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INTEGRATION STRUCTURES FOR THE OPERATOR-VALUED FEYNMAN INTEGRAL

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Abstract. The analytic in mass operator-valued Feynman integral is related to integration with respect to unbounded set functions formed from the semigroups obtained by analytic continuation of the heat semigroup and the spectral measure of multiplication by characteristic functions.

Introduction

The purpose of this paper is to take the word "integral" in the expression "Feynman integral" seriously. This is not to say that any other interpretation of the "Feynman integral" should not be taken seriously just because it might not follow the prescription outlined here. Rather, the purpose here is to explore how a mathematical interpretation of the idea of a "Feynman integral" might interact with the subject of *integration theory* that lies at the core of mathematical analysis developed in this century. In view of the connections between functional integration and geometry that have been intimated over the last decade, one might expect that the key ideas of a successful integration theory of Feynman integrals in the realm of nonrelativistic quantum physics would also apply to a mathematical treatment of functional integrals arising in, say, "topological quantum field theory". However, this paper make no claims as to just what these key ideas might be. Due to the singular nature of Feynman integrals, an alternative mathematical viewpoint is suggested by distribution theory.

Integration theory has come to mean the integration of measurable functions defined on a set Ω with respect to an additive set function defined on a collection of subsets of Ω. The distinguishing feature of

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the set functions associated with the Feynman integral of nonrelativistic quantum mechanics is that they are *unbounded* on the underlying algebra of cylinder sets, even if they are analytically continued via a complex factor of time.

This was first noticed by R.H. Cameron [2] and Yu. L. Daletskii [4]. The calculation is as follows. Suppose that $t > 0$ and C^t is the collection of all continuous functions $\omega : [0,t] \to \mathbb{R}$. Set $X_s(\omega) = \omega(s)$ for every $\omega \in C^t$ and $0 \le s \le t$. Let $\phi, \psi \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Re\lambda > 0$. Then for every cylinder set

$$E = \{X_0 \in B_0,\ X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n,\ X_t \in B\ \} \tag{1}$$

with $0 < t_1 < \cdots < t_n < t$, set

$$\begin{aligned} &\mu^t_{\lambda,\phi,\psi}(E) \\ &= C_{n+1}\lambda^{\frac{d(n+1)}{2}} \int_B \int_{B_n} \cdots \int_{B_1} \int_{B_0} \overline{\psi}(x_{n+1}) e^{-\frac{\lambda|x_{n+1}-x_n|^2}{2(t-t_n)}} e^{-\frac{\lambda|x_n - x_{n-1}|^2}{2(t_n-t_{n-1})}} \\ &\qquad \cdots e^{-\frac{\lambda|x_2-x_1|^2}{2(t_2-t_1)}} e^{-\frac{\lambda|x_1-x_0|^2}{2t_1}} \phi(x_0)\, dx_0 dx_1 \cdots dx_n dx_{n+1}. \end{aligned} \tag{2}$$

Here $C_{n+1} = (2\pi(t-t_n))^{-d/2} \ldots (2\pi t_1)^{-d/2}$. Then $\mu^t_{\lambda,\phi,\psi}$ defines an additive set function on the algebra $\mathcal{S}^t$ generated by all cylinder sets E of the form (1) as the times $0 \le t_1 < \cdots < t_n \le t$ vary, the Borel subsets $B_0, \ldots, B_n, B$ of $\mathbb{R}$ vary, and the index $n = 1, 2, \ldots$ varies. Here additivity means that if $E \in \mathcal{S}^t$ and $F \in \mathcal{S}^t$ are disjoint sets, then

$$\mu^t_{\lambda,\phi,\psi}(E \cup F) = \mu^t_{\lambda,\phi,\psi}(E) + \mu^t_{\lambda,\phi,\psi}(F).$$

In the limiting case with $\Re\lambda = 0$, the integrals (2) converge as improper iterated integrals.

Now fix the times $t_1, \ldots, t_n$ and consider the algebra $\mathcal{S}^{t_1,\ldots,t_n}$ generated by the sets of the form (1), just as the Borel subsets $B_0, \ldots, B_n, B$ of $\mathbb{R}$ vary. A calculation shows that the total variation $\|\mu^t_{\lambda,\phi,\psi}\|_{\mathcal{S}^{t_1,\ldots,t_n}}$ of $\mu^t_{\lambda,\phi,\psi}$ over the algebra $\mathcal{S}^{t_1,\ldots,t_n}$ is given by

$$\begin{aligned} &\|\mu^t_{\lambda,\phi,\psi}\|_{\mathcal{S}^{t_1,\ldots,t_n}} \\ &= \left(\frac{\Re\lambda}{2\pi t}\right)^{d/2} \left(\frac{|\lambda|}{\Re\lambda}\right)^{\frac{d(n+1)}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\psi(x_1)| e^{-\frac{\Re\lambda|x_1-x_0|^2}{2t}} |\phi(x_0)|\, dx_0 dx_1. \end{aligned} \tag{3}$$

The total variation $\|\mu^t_{\lambda,\phi,\psi}\|$ of $\mu^t_{\lambda,\phi,\psi}$ over the whole algebra $\mathcal{S}^t$ is necessarily greater than the total variation $\|\mu^t_{\lambda,\phi,\psi}\|_{\mathcal{S}^{t_1,\ldots,t_n}}$ for any choice of times $0 < t_1 < \cdots < t_n < t$ and any $n = 1, 2, \ldots$. Hence, if $|\lambda| > \Re\lambda$ then either $\|\mu^t_{\lambda,\phi,\psi}\| = +\infty$, or $\|\mu^t_{\lambda,\phi,\psi}\| = 0$, the last case occurring when either ϕ or ψ is zero almost everywhere. In the case $\Im\lambda \neq 0$, the additive set function $\mu^t_{\lambda,\phi,\psi}$ is highly singular.

