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NORM CONVERGENCE OF THE LIE–TROTTER–KATO PRODUCT FORMULA AND IMAGINARY-TIME PATH INTEGRAL

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ABSTRACT. The unitary Lie–Trotter–Kato product formula gives in a simplest way a meaning to the Feynman path integral for the Schröding-er equation. In this note we want to survey some of recent results on the *norm convergence* of the selfadjoint Lie–Trotter-Kato product formula for the Schrödinger operator $-\frac{1}{2}\Delta + V(x)$ and for the sum of two selfadjoint operators A and B. As one of the applications, a remark is mentioned about an approximation therewith to the fundamental solution for the imaginary-time Schrödinger equation.

1. Introduction

The Lie–Trotter–Kato product formula serves, as Nelson [22] noticed, as a simplest general way to give a meaning of the Feynman path integral for the imaginary-time as well as real-time Schrödinger equation. In this note we are interested in the one corresponding to the imaginary-time Schrödinger equation, i.e. heat equation

$$\frac{\partial}{\partial t}u(t,x) = -Hu(t,x), \quad t > 0; \quad u(0,x) = f(x), \quad x \in \mathbf{R}^d, \tag{1.1}$$

where $H = H_0 + V \equiv -\frac{1}{2}\Delta + V(x)$ is the nonrelativistic Schrödinger operator with mass 1 and scalar potential $V(x)$, a real–valued continuous function bounded below. By Kato's theorem [18], H is essentially

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selfadjoint on $C_0^\infty(\mathbf{R}^d)$ in $L^2 = L^2(\mathbf{R}^d)$, and so its unique selfadjoint extension is also denoted by the same H. The semigroup e^{-tH} solves (1.1) by $u(t,x) = e^{-tH}f(x)$, and the Lie–Trotter–Kato product formula

$$(1.2) \quad \lim_{n\to\infty}(e^{-tV/2n}e^{-tH_0/n}e^{-tV/2n})^n = \lim_{n\to\infty}(e^{-tV/n}e^{-tH_0/n})^n = e^{-tH}$$

in strong operator topology holds, locally uniformly in $t \geq 0$.

The aim of this note is to survey some recent results about *opeartor-norm* convergence of the Lie–Trotter–Kato product formula (1.2) with error estimates and its abstract version, mainly based on our joint works with Takanobu [9], [10], [11], with Doumeki and Hideo Tamura [2], and with Hideo Tamura [13], [12], [14]. A survey of an early version was briefly given in [8].

To describe the results, in general, for the sum of two nonnegative selfadjoint operators A and B in a Hilbert space with domains $D[A]$ and $D[B]$, let $H = A\dot{+}B$ also denote the form sum of A and B, where we assume for simplicity that the form domain, $D[H^{1/2}] = D[A^{1/2}] \cap D[B^{1/2}]$, is dense. If the operator sum $A + B$ is selfadjoint (resp. essentially selfadjoint) on $D[A] \cap D[B]$, H coincides with $A + B$ (resp. the closure of $A + B$). We shall use the following notations:

$$(1.3) \qquad K(t; A, B) = e^{-tB/2}e^{-tA}e^{-tB/2}, \quad G(t; B, A) = e^{-tB}e^{-tA}.$$

Then it is well-known that the Lie–Trotter–Kato product formula

$$(1.4)$$
$$\lim_{n\to\infty} K(t/n; A, B)^n = \lim_{n\to\infty} K(t/n; B, A)^n = \lim_{n\to\infty} G(t/n; B, A)^n = e^{-tH}$$

in strong operator topology holds, locally uniformly in $t \geq 0$ (e.g. [23]).

