for the present that $q \geq 0$. Then we shall show that there exists a continuous function x_σ on $[0, T]$ such that $x_\sigma \in BC$, $x(t_i) = \xi_i$, and such that on the intervals (t_{i-1}, t_i) we have $Dx_\sigma = 0$.

Then, since x_σ depends linearly on ξ, there exists a positive definite symmetric matrix $A_\sigma = (A_{ij})$, $1 \leq j \leq n$, such that

$$S_\sigma(\xi) = S(x_\sigma) = \frac{1}{2}(A_\sigma\xi, \xi) = \frac{1}{2}\sum_{ij} A_{ij}\xi_i\xi_j \tag{5}$$

Let K_σ be the n-by-n matrix $K_{ij} = K(t_i, t_j)$.

THEOREM 1. *Let $q \geq 0$ and assume $Dx = 0$ and $x \in BC$ implies $x = 0$. The matrices K_σ and A_σ are well defined and inverses of each other. In other terms: the quadratic form $\xi \mapsto S_\sigma(\xi)$ is reciprocal to the quadratic form defined by K_σ.*

EXAMPLE 1. In the Wiener case: $Dx = -x''$, $x(0) = 0$, $x'(T) = 0$, $K(t, s) = \min(t, s)$. We have

$$(A_\sigma\xi, \xi) = \frac{(\xi_n - \xi_{n-1})^2}{t_n - t_{n-1}} + \cdots + \frac{(\xi_2 - \xi_1)^2}{t_2 - t_1} + \frac{\xi_1^2}{t_1} \tag{6}$$

and

$$(K_\sigma\xi, \xi) = \sum_{1 \leq i,j \leq n} \min(t_i, t_j)\xi_i\xi_j \tag{7}$$

Thus, as is well known, these quadratic forms are reciprocal to each other, i.e., the corresponding matrices are each others inverse.

EXAMPLE 2. In the Dirichlet case $x(0) = 0$, $x(T) = 0$, $K(t, s) = t(T - s)/T$, $0 \leq t \leq s \leq T$.

$$(A_\sigma\xi, \xi) = \frac{\xi_n^2}{T - t_n} + \frac{(\xi_n - \xi_{n-1})^2}{t_n - t_{n-1}} + \cdots + \frac{(\xi_2 - \xi_1)^2}{t_2 - t_1} + \frac{\xi_1^2}{t_1} \tag{8}$$

$$(K_\sigma \xi, \xi) = \sum_{1 \le i,j \le n} \frac{t_i(T - t_j)}{T} \xi_i \xi_j \tag{9}$$

Let D_D be the set of functions x such that $x'' \in L^2$, satisfying the boundary conditions BC (periodic or separated), equipped with the norm $||x||_{\mathcal{H}}$ defined by

$$||x||^2_{\mathcal{H}} = \int_0^1 p\dot{x}^2 + qx^2 dt \tag{10}$$

Let $(x, y)_{\mathcal{H}}$ denote the corresponding inner product for which $||x||_{\mathcal{H}} = \sqrt{(x,x)_{\mathcal{H}}}$.

LEMMA 1. *Let $\mathcal{H}$ be the completion of D_D. Then we have the continuous injection $\mathcal{H} \hookrightarrow L^2(0,T)$.*

Proof. Let $\mathcal{H}_0$ be the domain D_D equipped with the norm in (10). Then the inclusion $\mathcal{H}_0 \hookrightarrow L^2$ is continuous. Since the operator K is strictly positive there exist constants $k > 0$ and $c > 0$ such that if $x = Kz$, we have, $(Dx, x) = (z, Kz) \ge k(z, z) \ge c(Kz, Kz) = c||x||^2$. To prove that the continuous extension to $\mathcal{H}$ of the injection $\mathcal{H}_0 \hookrightarrow L^2$ is injective we have to prove the following: If $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}_0$ and converges to 0 in the space L^2, then (x_n) goes to zero in $\mathcal{H}_0$. Proof: p being bounded below by a positive constant, $(\dot{x}_n)$ is Cauchy in $L^2(0,T)$ and so converges to $y \in L^2(0,T)$ in the space $L^2(0,T)$ and therefore in the space of distributions $\mathcal{D}'(0,T)$. But since x_n goes to 0 in $\mathcal{D}'(0,T)$ it follows that $\dot{x}_n$ goes to zero in $\mathcal{D}'(0,T)$ which implies that $y = 0$. Then p and q being bounded, (x_n) goes to zero in $\mathcal{H}_0$. □

LEMMA 2. *If $x \in \mathcal{H}$ and $y \in D_D$ then*

$$(x, y)_{\mathcal{H}} = (x, Dy)_{L^2} \tag{11}$$

Proof. For $x \in D_D$ this follows from (3) and both sides of the equation are continuous on $\mathcal{H}$. □