Integration with respect to unbounded set functions lies disguised in many subjects of modern analysis. For example, if Q is the spectral measure associated with the position operator in quantum mechanics of a particle moving on the line and P is the spectral measure of the momentum operator D, then the operator-valued set function $m : A \times B \mapsto Q(A)P(B)$ is unbounded on the algebra $\mathcal{E}$ generated by products $A \times B$ of Borel subsets of $\mathbb{R}$. Let $\hat{f}$ be the Fourier-Plancherel transform of $f \in L^2(\mathbb{R})$. For $\phi, \psi \in L^2(\mathbb{R})$, we have

$$\langle m(A\times B)\phi, \psi\rangle = (2\pi)^{-1} \lim_{n\to\infty} \int_{A\cap[-n,n]} \left(\int_{B\cap[-n,n]} e^{ixy}\hat{\phi}(y)\, dy \right) \overline{\psi}(x)\, dx.$$

Let $m_{\phi,\psi}(A \times B) = \langle m(A \times B)\phi, \psi\rangle$ for all Borel subsets A, B of $\mathbb{R}$. The total variation $\|m_{\phi,\psi}\|$ of the set function $m_{\phi,\psi}$ on the algebra $\mathcal{E}$ is $\|m_{\phi,\psi}\| = \|\hat{\phi}\|_1 \|\psi\|_1/(2\pi)$. The functions ϕ and ψ are just elements of $L^2(\mathbb{R})$, so $\|m_{\phi,\psi}\|$ may be infinite. Nevertheless, m-integrable functions $a : \mathbb{R}^2 \to \mathbb{C}$ correspond to bounded pseudodifferential operators $a(x, D)$ acting on $L^2(\mathbb{R})$. The subject of harmonic analysis on phase space and the Weyl functional calculus [5] is intimately connected with the mathematics of quantum theory.

It is worth noting in the present context that the bounded pseudodifferential $a(x, D)$ operator has two complementary representations—the traditional one of real variable harmonic analysis via distribution theory and another one as a bona-fide integral $a(x, D) = \int_{\mathbb{R}^2} a(x,\xi)\, m(dx, d\xi)$ [9]. The same may be said of the Feynman integral in quantum mechanics.

The variation of $m_{\phi,\psi}$ is finite on compact product sets, but the variation of the additive set function $\mu^t_{\lambda,\phi,\psi}$ on cylinder sets takes only the values $+\infty$ and 0 if $\Im\lambda \neq 0$. The variation of $m_{\phi,\psi}$ on compact subsets of $\mathbb{R}^2$ may be used to control the convergence of integrable functions, an approach precluded by the singularity of the additive set functions $\mu^t_{\lambda,\phi,\psi}$. In Section 2, analytic continuation in the parameter λ from the case $\Im\lambda = 0$ in which $\mu^t_{\lambda,\phi,\psi}$ is a σ-additive measure is used to control the convergence of integrable functions. This connects with the analytic in mass operator-valued Feynman integral studied by E. Nelson [18] and R. Cameron and D. Storvick [3] and given in Section 1.

The reinterpretation of the Feynman-Kac formula in terms of an integral with respect to operator-valued set functions associated with the unperturbed semigroup is not new [13-17], [7]. A treatment of the case of matrix semigroups is given in [20, 21]. I. Kluvánek developed a general integration theory [15] to treat a large class of integrals with respect to unbounded set functions, such as conditionally summable sequences and spectral resolutions. The integrals below connected with the analytic in mass operator-valued Feynman integral do not quite fit into Kluvánek's scheme in [15], but they are closely related. As a point of orientation, the ideas outlined in this paper combine the elements of the viewpoints of E. Nelson [18], R. Cameron and D. Storvick [3] and I. Kluvánek [13]. The treatment of additive functionals of Brownian motion mentioned below is related to recent work of S. Albeverio, G. Johnson, and Z. Ma [1]. Further connections are elaborated in the book [7]. Many of the issues in operator theory mentioned in passing below are discussed in greater detail in the monograph [11].

1. Operator-valued Feynman integral

Fix a positive integer d and a time $t > 0$. Set $C^t = C([0,t], \mathbb{R}^d)$. Let w be Wiener measure on the space C_0^t of continuous functions $\omega : [0,t] \to \mathbb{R}^d$ for which $\omega(0) = 0$. The analytic in mass operator-valued Feynman integral of a function $F : C^t \to \mathbb{C}$ is defined in the following fashion.