Now, to describe the result for the Schrödinger operator H, we see first the Schrödinger semigroup e^{-tH} makes sense in L^p. In fact, the solution $u(t,x)$ of (1.1) is represented by *imaginary-time path integral*, i.e. the Feynman–Kac formula (e.g. Simon [25])

$$(1.5) \qquad u(t,x) = (e^{-tH}f)(x) = \int e^{-\int_0^t V(X(s))ds} f(X(t)) d\mu_x(X),$$

for $f \in L^2$, where μ_x is the Wiener measure on the space $C_x([0,\infty) \to \mathbf{R}^d)$ of the continuous paths $X : [0,\infty) \to \mathbf{R}^d$ starting at $X(0) = x$. In passing, this formula can also be proved by using the Lie–Trotter–Kato product formula (1.2) ([22], [25]).

The formula (1.5) shows that e^{-tH} defines a bounded operator on L^2, and further a strongly continuous semigroup on $L^p = L^p(\mathbf{R}^d)$, for $1 \leq p < \infty$, and for $p = \infty$, on the Banach space $C_\infty = C_\infty(\mathbf{R}^d)$ of the continuous functions in $\mathbf{R}^d$ vanishing at infinity, equipped with L^∞ norm.

We have obtained the L^p results for Schrödinger operators by use of the Feynman–Kac formula in [9], [10], and the L^2 results with operator-theoretic methods, in Doumeki–Ichinose–Tamura [2] and Ichinose–Tamura [13], where the problem in the trace norm is also treated.

We also give analogous results [10], [11] for the relativistic Schrödinger operator $H^r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V(x)$. Note that the Feynman–Kac formula also holds for the imaginary–time relativistic Schrödinger equation (1.1) with this H^r in place of H (e.g. [7], [15]).

In Section 2 we state these results, and give in Section 3 a sketch of proof. In Section 4, a remark is given about how good a approximation the result can give to the fundamental solution of the heat equation.

2. Lie–Trotter–Kato product formula in operator norm

The norm convergence of the Lie–Trotter–Kato product formula (1.4) is trivial, if both operators A and B are bounded. So it is when two operators are not necessarily bounded that we are going to deal with this formula. In the following, results are described first for Schrödinger operators, and second, abstract ones for the sum of two nonnegative selfadjoint operators.

2.1. For Schrödinger operators

We first estimate in L^p operator norm the difference between $K(t; H_0, V) = e^{-tV/2}e^{-tH_0}e^{-tV/2}$ and e^{-tH} by a power of small $t > 0$ with order greater than 1.

It is B. Helffer [5] (cf. [4], [6]) who first did in L^2 operator norm to get $O(t^2)$ by pseudo-differential operator calculus, when $V(x)$ is a C^∞–function bounded below by b and satisfying $|\partial^\alpha V(x)| \leq C_\alpha(1 + |x|^2)^{(2-|\alpha|)_+/2}$ for every multi-index α with constant C_α, where $a_+ = \max\{a, 0\}$. He has called $K(t; H_0, V)$ the *Kac operator*, for it turns out to be the transfer operator/matrix for a Kac's lattice model [16], [17], [28] in statistical mechanics. He wanted to observe with this kind of error estimate the asymptotic eigenvalue splitting of $K(t; H_0, V)$, comparing with the eigenvalues of the Schrödinger operator H.

In the following, $\|\cdot\|_p$ stands for the L^p operator norm, i.e. the operator norm of bounded operators on L^p, $1 \le p < \infty$, or on C_∞, for $p = \infty$.

THEOREM 1. *(The nonrelativistic case) Let $0 < \delta \le 1$. Let m be a nonnegative integer such that $m\delta \le 1$ and $(m+1)\delta \ge 1$. Suppose that $V(x)$ is a C^m-function in $\mathbf{R}^d$ bounded below by a constant b which satisfies that*

$$(2.1a) \qquad |\partial^\alpha V(x)| \le C(V(x) - b + 1)^{1-|\alpha|\delta}, \quad 0 \le |\alpha| \le m,$$

with a constant $C > 0$, and further that $\partial^\alpha V(x), |\alpha| = m$, are Hölder-continuous:

$$(2.1b) \qquad |\partial^\alpha V(x) - \partial^\alpha V(y)| \le C|x-y|^\kappa, \quad x, y \in \mathbf{R}^d,$$

with constants $C > 0$ and $0 \le \kappa \le 1$. Then it holds that, as $t \downarrow 0$,

$$(2.2) \qquad \|K(t; H_0, V) - e^{-tH}\|_p = \begin{cases} O(t^{1+(m+\kappa)/2}), & m = 0, 1, \\ O(t^{1+2\delta}), & m \ge 2. \end{cases}$$