LEMMA 3. *The reproducing operator*[4] *of $\mathcal{H}$ as a subspace of L^2 is the integral operator with kernel K. In particular, we have the continuous inclusion*

$$\mathcal{H} \hookrightarrow C[0,T] \tag{12}$$

[4] We follow [13] p.151 where the reproducing operator is called 'noyau reproduisant'

Proof. For $f \in L^2$ we have $Kf \in D_D$ and $DKf = f$. Thus, by the previous lemma

$$(x, Kf)_{\mathcal{H}} = (x, DKf)_{L^2} = (x, f)_{L^2} \tag{13}$$

□

Now by (12) we have the continuous linear map $\pi_\sigma : \mathcal{H} \longrightarrow \mathbb{R}^\sigma$ defined by $\pi_\sigma(x) = (x(t_1), \ldots, x(t_n))$. We consider the image Hilbert space $\mathcal{H}_\sigma = \pi_\sigma(\mathcal{H})$ which is $\mathbb{R}^\sigma$ equipped with a certain Hilbert structure. If $\xi \in \mathbb{R}^\sigma$ we have

$$||\xi||_{\mathcal{H}_\sigma} = \min_{\pi_\sigma(x)=\xi} ||x||_{\mathcal{H}} = ||x_\sigma||_{\mathcal{H}} \tag{14}$$

the minimum being attained by the unique element $x_\sigma \in \mathcal{H}$ orthogonal to the kernel N_σ of π_σ restricted to $\mathcal{H}$ ([13, p. 176]). If $\varphi \in \mathcal{D}(0,T)$ with support in the complement of σ it is obvious that $D\varphi \in N_\sigma$. Thus $(Dx_\sigma, \varphi) = (x_\sigma, D\varphi) = 0$ which means that $Dx_\sigma = 0$ on the complement of σ, i.e. in between the points t_i. Thus x_σ is the element previously denoted as such. It follows that

$$||\xi||^2_{\mathcal{H}_\sigma} = \sum_{1 \le i,j \le n} A_{ij}\xi_i\xi_j \tag{15}$$

LEMMA 4. *Let $\mathcal{K} = \mathbb{R}^n$ equipped with the norm defined by the positive definite matrix A as in (15). Then the reproducing operator of $\mathcal{K}$ is the inverse matrix $K = A^{-1}$.*

Proof. Let K be the reproducing operator. Then

$$(x, K\xi)_{\mathcal{K}} = < x, \xi >, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \tag{16}$$

the latter bracket denoting the duality between $\mathbb{R}^n$ and its dual $\mathbb{R}^n$, or the standard inner product on $\mathbb{R}^n$. Now $(x,y)_{\mathcal{K}} = < Ax, y > = < x, Ay >$. Thus we have

$$< x, AK\xi > = < x, \xi > \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \tag{17}$$

proving that $K = A^{-1}$. □

The proof of the theorem now follows from the fact that, by the Schwartz calculus of Hilbert subspaces ([13] p. 176, Prop. 21), the reproducing operator of the image $\pi_\sigma(\mathcal{H})$ is $\pi_\sigma K \pi_\sigma^*$. Since $\pi_\sigma^*(\alpha) = \sum \alpha_i \delta_{t_i}$ the corresponding matrix is just $K(t_i, t_j)$. By Lemma 4 this matrix equals A^{-1}.

REMARK. The proof shows that under the assumptions of Theorem 1 the element x_σ is characterized by the relation

$$S(x_\sigma) = \min_{\substack{\pi_\sigma(x)=\xi \\ x \in BC}} S(x) \tag{18}$$

The proof also shows that instead of assuming $q \geq 0$ it is enough to assume that the operator D is strictly positive, i.e. the integral operator associated to the Green function is positive.

2. The harmonic oscillator

Here we have

$$H_\omega x = -x'' - \omega^2 x \tag{1}$$

with $\omega \in \mathbb{R}$ and with Dirichlet boundary conditions.

The operator being in general not positive, the previous analysis does not apply.

If ωT does not belong to $\pi\mathbb{Z}$ the conditions $H_\omega x = 0$, $x(0) = x(T) = 0$ imply $x = 0$, so there exists a Green function for H_ω which is:

$$K^\omega(t,s) = \frac{\sin\omega(T-t)\sin\omega s}{\omega \sin \omega T}, \quad 0 \leq s \leq t \leq T \tag{3}$$

It is a meromorphic function of ω with poles in the set $\pi\mathbb{Z}/T$.