Suppose that for each $\lambda > 0$ and $\xi \in \mathbb{R}$, the function $\omega \mapsto F(\lambda^{-1/2}\omega + \xi)$, $\omega \in \mathbb{C}_0^t$, is w-measurable and there exists an operator $K_\lambda^t(F) \in \mathcal{L}(L^2(\mathbb{R}^d))$ such that for every $\psi \in L^2(\mathbb{R}^d)$, the function

$$\omega \mapsto F(\lambda^{-1/2}\omega + \xi)\psi(\lambda^{-1/2}\omega(t) + \xi), \quad \omega \in C_0^t,$$

is w-integrable for almost all $\xi \in \mathbb{R}^d$ and the equality

$$(4) \qquad (K_\lambda^t(F)\psi)(\xi) = \int_{C_0^t} F(\lambda^{-1/2}\omega + \xi)\psi(\lambda^{-1/2}\omega(t) + \xi)\, dw(\omega)$$

holds for almost all $\xi \in \mathbb{R}^d$. If $\lambda \mapsto K_\lambda^t(F)$, $\lambda > 0$, is the restriction to the positive real axis of an $\mathcal{L}(L^2(\mathbb{R}^d))$-valued function analytic in the region $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$, then we call the operator $K_{-iq_0}^t(F) = \lim_{\lambda \to -iq_0} K_\lambda^t(F)$ the *analytic in mass operator-valued Feynman integral of F with parameter $-iq_0 \neq 0$* if the limit exists in the strong operator topology. Other types of limits are possible. For example, a non-tangential limit as $\lambda \to -iq_0$, or the limit through a wedge with apex $-iq_0$ is possible.

Let $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ be the selfadjoint Laplacian acting in $L^2(\mathbb{R}^d)$ and set $S_\lambda(t) = e^{t\Delta/(2\lambda)}$ for all $t \geq 0$ and $\lambda > 0$. The exponential is defined by the functional calculus for selfadjoint operators. Then for each $\lambda > 0$, the C_0-semigroup S_λ is given by

$$(S_\lambda(t)\phi)(x) = \left(\frac{\lambda}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{\lambda}{2t}|x-y|^2} \phi(y)\, dy \quad a.e. \tag{5}$$

for every $\phi \in L^2(\mathbb{R}^d)$ and $t > 0$.

Let $Q(B) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator of multiplication by the characteristic function of the Borel subset B of $\mathbb{R}^d$. For a bounded Borel measurable function f, the operator of multiplication by f is written as $Q(f)$. The operator $Q(f)$ is actually the integral $\int_{\mathbb{R}^d} f\, dQ$ of the bounded function f with respect to the spectral measure $Q : B \mapsto Q(B)$.

For the cylinder set

$$E = \{\omega \in C^t : \omega(t_1) \in B_1, \ldots, \omega(t_n) \in B_n\} \tag{6}$$

with $0 \leq t_1 < \cdots < t_n \leq t$, set $M_\lambda^t(E)$ equal to the operator

$$S_\lambda(t - t_n)Q(B_n)S_\lambda(t_n - t_{n-1}) \cdots Q(B_1)S_\lambda(t_1). \tag{7}$$

The algebra generated by all such cylinder sets E as the times $0 \leq t_1 < \cdots < t_n \leq t$, the Borel sets $B_1, \ldots, B_n$ and the positive integer n vary, is denoted by $\mathcal{S}^t$. Then $M_\lambda^t : E \mapsto M_\lambda^t(E)$ is well defined and has a unique additive extension to the algebra $\mathcal{S}^t$ of subsets of C^t. Moreover, for each $\lambda > 0$, the additive set function M_λ^t is the restriction to $\mathcal{S}^t$ of a unique $\mathcal{L}(L^2(\mathbb{R}^d))$-valued measure defined on the σ-algebra $\sigma(\mathcal{S}^t)$ generated by $\mathcal{S}^t$. We call M_λ^t the (S_λ, Q)-measure on $\sigma(\mathcal{S}^t)$.

To verify that such a measure exists, we write it in terms of the integral (4). Let $f_1, \ldots, f_n$ be bounded complex-valued Borel measurable functions on $\mathbb{R}^d$. Let $0 < t_1 < \cdots < t_n < t$ be n distinct times before t.

Set $F(\omega) = f_1(\omega(t_1)) \cdots f_n(\omega(t_n))$ for every $\omega \in C^t$. Then

$$
\begin{aligned}
&(K^t_\lambda(F)\psi)(\xi) \\
&= \int_{C_0^t} F(\lambda^{-1/2}\omega + \xi)\psi(\lambda^{-1/2}\omega(t) + \xi)\, dw(\omega) \\
&= \int_{C_0^t} f_1(\lambda^{-1/2}\omega(t_1) + \xi) \cdots f_n(\lambda^{-1/2}\omega(t_n) + \xi)\psi(\lambda^{-1/2}\omega(t) + \xi)\, dw(\omega) \\
&= C_{n+1} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_1(\lambda^{-1/2}x_1 + \xi) \cdots f_n(\lambda^{-1/2}x_n + \xi)\psi(\lambda^{-1/2}x_{n+1} + \xi) \\
&\qquad \times e^{\frac{-|x_{n+1}-x_n|^2}{2(t-t_n)}} e^{\frac{-|x_n-x_{n-1}|^2}{2(t_n-t_{n-1})}} \cdots e^{\frac{-|x_2-x_1|^2}{2(t_2-t_1)}} e^{\frac{-|x_1|^2}{2t_1}}\, dx_1 \cdots dx_{n+1} \\
&= C_{n+1}\lambda^{\frac{d(n+1)}{2}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_1(x_1) \cdots f_n(x_n)\psi(x_{n+1}) \\
&\qquad \times e^{-\frac{\lambda|x_{n+1}-x_n|^2}{2(t-t_n)}} e^{-\frac{\lambda|x_n-x_{n-1}|^2}{2(t_n-t_{n-1})}} \cdots e^{-\frac{\lambda|x_2-x_1|^2}{2(t_2-t_1)}} e^{-\frac{\lambda|x_1-\xi|^2}{2t_1}}\, dx_1 \cdots dx_{n+1}.
\end{aligned}
$$