Note that condition (2.1b) with $\kappa = 1$ is equivalent to that $\partial^\alpha V(x)$, $|\alpha| = m+1$, are essentially bounded. By $\kappa = 0$ we understand $\partial^\alpha V(x)$, $|\alpha| = m$, bounded.

Then we have from Theorem 1 the following Lie–Trotter–Kato product formula for the Schrödinger operator $H = H_0 + V$.

THEOREM 2. *(The nonrelativistic case) For the same function $V(x)$ as in Theorem 1, it holds that, as $n \to \infty$,*

(2.3)

$$\|(K(t/n; H_0, V)^n - e^{-tH}\|_p, \quad \|(G(t/n; V, H_0)^n - e^{-tH}\|_p$$
$$= \begin{cases} n^{-\kappa/2} O(t^{1+\kappa/2}), & m = 0, 0 < \kappa \le 1, \\ n^{-(1+\kappa)/2} O(t^{1+(1+\kappa)/2}), & m = 1, 0 \le \kappa \le 1, \\ n^{-2\delta} O(t^{1+2\delta}), & m \ge 2. \end{cases}$$

A few words for the proof: The first part is a direct consequence of Theorem 1 with telescoping. But for the second one we have to elaborate more.

EXAMPLES. The function $|x|^2$ (harmonic oscillator potential) satisfies conditions (2.1ab) for $V(x)$ in Theorem 1 with $(\delta, m, \kappa) = (\frac{1}{2}, 1, 1)$ or

$(\frac{1}{2}, 2, 0)$, $|x|^4 - |x|^2$ (double well potential) with $(\delta, m, \kappa) = (\frac{1}{4}, 3, 1)$ or $(\frac{1}{4}, 4, 0)$, and $|x|$ with $(\delta, m, \kappa) = (1, 0, 1)$, while $|x|^{1999}$ and $|x|^{7.14}$ satisfy conditions (2.1ab) with $(\delta, m, \kappa) = (1/1999, 1998, 1)$ or $(1/1999, 1999, 0)$ and $(1/7.14, 7, 0.14)$, respectively.

REMARK 1. Helffer's result [5] is included in Theorem 1 with $p = 2$ and $(\delta, m, \kappa) = (1/2, 1, 1)$ or $(\delta, m, \kappa) = (1/2, 2, 0)$, because his condition implies that

$$|\partial^\alpha V(x)| \leq C(V(x) - b + 1)^{(1-|\alpha|/2)_+} \tag{2.4}$$

for every multi-index α.

Dia–Schatzman [1] has also given an operator–theoretic proof of Helffer's result.

REMARK 2. Theorems 1 and 2 are valid with the operator H_0 replaced by the magnetic Schrödinger operator $H_0(A) = \frac{1}{2}(-i\partial - A(x))^2$ with vector potential $A(x)$ including the case of constant magnetic fields (see [9], cf. [2]).

THEOREM 3. *(The relativistic case) Let $V(x)$ be the same function as in Theorem 1. Then it holds that, as $t \downarrow 0$,*

$$\|K(t; H_0^r, V) - e^{-tH^r}\|_p = \begin{cases} O(t^{1+\kappa}), & m = 0, 0 \leq \kappa < 1, \\ O(t^2|\ln t|), & (m,\kappa) = (0,1) \text{ or } (1,0), \\ O(t^2), & m = 1, 0 < \kappa \leq 1, \\ O(t^{1+2\delta}), & m \geq 2. \end{cases} \tag{2.5}$$

Then we have from Theorem 3 the following Lie–Trotter–Kato product formula.