On the other hand if

$$\sin(\omega(t_i - t_{i-1})) \neq 0, \ i = 1, \ldots, n \tag{4}$$

then the continuous function x_σ which satisfies the Dirichlet boundary conditions, the condition $x(t_i) = \xi_i$, $i = 1, \ldots, n$, and on the intervals (t_{i-1}, t_i) the condition $H_\omega x_\sigma = 0$, is well defined. The condition (4) is satisfied for all non-real complex ω. For real ω (4) is satisfied for almost all choices of σ, in particular for those partitions σ such that the modulus $|\sigma|$ satisfies $|\omega||\sigma| < \pi$.

Under those conditions, if we put

$$S_\omega(x) = \frac{1}{2}\int_0^1 \dot{x}^2 - \omega^2 x^2 dt \tag{5}$$

the matrix A_σ^ω such that

$$S_\omega(x_\sigma) = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}^\omega \xi_i \xi_j \tag{6}$$

is well defined and is for fixed σ a holomorphic function on the set Ω of elements ω such that $\omega \notin \pi\mathbb{Z}/T$ and such that the conditions (4) are satisfied.

On the other hand the the Green function (3) is holomorphic on $\complement\pi\mathbb{Z}/T$.

As a consequence we have, if $K_{ij}^\omega = K^\omega(t_i, t_j)$,

THEOREM 2. *Given $\sigma = \{t_1, \ldots, t_n\}$ the matrices A_{ij}^ω and K_{ij}^ω are inverses of each other for all $\omega \in \Omega$.*

Proof. The function $\omega \mapsto A_\sigma^\omega K_\sigma^\omega$ is holomorphic on Ω and equal to the identity matrix I if ω is purely imaginary. Thus, Ω being connected, the identity $A_\sigma^\omega K_\sigma^\omega = I$ holds throughout Ω. □

3. The general case

We now consider a general symmetric Sturm-Liouville operator

$$Dx = -\frac{d}{dt}(p\frac{d}{dt}x) + qx \tag{1}$$

where q is continuous and p is continuously differentiable on $[0, T]$, with $p(t) > 0$ and $q(t) \in \mathbb{R}$ for all $t \in [0, T]$.

As before we assume the boundary condition (1.2, 1.2') is such that $p(T)x(T)x'(T) = p(0)x(0)x'(0)$, so that we have

$$(Dx, x) = \int_0^1 p\dot{x}^2 + qx^2 dt \tag{2}$$

and we assume that the only solution of the homogeneous boundary value problem is zero. This implies that the operator has a well defined Green function K, which defines an invertible self-adjoint integral operator denoted K also.

THEOREM 3. *1. There exists $\delta > 0$ such that if $|\sigma| \leq \delta$ the continuous function x_σ, satisfying the boundary conditions, taking the values $x(t_i) = \xi_i$, $i = 1, \ldots, n$ and satisfying the equation $Dx = 0$ in the intervals (t_{i-1}, t_i), $i = 1, \ldots, n$, is uniquely defined.*

2. If $|\sigma| \leq \delta$ and $A = A_\sigma$ is the matrix such that

$$S(x_\sigma) = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j \tag{3}$$

and K_σ is the matrix $(K(t_i, t_j))$, then A_σ and K_σ are each others inverse.

Proof. Let $S = \{x \in C^1[0,T] : Dx = 0\}$. The space S is two-dimensional, and so its topology coincides with the topology induced by $C^1[0,T]$. We first assume we have separated boundary conditions. For $0 < a < T$ let $N_a = \{x \in S : x(a) = 0\}$. For $a = 0$ or $a = T$ let $N_a = \{x \in S : x \in BC_a\}$ BC_a being the set of functions satisfying the boundary conditions at a. Then, for all $a \in [0,T]$ the space N_a is one-dimensional. □

LEMMA 1. *There exists $\delta > 0$ such that for $a, a' \in [0,T]$ and $0 < |a - a'| \leq \delta$, we have $N_a \cap N_{a'} = \{0\}$.*

Proof. If not there exist sequences a_n and a'_n in $[0,T]$ such that $0 < |a_n - a'_n| \to 0$ and $N_{a_n} \cap N_{a'_n} \neq \{0\}$. Passing to a subsequence we may assume a_n and a'_n converge to a limit a. Let $x_n \in N_{a_n} \cap N_{a'_n} \subset S$ be chosen such that $||x_n||_\infty = 1$. Passing to a subsequence we may assume it converges to $x \in S$, with $||x||_\infty = 1$. We may also assume that $0 < a_n < T$, $0 < a'_n < T$ (if from a certain point on $a_n = 0$ or T the details are somewhat simpler). Then since $\frac{x_n(a'_n) - x_n(a_n)}{a'_n - a_n} = \frac{1}{a'_n - a_n} \int_{a_m}^{a'_n} x'_n(s) ds \to x'(a)$ we have $x'(a) = 0$ and $x(a) = 0$. This is true whether a is an endpoint or not. By the uniqueness of the Cauchy problem this implies that $x = 0$ which gives a contradiction. In the case where $p(0) = p(T)$ with periodic boundary conditions, the proof of the lemma is similar, the interval being replaced by a circle. □

For $0 < a < T$ let $L_a(x) = x(a)$ (cf. (1.2)). In the case of periodic boundary conditions let $L_a(x) = x(a)$ for all a.