Here $C_{n+1} = (2\pi(t-t_n))^{-d/2} \ldots (2\pi t_1)^{-d/2}$. On the other hand, if $\phi \in L^2(\mathbb{R}^d)$, then

$$
\begin{aligned}
&\left(\left(\int_{C^t} F(\omega)\, M^t_\lambda(d\omega)\right)\phi\right)(\xi) \\
&= \left(\int_{C^t} F(\omega)\,(M^t_\lambda\phi)(d\omega)\right)(\xi) \\
&= \left(\int_{C^t} f_1(\omega(t_1)) \cdots f_n(\omega(t_n))\,(M^t_\lambda\phi)(d\omega)\right)(\xi) \\
&= (S_\lambda(t-t_n)Q(f_n)S_\lambda(t_n - t_{n-1}) \cdots Q(f_1)S_\lambda(t_1)\phi)\,(\xi) \\
&= C_{n+1}\lambda^{\frac{d(n+1)}{2}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f_1(x_1) \cdots f_n(x_n) \\
&\qquad \times e^{-\frac{\lambda|\xi-x_n|^2}{2(t-t_n)}} e^{-\frac{\lambda|x_n-x_{n-1}|^2}{2(t_n-t_{n-1})}} \cdots e^{-\frac{\lambda|x_2-x_1|^2}{2(t_2-t_1)}} e^{-\frac{\lambda|x_1-x|^2}{2t_1}}\, \phi(x)\, dx dx_1 \cdots dx_n.
\end{aligned}
$$

The integral with respect to $M^t_\lambda\phi$ is to be interpreted in the sense of approximation by cylinder functions. Comparison between the expressions above shows that

$$\left\langle \left(\int_{C^t} F(\omega)\, M^t_\lambda(d\omega)\right)\phi, \overline{\psi} \right\rangle = \langle K^t_\lambda(F)\psi, \overline{\phi}\rangle = \langle \phi, K^t_\lambda(\overline{F})\overline{\psi}\rangle$$

Hence, $\int_{C^t} F(\omega)\, M^t_\lambda(d\omega) = K^t_\lambda(\overline{F})^*$. The σ-additive extension of M^t_λ from $\mathcal{S}^t$ to $\sigma(\mathcal{S}^t)$ is equal to the operator-valued measure $A \mapsto K^t_\lambda(\chi_A)^*$

for all $A \in \sigma(\mathcal{S}^t)$. As is customary, the unique extension is denoted, again, by M_λ^t. To check that the set function

$$A \mapsto K_\lambda^t(\chi_A)^*, \quad A \in \sigma(\mathcal{S}^t)$$

is actually a measure for each $\lambda > 0$, inspection of (4) shows that

$$\begin{aligned}
&\|K_\lambda^t(\chi_A)^*\phi\|_2 \\
&= \sup_{\|\psi\|_2 \le 1} \left| \int_{\mathbb{R}^d} \left(\int_{C_0^t} \chi_A(\lambda^{-1/2}\omega + \xi)\overline{\psi}(\lambda^{-1/2}\omega(t) + \xi)\, dw(\omega) \right) \phi(\xi)\, d\xi \right| \\
&\le \sup_{\|\psi\|_2 \le 1} \int_{C_0^t} \left(\int_{\mathbb{R}^d} \chi_A(\lambda^{-1/2}\omega + \xi)|\psi(\lambda^{-1/2}\omega(t) + \xi)|\,|\phi(\xi)|\, d\xi \right) dw(\omega).
\end{aligned}$$

Each semigroup S_λ, $\lambda > 0$, has a unique analytic extension from the set of all positive real numbers λ to the set $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$. Formula (5) is valid for all $\lambda \in \mathbb{C}_+$ provided we take the branch of $\lambda \mapsto \lambda^{1/2}$ such that $\Re(\lambda^{1/2}) > 0$ on $\mathbb{C}_+$. Then the operator-valued function $\lambda \mapsto M_\lambda^t(E)$, $\lambda > 0$, has a unique analytic extension to $\mathbb{C}_+$ for each cylinder set (6). In this fashion, we obtain a family $\langle M_\lambda^t \rangle_{\lambda \in \mathbb{C}_+}$ of additive operator-valued functions $M_\lambda^t : \mathcal{S}^t \to \mathcal{L}(L^2(\mathbb{R}^d))$.

Comparison with formula (2) shows that $\langle M_\lambda^t(E)\phi, \psi\rangle = \mu_{\lambda,\phi,\psi}^t(E)$ for each cylinder set $E \in \mathcal{S}^t$. As mentioned in the Introduction, if $\Im\lambda \ne 0$ then $\|\mu_{\lambda,\phi,\psi}^t\| = +\infty$ for any nonzero elements ϕ and ψ of $L^2(\mathbb{R})$, so the collection

$$\{M_\lambda^t(A) : A \in \mathcal{S}^t\}$$

of bounded linear operators is *unbounded* in the operator norm. Note here that $\mathcal{S}^t$ is the *algebra* generated by cylinder sets (6). Each operator (7) is a contraction on $L^2(\mathbb{R}^d)$ for $\lambda \in \mathbb{C}_+$, but the collection of all cylinder sets (6) is only a *semi-algebra.*

2. Integration

Integration with respect to the family $\langle M_\lambda^t \rangle_{\lambda>0}$ of operator-valued measures is used to control the convergence of integrals with respect to the set functions $\langle M_\lambda^t \rangle_{\lambda \in \mathbb{C}_+}$. First we have to make precise the idea of integrating with respect to a family of operator-valued measures.

The space $L^1(\langle M_\lambda^t \rangle_{\lambda>0})$ of equivalence classes of functions integrable with respect to each operator-valued measure M_λ^t, $\lambda > 0$, is equipped with a natural locally convex topology with respect to which it is a sequentially complete lcs. The completeness is a consequence of the operator-valued measures M^t_λ and M^t_ν having disjoint support for all $\lambda > 0$ and $\nu > 0$ such that $\lambda \neq \nu$, that is, the operator-valued measures live on spaces of paths with distinct quadratic variation according to a result of P. Lévy. The measurability of functions belonging to $L^1(\langle M^t_\lambda\rangle_{\lambda>0})$ is precisely the scale-invariant measurability studied in [10]. The seminorms defining the topology of $L^1(\langle M^t_\lambda\rangle_{\lambda>0})$ are given by

$$f \mapsto \sup\left\{\int_{C^t} |f|\, d\mu^t_{\lambda,|\phi|,|\psi|} : \psi \in L^2(\mathbb{R}^d),\ \|\psi\|_2 \leq 1\right\}$$

for every $\phi \in L^2(\mathbb{R}^d)$ and $\lambda > 0$.

Suppose that $H(\lambda)$ is the quadratic form sum of $-\Delta/(2\lambda)$ and V. Then

$$e^{-H(\lambda)t} = \int_{C^t} e^{-\int_0^t V(\omega(s)\, ds}\, M^t_\lambda(d\omega)$$

for all $t \geq 0$. This is a reinterpretation of the Feynman-Kac formula [19, Theorem X.68] in terms of operator-valued measures. The proof is by now standard: first establish the equality for V bounded and then use monotone convergence for quadratic forms on the left and monotone convergence for integrals on the right.

How can we integrate with respect to the operator-valued set functions $M^t_\lambda : \mathcal{S}^t \to \mathcal{L}(L^2(\mathbb{R}^d))$ in the case that $\Im\lambda \neq 0$? As is usual in integration theory, one starts with *simple functions*, in this case, a finite linear combination $s = \sum_{j=1}^k c_k \chi_{E_j}$ of characteristic functions of sets $E_j \in \mathcal{S}^t$, for $j = 1, \ldots, k$. Then linearity gives

$$\int_{C^t} s\, dM^t_\lambda = \sum_{j=1}^k c_k M^t_\lambda(E_k).$$

One idea is to give the topology on the space of simple functions s based on the algebra $\mathcal{S}^t$ so that a net $\langle s_\alpha\rangle_{\alpha\in A}$ of simple functions converges to a function f if and only if it converges to f in the quasi-complete space $L^1(\langle M^t_\lambda\rangle_{\lambda>0})$ and the net is also Cauchy with respect to the seminorms

$$p_{E,K,\phi} : s \mapsto \sup_{\lambda\in K}\left\|\int_E s(\omega)\,(M^t_\lambda\phi)(d\omega)\right\|_2 \tag{8}$$

as E varies over cylinder sets (6), the function ϕ varies over $L^2(\mathbb{R}^d)$ and K varies over compact subsets of $\mathbb{C}_+$. Then we can *define*

$$\int_E f(\omega)\,(M_\lambda^t\phi)(d\omega) := \lim_{\alpha\in A}\int_E s_\alpha(\omega)\,(M_\lambda^t\phi)(d\omega)$$

so that the convergence is uniform in the strong operator topology as λ varies over compact subsets of $\mathbb{C}_+$. It follows that the operator-valued function

$$(E,\lambda)\mapsto \int_E f(\omega)\,(M_\lambda^t\phi)(d\omega)$$

is additive in $E\in\mathcal{S}^t$ and analytic in $\lambda\in\mathbb{C}_+$.

For the situation of interest—quantum mechanics—λ is interpreted as $-i$ times a mass parameter m. It is not unreasonable to expect that the dynamics of a quantum system should exhibit continuous dependence upon nonzero (and positive) mass. Then analytic continuation in λ from the boundary values on $(i\mathbb{R})\setminus\{0\}$ to $\mathbb{C}_+$ can be achieved by the Poisson integral formula.

Let us look at two possibilities for going from the set $\mathbb{C}_+$ to the boundary $\partial\mathbb{C}_+ = i\mathbb{R}$ of $\mathbb{C}_+$. Let $H(\mathbb{C}_+)$ denote the space of all functions which are analytic in $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}}_+\setminus\{0\}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}_+$. The space $H(\mathbb{C}_+)$ endowed with the topology of uniform convergence on compact subsets of $\overline{\mathbb{C}}_+\setminus\{0\}$ is written as $\overline{H}(\mathbb{C}_+)$. The two locally convex spaces $\overline{H}(\mathbb{C}_+)$ and $H(\mathbb{C}_+)$ have the same underlying sets, only the topologies differ. The space $\overline{H}(\mathbb{C}_+)$ is complete and metrisable. Although $H(\mathbb{C}_+)$ is metrisable, it is not a complete locally convex space.

Of course, we could equally use $\lambda=-im$ with some smaller interval I of the mass parameter m, say, all positive real values. Then we would look at analytic functions in $\mathbb{C}_+$ with continuous boundary values on $-iI$.

We consider two types of integrability for a function $f: C^t\to\mathbb{C}$.