LEMMA 2. *Let $0 \leq a < a' \leq T$ be such that $N_a \cap N_{a'} = \{0\}$. Then for any $\xi, \xi' \in \mathbb{R}$ there exists a unique element $x = x_{a,a'} \in S$ such that, $L_a(x) = \xi$, $L_{a'}(x) = \xi'$.*

Proof. The map $x \mapsto (L_a(x), L_{a'}(x)) \in \mathbb{R}^2$ being one-to-one on the two-dimensional space S, it is a bijection. $\square$

LEMMA 3. *Let $\delta > 0$ be as in Lemma 1. Let $\sigma = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T\}$ be a subdivision with $|\sigma| = \max(t_i - t_{i-1}) \leq \delta$. Let $(\xi_1, \ldots \xi_n) \in \mathbb{R}^n$. Then there exists a unique continuous function x_σ satisfying the boundary conditions and such that $x_\sigma(t_i) = \xi_i$, $i = 1, \ldots, n$.*

Proof. We define x_σ to be the function which on the interval $[t_{i-1}, t_i]$ coincides with the solution x_{t_{i-1},t_i} of Lemma 2, taking the value ξ_i at t_i, $i = 1, \ldots, n$. $\square$

This proves the first statement of the theorem.

To prove the second statement we proceed by analytic continuation. Observe that the first statement of theorem is valid if we replace the operator D by $D_\lambda = D - \lambda$ where λ belongs to the resolvent set $\varrho(D)$.[5] Let x_σ^λ be the corresponding function satisfying the boundary conditions and such that $x_\sigma^\lambda(t_i) = \xi_i$. We wish to prove that it is holomorphic in λ. Actually we have:

PROPOSITION. *Let C be a compact subset of the resolvent set $\varrho(D)$. Then there exists $\delta > 0$ such that for $|\sigma| \leq \delta$ and $\lambda \in C$ the function x_σ^λ satisfying the boundary conditions (BC), such that $x_\sigma^\lambda(t_i) = \xi_i$, $i = 1, \ldots, n$, and satisfying the equation $D_\lambda x_\sigma^\lambda = 0$ on the intervals (t_{i-1}, t_i), is well defined. Moreover it is holomorphic in $\lambda \in \overset{\circ}{C}$.*

The proof is almost the same as previously:

We denote $S^\lambda = \{x : D_\lambda x = 0\}$, $N_a^\lambda = \{x \in S^\lambda : L_a(x) = 0\}$.

LEMMA 1'. *Let C be a compact subset of $\varrho(D)$. Then there exists $\delta > 0$ such that $0 < |a - a'| \leq \delta$ and $\lambda \in C$ implies $N_a^\lambda \cap N_{a'}^\lambda = \{0\}$.*

Proof. If there is no such $\delta > 0$ there exist sequences $a_n, a'_n \in [0, T]$, $\lambda_n \in C$ such that $0 < |a_n - a'_n| \to 0$, such that $N_{a_n}^{\lambda_n} \cap N_{a'_n}^{\lambda_n} \neq \{0\}$. We can take $x_n \in N_{a_n}^{\lambda_n} \cap N_{a'_n}^{\lambda_n}$ such that $||x_n||_\infty = 1$ and, using the fact that the solution of the Cauchy problem $x \in S^\lambda$ is obtained as the solution of a Volterra integral equation, depending on the holomorphically on the parameter λ, we may, passing to a subsequence if necessary, assume

[5] We denote $\varrho(D)$ the resolvent set of the self-adjoint operator $D = K^{-1}$.

$\lambda_n \to \lambda$ and $x_n \to x \in S^\lambda$. This leads to a contradiction in the same way as in the proof of Lemma 1. □

LEMMA 2'. *Let $0 \leq a < a' \leq T$ be such that $N_a^\lambda \cap N_{a'}^\lambda = \{0\}$. Then for any $\xi, \xi' \in \mathbb{R}$ there exists a unique element $x = x_{a,a'} \in S^\lambda$ such that, $L_a(x) = \xi$, $L_{a'}(x) = \xi'$.*

The proof is the same as for Lemma 2.

LEMMA 3'. *Let $\delta > 0$ be as in Lemma 1'. Let σ be a subdivision with $|\sigma| \leq \delta$. Let $(\xi_1, \dots \xi_n) \in \mathbb{R}^n$. Let $\lambda \in C \subset \varrho(D)$. Then there exists a unique continuous function x_σ^λ satisfying the boundary conditions and such that $x_\sigma^\lambda(t_i) = \xi_i$, $i = 1, \dots, n$. Moreover, the map $\lambda \mapsto x_\sigma^\lambda \in C[0,T]$ is holomorphic on the interior of C.*

Proof. The proof of the first statement is similar to that of Lemma 3. For the proof of the holomorphy it is sufficient to prove that the maps $\lambda \mapsto x^\lambda_{\sigma|_{[t_{i-1},t_i]}}$ are holomorphic. Since the interval is such that the function x_σ^λ is uniquely defined, for $\lambda \in C$, there is a Green function K_i^λ for this interval with the boundary conditions L_{t_i} and $L_{t_{i-1}}$, which is obviously holomorphic in $\lambda \in \overset{\circ}{C}$. Let y_i be function in $C[t_{i-1}, t_i]$ satisfying the boundary conditions. Then $D_\lambda(x_\sigma^\lambda - y_i) = -D_\lambda(y_i)$ so that $x_\sigma^\lambda = -K_i^\lambda(D_\lambda y_i) + y_i$, which is a holomorphic function of $\lambda \in \overset{\circ}{C}$.□

Now we can finish the proof of Theorem 3. Let C be a compact subset of $\varrho(D)$ such that the interior $\overset{\circ}{C}$ is connected, contains 0 and contains some $\lambda \in \mathbb{R}$ with $\lambda < \min q$. Let $\delta > 0$ be such as in the proposition and let σ be such that $|\sigma| \leq \delta$. For $\lambda \in C$ let K_σ^λ and A_σ^λ be the matrices corresponding to D_λ. Then if $\lambda < \min q$ the matrices K_σ^λ and A_σ^λ are, by Theorem 1, inverse to each other. Since they depend holomorphically on $\lambda \in \overset{\circ}{C}$, it follows that also for $\lambda = 0$ we have $A^0 K^0 = I$.

REMARK. Under the assumptions of Theorem 3, and although the operator D is bounded below, we do not in general have the minimum property (1.18).

In fact, if $\varphi \in \mathcal{D}(0,1)$ has its support disjoint from σ, and is such that $S(\varphi) < 0$, which for a given φ can always be accomplished by taking q appropriately negative, we have $S(x_\sigma + \varphi) = S(x_\sigma) + S(\varphi) < S(x_\sigma)$ while $x_\sigma + \varphi$ satisfies the same boundary conditions and takes the same values at the t_i as x_σ.

4. Gaussian integrals

In this short section we assume that the operator D with its domain D_D defined by the boundary conditions, is ≥ 0.

We denote $M[0,T]$ the set of Radon measures μ on $[0,T]$ and we write $< x, \mu >$ for the integral of the function x with respect to μ. We denote $K\mu$ the function $t \mapsto < K_t, \mu >$, so that $< K\mu, \mu >= \int\int K(t,s)\mu(dt)\mu(ds)$.

THEOREM 4. *Let $K \geq 0$ be the Green operator associated to the operator $D \geq 0$. Then there exists a unique probability measure G_K on $C[0,T]$ such that*

$$\int_{C[0,T]} e^{i<x,\mu>} G_K(dx) = e^{-\frac{1}{2}<K\mu,\mu>}, \quad \mu \in M[0,T]. \tag{1}$$

For $\sigma = \{t_1, \ldots, t_n\}$ the image of G_K under the map π_σ is the Gaussian measure on $\mathbb{R}^n$

$$G_{K_\sigma} = \sqrt{\frac{\det A_\sigma}{(2\pi)^n}} e^{-\frac{1}{2}(A_\sigma \xi, \xi)} d\xi \tag{2}$$

whose covariance kernel is K_σ. Briefly, G_K equals the projective limit

$$G_K = \varprojlim_{\sigma} G_{K_\sigma} \tag{3}$$

Proof. The main thing to prove is that the centered Gaussian process, with covariance K, has continuous paths. Since K is a Green function, it follows from the construction [5 p. 355] that the partial derivatives $\frac{\partial}{\partial t}K(t,s)$ and $\frac{\partial}{\partial s}K(t,s)$ exist for $t \neq s$ and are bounded in absolute value by some number M. This implies that