[$H(\mathbb{C}_+)$] the function $\lambda\mapsto\int_E f(\omega)\,(M_\lambda^t\phi)(d\omega)$ defined above belongs to $H(\mathbb{C}_+)$ for each $E\in\mathcal{S}^t$ and all $\phi\in L^2(\mathbb{R}^d)$;

[$\overline{H}(\mathbb{C}_+)$] the net $\langle s_\alpha\rangle_{\alpha\in A}$ of $\mathcal{S}^t$-simple functions mentioned above converges to f in the quasicomplete space $L^1(\langle M_\lambda^t\rangle_{\lambda>0})$ and the net is also Cauchy with respect to the seminorms $p_{E,K,\phi}$ defined in formula (8) as E varies over cylinder sets (2), the function ϕ varies over $L^2(\mathbb{R}^d)$ and K varies over compact subsets of $\overline{\mathbb{C}}_+\setminus\{0\}$.

We now look at the distinction between $H(\mathbb{C}_+)$-integrability and $\overline{H}(\mathbb{C}_+)$-integrability. Every $\overline{H}(\mathbb{C}_+)$-integrable function is $H(\mathbb{C}_+)$-integrable because the function

$$\lambda \mapsto \int_E f(\omega)\,(M_\lambda^t \phi)(d\omega) \tag{9}$$

is the uniform limit of functions

$$\lambda \mapsto \int_E s_\alpha(\omega)\,(M_\lambda^t \phi)(d\omega), \quad \alpha \in A,$$

on compact subsets of $\overline{\mathbb{C}}_+ \setminus \{0\}$, with $\langle s_\alpha \rangle_{\alpha \in A}$ a net of simple functions. Hence, (9) is analytic in $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}}_+ \setminus \{0\}$. A few examples of functions integrable in the above senses follow.

$\overline{H}(\mathbb{C}_+)$-integrable functions

i) $F : \omega \mapsto f_1(\omega(t_1)) \cdots f_n(\omega(t_n))$, $\omega \in C^t$, with $f_1, \dots, f_n$ bounded and Borel measurable on $\mathbb{R}^d$.

ii) $e^{-i\int_0^t V \circ X_s\, ds}$ with $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, $p > d/2$ for $d \geq 3$.

In Example (2), we have the representation

$$\begin{aligned} e^{-itH(m)} &= \int_{C^t} e^{-i\int_0^t V \circ X_s\, ds}\, dM_\lambda^t \\ &= K_{-\lambda}^t \left(e^{i\int_0^t V \circ X_s\, ds}\right)^*, \quad \lambda = -im,\ m \in \mathbb{R},\ m \neq 0, \end{aligned}$$

relative to the operator $H(m) = -\Delta/(2m) + V$. Here $X_s : \omega \mapsto \omega(s)$ for all $\omega \in C^t$ and $0 \leq s \leq t$.

To check what is involved in proving $\overline{H}(\mathbb{C}_+)$-integrability for the function i) and ii) above, the calculations are given below.

Proof of $\overline{H}(\mathbb{C}_+)$-integrability. i) Let $s_{k,j} : \mathbb{R}^d \to \mathbb{C}$ be Borel measurable simple functions defined for all $k = 1, 2, \dots$ and $j = 1, \dots, n$, such that $\|s_{k,j}\|_\infty \leq \|f_j\|_\infty$ for all $k = 1, 2 \dots$, and with the property that for each $x \in \mathbb{R}^d$ and $j = 1, \dots, n$, we have $s_{k,j}(x) \to f_j(x)$ as $k \to \infty$. Then by dominated convergence $(s_{k,1} \circ X_{t_1}) \cdots (s_{k,n} \circ X_{t_n}) \to F$ in $L^1(\langle M_\lambda^t \rangle_{\lambda > 0})$ as $k \to \infty$.

We need to look at the convergence in $L^2(\mathbb{R}^d)$ of

$$\int_E (s_{k,1} \circ X_{t_1}) \cdots (s_{k,n} \circ X_{t_n})\, d(M_\lambda^t \phi) \tag{10}$$

as $k \to \infty$, E varies over cylinder sets and λ varies over compact subsets of $\overline{\mathbb{C}}_+ \setminus \{0\}$. The proof of convergence for $E = C^t$ gives the idea, for then (10) is equal to

$$S_\lambda(t - t_n)Q(s_{k,n})S_\lambda(t_n - t_{n-1}) \cdots Q(s_{k,1})S_\lambda(t_1)\phi \tag{11}$$

for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$. The operator of multiplication by a Borel measurable function f on $\mathbb{R}^d$ is written as $Q(f)$. Because $Q(s_{k,j}) \to Q(f_j)$ in the strong operator topology of $\mathcal{L}(L^2(\mathbb{R}^d))$ as $k \to \infty$, convergence also holds for the topology of precompact convergence, that is, uniform convergence on precompact sets of $L^2(\mathbb{R}^d)$. Now for $\tau \geq 0$ fixed, K a compact subset of $\overline{\mathbb{C}}_+ \setminus \{0\}$ and C a precompact subset of $L^2(\mathbb{R}^d)$, the set

$$\{S_\lambda(\tau)g : \lambda \in K,\ g \in C\}$$

is a precompact subset of $L^2(\mathbb{R}^d)$ because the mapping $(\lambda, g) \mapsto S_\lambda(\tau)g$ is continuous from $(\overline{\mathbb{C}}_+ \setminus \{0\}) \times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$. It follows that (11) converges to

$$S_\lambda(t - t_n)Q(f_n)S_\lambda(t_n - t_{n-1}) \cdots Q(f_1)S_\lambda(t_1)\phi \tag{12}$$

in $L^2(\mathbb{R}^d)$ uniformly for $\lambda \in K$ as $k \to \infty$.