$$\begin{aligned} |K(s,s) - 2K(t,s) + K(t,t)| &\leq |K(s,s) - K(t,s)| + |K(t,s) - K(t,t)| \\ &\leq 2M|t-s| \end{aligned}$$

for all $t, s \in [0,T]$. The continuity of the paths then follows from the known criteria for continuity cf. [9, Thm 4.1.1, p. 48] or [11, Ch IV Thm 5, p. 172]. The space $C[0,T]$ being separable, this implies the existence of the measure G_K with covariance kernel K. The image under the map π_σ is the Gaussian measure whose covariance kernel is the matrix K_σ such that for $\xi \in \mathbb{R}^n$ one has $< K_\sigma\xi, \xi >=< K\pi_\sigma^t(\xi), \pi_\sigma^t(\xi) >$, obtained by transposing π_σ. □

COROLLARY. *In the case of Dirichlet boundary conditions $x(0) = 0$ or $x(T) = 0$ it follows that G_K is concentrated on the closed subspace of functions $x \in C[0,T]$ vanishing at 0 or at T.*

Proof. If $x(0) = 0$ is one of the boundary conditions we have $K(0,t) = 0$ for all $t \in [0,T]$, which implies that the Hilbert space $\mathcal{H}$ whose reproducing kernel is K, is contained in the closed subspace $C_0[0,T] = \{x \in C[0,T] : x(0) = 0\}$. Then G_K being the image under the injection $\mathcal{H} \hookrightarrow C[0,T]$ of the canonical normal cylinder measure on $\mathcal{H}$, it follows that G_K is concentrated on $C_0[0,T]$, the support of G_K being the closure of $\mathcal{H}$. More simply, if $\ell(x) = x(0)$ the relation $\int < \ell, x >^2 G_K(dx) =< K\ell, \ell >= 0$ implies that G_K is concentrated on the set $\{x : \ell(x) = 0\}$. The reasoning for the other end-point is similar. □

REMARK. From the case of Brownian motion, $Dx = -x''$, $x(0) = 0$, $x'(T) = 0$, we see that there is in general no analogue to this Corollary when the boundary conditions involve a derivative. The above argument is not quite valid with the discontinuous linear form $\ell(x) = x'(T)$

5. Summable distributions

We recall the elements of the theory of summable distributions, adopting the standard notations of the theory of distributions [14]. For more details see [14], [15], [16].

Let $\mathcal{B}(\mathbb{R}^n)$ be the space of functions φ of class C^∞ which are bounded as well as all their derivatives $D^k\varphi$ (k_i times with respect to x_i, $i = 1, \dots, n$). As usual $|k| = k_1 + \cdots + k_n$ denotes the order of differentiation. We put $D^0\varphi = \varphi$. For $\varphi \in \mathcal{B}(\mathbb{R}^n)$ let

$$p_m(\varphi) = \sup_{|k| \le m} ||D^k\varphi||_\infty. \tag{1}$$

Equipped with these seminorms $\mathcal{B}(\mathbb{R}^n)$ naturally becomes a Fréchet space.

A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is *summable* if there exist constants $M \ge 0$ and an integer $m \ge 0$ such that

$$| < T, \varphi > | \le M p_m(\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \tag{2}$$

The smallest possible m will be called the summability order of T.

A distribution of sum-order 0 is just a bounded measure $\mu \in \mathcal{M}_b$. More generally it follows from the estimate (2) and by the Hahn-Banach theorem, that a summable distribution is a finite sum of derivatives of bounded measures:

$$T = \sum_{|k| \leq m} D^k \mu_k \tag{3}$$

This shows that T may be extended to the space $\mathcal{B}(\mathbb{R}^n)$ by putting

$$< T, \varphi >= \sum_{|k| \leq m} < \mu_k, (-1)^{|k|} D^k \varphi > \tag{4}$$

and that the extension has the *bounded convergence property*, i.e. :

If $\varphi_i \rightarrow \varphi$ in the C^∞ topology (convergence uniformly on compact sets for the functions and their derivatives) and if the φ_i remain bounded in the space $\mathcal{B}(\mathbb{R}^n)$ ($\sup_i p_m(\varphi_i) < +\infty$ for all m), then $< T, \varphi_i > \longrightarrow < T, \varphi >$.

In particular, if $\alpha \in \mathcal{D}(\mathbb{R}^n)$ and $\alpha(x) = 1$ for x in the unit ball, if $\alpha_n(x) = \alpha(x/n)$ are the usual cutoff functions, we have, for $\varphi \in \mathcal{B}(\mathbb{R}^n)$

$$< T, \varphi >= \lim_n < T, \alpha_n \varphi > \tag{5}$$

which shows that the extension of T to $\mathcal{B}$ does not depend on the representation (3). We call it the *canonical extension.*

In particular, the total mass $< T, 1 >$ is canonically defined, which accounts for the name *summable distribution.*

It can be shown that conversely, a linear map $T : \mathcal{B}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ having the bounded convergence property, is the canonical extension of a summable distribution, its restriction to $\mathcal{D}$. Thus , we may as well define summable distributions as linear maps having the bounded convergence property. With this definition the notion of summable distribution also makes sense in Banach spaces (In the PhD thesis of E. Cator summable distributions on locally convex spaces are studied cf. [3]).

The following theorem gives a number of characterizations of summable distributions:

THEOREM 5. [Schwartz, 14] *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. The following are equivalent:*

1. T is summable.

2. For all $\alpha \in \mathcal{D}(\mathbb{R}^n)$ the convolution product $\alpha * T$ is a bounded measure.

3. For all $\alpha \in \mathcal{D}(\mathbb{R}^n)$ the convolution product $\alpha * T$ is an L^1 function.

4. $T = \sum_k D^k \mu_k$ is a finite sum of derivatives of bounded measures.

5. $T = \sum_k D^k f_k$ is a finite sum of derivatives of L^1 functions.

A number of the usual operations on bounded measures also make sense for summable distributions. In particular: direct products, images under linear maps, convolution products, Fourier transforms.

For instance the image of the summable distribution $T \in \mathcal{D}'_L(\mathbb{R}^n)$ under the linear map $u : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is the summable distribution $u(T)$ defined for $\varphi \in \mathcal{B}(\mathbb{R}^k)$ by

$$< u(T), \varphi > = < T, \varphi \circ u > \tag{6}$$

We then have

$$\text{sum-order}(u(T)) \leq \text{sum-order(T)} \tag{7}$$

We usually denote the effect of T on $\varphi \in \mathcal{B}$ as in the case of measures by

$$\int \varphi(x) T(dx) \tag{8}$$

A summable distribution T is temperate, its Fourier transform being the continuous function of polynomial growth $\widehat{T}$ defined directly by

$$\widehat{T}(y) = \int e^{-ixy}\, T(dx) \tag{9}$$

where $xy = x_1 y_1 + \cdots + x_n y_n$.

PROPOSITION[6]. *Let A be a invertible symmetric n-by-n matrix and let*
$S(x) = \frac{1}{2} < Ax, x >$. Then the Fresnel distribution e^{iS} is summable and its Fourier transform equals $\sqrt{\frac{(2\pi i)^n}{\det A_\sigma}} \exp(-\frac{i}{2} < Ky, y >)$ where $K = A^{-1}$. More generally, for every polynomial P the product Pe^{iS} is summable.

[6]The fact that $e^{i\pi x^2}$ is summable is observed in [14] p. 271.

Proof. Since $S(x)$ is real the function $E = e^{iS}$ is bounded, hence temperate. The calculation of the Fourier transform, with the help of a diagonalization of A , is elementary. The result shows that the Fourier transform of $\alpha * E$ or $\alpha * PE$ belongs to the Schwartz space $\mathcal{S}$. Hence $\alpha * (PE)$ belongs to $\mathcal{S}$, and a fortiori to L^1. Thus, PE is summable by Schwartz' theorem. □

6. Cylindrical Fresnel distributions

Beside the evaluation maps $\pi_\sigma : C[0,T] \longrightarrow \mathbb{R}^\sigma$ we have, for $\sigma \leq \sigma'$ (i.e. $\sigma \subset \sigma'$) the projection maps

(1) $$\pi_{\sigma\sigma'} : \mathbb{R}^{\sigma'} \longrightarrow \mathbb{R}^\sigma$$

which are such that for $\sigma \leq \sigma'$

(2) $$\pi_\sigma = \pi_{\sigma\sigma'}\pi_{\sigma'}$$

Let $\mathfrak{S}$ be the set of subdivisions $\sigma = \{t_1, \ldots, t_n\}$ of $[0, T]$, with $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$. If $\delta > 0$ let $\mathfrak{S}_\delta = \{\sigma \in \mathfrak{S} : |\sigma| \leq \delta\}$.

DEFINITION A *cylindrical path distribution* is a family of summable distributions $(T_\sigma)_{\sigma\in\mathfrak{S}_\delta}$ such that, for $\sigma \leq \sigma'$, we have the coherence condition

(3) $$T_\sigma = \pi_{\sigma\sigma'}(T_{\sigma'})$$

This notion is very close to the notion of prodistribution developed in [6]. The main difference is that we take $\mathfrak{S}$ or $\mathfrak{S}_\delta$ as index set, rather than the set of finite-dimensional quotients. Also we assume that the finite dimensional marginals are summable distributions. This implies, but is not equivalent to, the property that their Fourier transform is a continuous function of polynomial growth (cf. [6], p. 61).