Repeating the argument with C^t replaced by a cylinder set E, we see that F is $\overline{H}(\mathbb{C}_+)$-integrable and

$$\int_{C^t} F\, dM^t_\lambda = S_\lambda(t - t_n)Q(f_n)S_\lambda(t_n - t_{n-1}) \cdots Q(f_1)S_\lambda(t_1) \tag{13}$$

for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$.

ii) The argument here appears in [6], [7]. We first consider the case of bounded continuous functions $V : \mathbb{R}^d \to \mathbb{R}$ for any $d = 1, 2, \ldots$. Let

$$F_n = \prod_{j=1}^{n} e^{-it(V \circ X_{jt/n})/n}$$

for each $n = 1, 2, \ldots$. Then $F_n : C^t \to \mathbb{C}$ is a uniformly bounded function of the form i), so F_n is $\overline{H}(\mathbb{C}_+)$-integrable and from (13) we have

$$\int_{C^t} F_n\, d(M^t_\lambda \phi) = \prod_{j=1}^{n} \left[e^{-itQ(V)/n} S_\lambda(t/n) \right] \phi \tag{14}$$

for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ and $\phi \in L^2(\mathbb{R}^d)$. The operator $Q(e^{-itV/n})$ of multiplication by the function $e^{-itV/n}$ is just the unitary operator $e^{-itQ(V)/n} = \sum_{k=0}^{\infty}(-itQ(V)/n)^k/k!$.

By the continuity of V, the continuity of paths in the sample space C^t and dominated convergence, it follows that $F_n \to e^{-i\int_0^t V\circ X_s\,ds}$ in $L^1(\langle M_\lambda^t\rangle_{\lambda>0})$ as $n \to \infty$. We need to show that (14) converges in $L^2(\mathbb{R}^d)$ uniformly for λ in compact subsets of $\overline{\mathbb{C}}_+ \setminus \{0\}$ as $n \to \infty$. The argument with C^t replaced by a cylinder set E is similar.

Clearly, we are looking at convergence of the type of the Lie-Kato-Trotter product formula. The infinitesimal generator of S_λ is $\Delta/(2\lambda)$ for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$. Let K be a compact subset of $\overline{\mathbb{C}}_+ \setminus \{0\}$ and let $C(K, L^2(\mathbb{R}^d))$ be the Banach space of all continuous functions $f : K \to L^2(\mathbb{R}^d)$ with the uniform norm $\|f\|_\infty = \sup_{\lambda\in K}\|f(\lambda)\|_2$. Then

$$S_\lambda(t+s)f(\lambda) = S_\lambda(t)(S_\lambda(s)f(\lambda)), \quad \text{for all } \lambda \in K,$$

so the operator $f \mapsto S_{(\cdot)}(t)(f(\cdot))$, $f \in C(K, L^2(\mathbb{R}^d))$, defines a contraction semigroup $\tilde{S}$ of bounded linear operators for each $t \geq 0$. Because the linear subspace $C(K) \otimes L^2(\mathbb{R}^d)$ is dense in $C(K, L^2(\mathbb{R}^d))$, the semigroup $\tilde{S}$ is continuous at zero and its generator is the application of the operator $\Delta/(2\lambda)$ to $f(\lambda)$ for a dense set of functions $f \in C(K, L^2(\mathbb{R}^d))$. But $-iQ(V)$ is a bounded perturbation of this generator, so appealing to the elementary version of the Lie-Kato-Trotter product formula applied to the Banach space $C(K, L^2(\mathbb{R}^d))$, the operators (14) converge in $L^2(\mathbb{R}^d)$ uniformly for $\lambda \in K$ as $n \to \infty$. It follows that $e^{-i\int_0^t V\circ X_s\,ds}$ is $\overline{H}(\mathbb{C}_+)$-integrable and

$$\int_{C^t} e^{-i\int_0^t V\circ X_s\,ds}\,dM_\lambda^t = e^{\frac{t}{2\lambda}\Delta - itQ(V)} \tag{15}$$

for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$.

Next, approximate $V \in L^\infty(\mathbb{R}^d)$ almost everywhere by continuous functions V_ϵ such that $\|V_\epsilon\|_\infty \leq \|V\|_\infty$ for all $\epsilon > 0$. Again, convergence in the Banach space $C(K, L^2(\mathbb{R}^d))$ yields $\overline{H}(\mathbb{C}_+)$-integrability and equation (15). For the general case $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, $p > d/2$ for $d \geq 3$, the selfadjoint operator $Q(V)$ is a small perturbation of Δ, that is, the operator $Q(V)$ is relatively bounded with respect to Δ and the relative bound is zero. Hence, we can approximate $Q(V)$ by bounded operators $Q(V_n)$ associated with cutoff potentials $V_n = V\chi_{\{|V|\leq n\}}$, $n = 1, 2, \ldots$, acting in the Banach space $C(K, L^2(\mathbb{R}^d))$. Again, we get uniform convergence in $\lambda \in K$ and equation (15) holds. $\square$

A critical part of the proof above is the condition that $Q(V)$ is a small perturbation of Δ. If this fails, it looks like we have to relax the conditions of $\overline{H}(\mathbb{C}_+)$-integrability.

$H(\mathbb{C}_+)$-integrable functions

iii) $e^{-i\int_0^t V\circ X_s\,ds}$ with $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, $d \geq 3$.

iv) $e^{-i\int_0^t V\circ X_s\,ds}$ with $V(x) = -c/|x|^2$, $c > 0$ in $\mathbb{R}^3$.

v) $f^t = e^{-i\alpha_t}$ with $\langle \alpha_t \rangle_{t\geq 0}$ an additive functional of Brownian motion [1].