If the T_σ are Fresnel distributions we shall call $(T_\sigma)_{\sigma\in\mathfrak{S}_\delta}$ a cylindrical Fresnel distribution.

THEOREM 6. *Under the assumptions of Theorem 3, there exists a unique cylindrical Fresnel distribution F_K such that one has*

(8) $$\int_{C[0,T]} e^{i<x,\mu>} F_K(dx) = e^{-\frac{i}{2}<K\mu,\mu>}, \quad \mu \in M[0,T]$$

at least if μ is a finite linear combination of point-masses.

For $\sigma = \{t_1, \ldots, t_n\}$ and $|\sigma| \le \delta$, the corresponding marginal distribution is the Fresnel distribution on $\mathbb{R}^\sigma$

$$F_{K_\sigma} = C_\sigma e^{iS_\sigma(\xi)} d\xi \tag{9}$$

where $S_\sigma(\xi) = \frac{1}{2}(A_\sigma \xi, \xi)$ is the stationary value of the action $S(x)$ for paths satisfying the boundary condition and such that $\pi_\sigma(x) = \xi$, and $C_\sigma = \sqrt{\frac{\det A_\sigma}{(2\pi i)^n}}$.

Proof. Given the preceding work it is sufficient to observe that a family $(T_\sigma)_\sigma$ is compatible iff the Fourier transforms $\widehat{T}_\sigma$ are restrictions of each other. □

REMARK. It can be shown that the sum-orders of the distributions F_{K_σ} with $\sigma = \{t_1, \ldots, t_n\}$, are at least $n/2$ and so unbounded [17].[7] This implies that one cannot define the Feynman integral as a summable distribution on the Banach space $E = C[0,T]$, i.e. as a functional on the space $\mathcal{B}(E)$, having the bounded convergence property.

Thus the problem remains to define an appropriate subspace $\mathcal{B}_{(\infty)}(E) \subset \mathcal{B}(E)$, containing the cylinder functions $\varphi \circ \pi_\sigma$ with $\varphi \in \mathcal{B}(\mathbb{R}^\sigma)$, on which cylindrical distributions such as F_K, but not only Fresnel distributions, are defined. One would expect $e^{i<x,\mu>}$ to belong to this space, so that formula (8) makes sense for all $\mu \in M[0,T]$. More generally, experience with the discrete time situation [17], leads one to expect the space to contain superpositions, $\Phi(x) = \int e^{i<x,\mu>}\Omega(d\mu)$, where Ω is a measure on $M[0,T]$ having exponential moments. This gives a link with the work of Albeverio Hoegh-Krohn [1].

7. Feynman's notation

Given the fact that the cylinder distribution F_K has as marginal distributions the functions $c_\sigma e^{iS_\sigma}$ where S_σ is a discrete version of the action functional $S(x) = \frac{1}{2}\int_0^T p\dot{x}^2 + qx^2 dt$ it seems natural to indicate the integral (6-8) as

$$\int_{P[0,T]} e^{iS(x)+i<\mu,x>}\mathcal{D}(dx) = e^{-\frac{i}{2}<K\mu,\mu>} \tag{1}$$

[7] F.Bijma, in her Undergraduate Thesis [2], has shown the exact summability order to be $n+1$.

where $P[0,T]$ denotes, in view of the result for Gaussian measures, the closure in $C[0,T]$ of the set of paths satisfying the boundary conditions.

In the case of the harmonic oscillator we get, on the interval $[t_a, t_b]$ instead of $[0,T]$ and $T = t_b - t_a$,

$$S(x) = \frac{m}{2}\int_{t_a}^{t_b} \dot{x}^2 - \omega^2 x^2 dt \tag{2}$$

$$\int_{P[t_a,t_b]} e^{iS(x)+i<\mu,x>}\mathcal{D}(dx) = e^{-\frac{i}{2}<K\mu,\mu>} \tag{3}$$

with, according to (2-3)

$$\frac{1}{2} < K\mu, \mu >= \iint\limits_{t_a<s<t<t_b} \frac{\sin\omega(t_b - t)\sin\omega(s - t_a)}{m\omega\sin\omega T}\mu(s)\mu(t)dtds \tag{4}$$

in conformity with the calculation in [12] p. 38, (6.42), with $\mu = e$, and where $x_a = 0$, $x_b = 0$, $P[t_a, t_b]$ being the set of paths beginning and ending at 0, i.e., subject to Dirichlet boundary conditions.

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