Proof of $H(\mathbb{C}_+)$-integrability. We just look at how to verify the conditions $[H(\mathbb{C}_+)]$ in the case that $t > 0$ and the cylinder set E is equal to the whole space C^t. For all of these functionals f^t, the approximation of $\int_{C^t} f^t\,dM_\lambda^t$ for $\lambda > 0$ converges uniformly for λ belonging to compact subsets of $\mathbb{C}_+$ by the operator version of Vitali's convergence theorem. The proofs of $H(\mathbb{C}_+)$-integrability of f^t in the cases above merely require the proof of continuity of $\lambda \mapsto \int_{C^t} f^t\,dM_\lambda^t$ in the strong operator topology for all $\lambda \in \partial\mathbb{C}_+ \setminus \{0\}$ and the identification of the function $\lambda \mapsto \int_{C^t} f^t\,dM_\lambda^t$, $\lambda \in \partial\mathbb{C}_+ \setminus \{0\}$, as the continuous boundary value of the operator-valued analytic function $\lambda \mapsto \int_{C^t} f^t\,dM_\lambda^t$, $\lambda \in \mathbb{C}_+ \setminus \{0\}$. This sort of property is known from perturbation theory. Example iii) is treated in [12, Lemma VI.4.8b, Remark 4.9a], Example iv) in [18] and a suitable modification of [1] gives Example v). □

More generally, Example iii) could be any measurable function $V : \mathbb{R}^d \to \mathbb{R}$ such that the form sum $-\Delta + aV$ is bounded below for all $a \in \mathbb{R}$ [12, Theorem IV.3.6], for example, if V is a small (zero relative bound) *form* perturbation of $-\Delta$.

The functionals $e^{-it(V\circ X_{jt/n})/n}$ cannot converge in the topology of $\overline{H}(\mathbb{C}_+)$ -integrable functions to $e^{-i\int_0^t V\circ X_s\,ds}$ in Example iv), otherwise $t \mapsto \int_{C^t} e^{-i\int_0^t V\circ X_s\,ds}\,dM_\lambda^t$ would be a unitary group for purely imaginary λ. For each $c > 0$, this is known not to be the case for sufficiently large positive values of the mass parameter $-\Im(\lambda)$ [18]. However, to establish that $e^{-i\int_0^t V\circ X_s\,ds}$ is *not* $\overline{H}(\mathbb{C}_+)$-integrable, we would need to show that it is not the limit in the seminorms (8) of *any* net of simple functions, rather than just this particular sequence of approximations. This looks hard to prove.

The space of all $\overline{H}(\mathbb{C}_+)$-integrable functions may be given a locally convex topology under which all Cauchy *sequences* converge in the lcs. This is not possible for the space of all $H(\mathbb{C}_+)$-integrable functions, because $H(\mathbb{C}_+)$ is not itself complete.

We can pursue an analogy here. Let X be an infinite dimensional separable Hilbert space. Let $(\Sigma, \mathcal{S}, \mu)$ be a σ-finite measure space. A function $f : \Sigma \to X$ for which the scalar function $\langle f, \xi\rangle : t \mapsto \langle f(t), \xi\rangle$ is μ-measurable for each $\xi \in X$ is called

(1) *Bochner integrable* if $\int_\Sigma \|f(t)\|_X \,\mu(dt) < \infty$,

(2) *Pettis integrable* if $\int_\Sigma |\langle f(t), \xi\rangle| \,\mu(dt) < \infty$ for ever $\xi \in X$.

In either case, we can define

$$\int_E f(t)\,\dot{\mu}(dt) = \lim_{n\to\infty} \int_E s_n(t)\,\mu(dt), \quad E \in \mathcal{S},$$

for a suitable sequence of $\mathcal{S}$-simple functions $\langle s_n\rangle$ converging μ-a.e. to f. The space of Bochner integrable functions is complete with respect to the norm $f \mapsto \int_\Sigma \|f\|_X \,d\mu$, but, in general, the space of Pettis integrable functions is *not* complete with respect to the norm $f \mapsto \sup_{\|\xi\|\le 1} \int_\Sigma |\langle f, \xi\rangle| \,d\mu$.

Nevertheless, the space of Pettis integrable functions is usually more interesting than the space of Bochner integrable functions. For example, singular integral operators can be associated with operator-valued Pettis integrable functions [8]. Bochner integrable functions are associated with integral operators with a regular kernel.

The lack of completeness of the space of $H(\mathbb{C}_+)$-integrable functions is compensated by the apparent ability to represent the dynamical group $e^{-it(H_0+V)}$ with more singular perturbations V of the free Hamiltonian $H_0 = \Delta/(2m)$ than the case of $\overline{H}(\mathbb{C}_+)$-integrable functions.

As mentioned above, if the form sum $-\Delta + aV$ is bounded below for all $a \in \mathbb{R}$, then the functional $e^{-i\int_0^t V\circ X_s\,ds}$ is $H(\mathbb{C}_+)$-integrable. By contrast, there seems to be a close connection between the $\overline{H}(\mathbb{C}_+)$-integrability of the functional $e^{-i\int_0^t V\circ X_s\,ds}$ and the sufficient condition that $\Delta + aV$ is essentially selfadjoint for all $a \in \mathbb{R}$. One might expect the dynamical group $e^{-it(H_0+V)}$, $t \in \mathbb{R}$, to exhibit nice stability properties under the class of perturbations V for which the multiplicative functional $e^{-i\int_0^t V\circ X_s\,ds}$, $t \ge 0$ is $\overline{H}(\mathbb{C}_+)$-integrable for all $t \ge 0$. To make these observations precise requires further study.

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