THEOREM 4. *(The relativistic case) For the same function $V(x)$ as in Theorem 1, it holds that, as $n \to \infty$,*

$$\begin{aligned} &\|(K(t/n; H_0^r, V)^n - e^{-tH^r}\|_p, \quad \|(G(t/n; V, H_0^r)^n - e^{-tH^r}\|_p \\ &\qquad = \begin{cases} n^{-\kappa} O(t^{1+\kappa}), & m = 0, 0 < \kappa < 1, \\ n^{-1} O(t^2|\ln(t/n)|), & (m,\kappa) = (0,1) \text{ or } (1,0), \\ n^{-1} O(t^2), & m = 1, 0 < \kappa \leq 1, \\ n^{-2\delta} O(t^{1+2\delta}), & m \geq 2. \end{cases} \end{aligned} \tag{2.6}$$

2.2. Abstract results

To describe the results of the abstract version of the Lie–Trotter–Kato product formula in operator norm, we keep those notations around (1.3) in Section 1, about nonnegative selfadjoint operators A and B in a Hilbert space.

It is Rogava [24] who first proved that if A is strictly positive, i.e. $A \geq \delta I$ for some $\delta > 0$ (I stands for the identity operator) with $D[A] \subseteq D[B]$, and $A + B$ is selfadjoint on $D[H] = D[A] \cap D[B] = D[A]$, then

$$(2.7) \quad \|K(t/n; B, A)^n - e^{-tH}\|, \quad \|G(t/n; B, A)^n - e^{-tH}\| = O(n^{-1/2} \ln n),$$

as $n \to \infty$, uniformly in $t \geq 0$. In this case, B is A–bounded. Notice that in our Theorems 2 and 4, neither V is H_0–bounded nor H_0 is V–bounded. So Theorems 2 and 4 are independent of this abstract result by Rogava, though, for instance, in Theorem 2, it can been shown that, if $V(x)$ is besides a C^1 function, then $H = H_0 + V$ is selfadjoint on $D[H] = D[H_0] \cap D[V]$.

Stimulated by Rogava's result, we have shown in Ichinose–Tamura [14] (cf. [12]) a better error bound though with a stronger condition: if $A \geq \delta I$, $B \geq \delta I$ for some $\delta > 0$, and $D[A^\alpha] \subseteq D[B]$ for some $0 < \alpha < 1$, then

$$(2.8) \quad \|K(t/n; B, A)^n - e^{-tH}\|, \quad \|G(t/n; B, A)^n - e^{-tH}\| = O(n^{-1} \ln n),$$

as $n \to \infty$, locally uniformly in $t \geq 0$. In fact, we have proved the case where the operator $B = B(t)$ may be t-dependent.

Around the same time, Neidhardt and Zagrebnov [19] have proved (2.8), but uniformly in $t \geq 0$, if $A \geq \delta I$, $B \geq \delta I$ for some $\delta > 0$, and $D[A] \subseteq D[B]$ and B is A-bounded with relative bound less than 1. In fact, they have considered more general functions $f(x)$, $g(x)$ than e^{-x} to define the products $f(tA)g(tB)$ instead of $G(t; A, B)$, etc. For further related results see Neidhardt–Zagrebnov [20], [21].

However, the Lie–Trotter–Kato product formula in operator norm does not in general hold for the form sum $H = A \dot{+} B$, even when B is A-form-bounded with relative bound less than 1. This can be seen by an example Hiroshi Tamura [27] has recently constructed. In his example, further, $A \geq \delta I$ for some $\delta > 0$, $B \geq 0$, and the operator sum $H = A + B$ is essentially selfadjoint on $D[A] \cap D[B]$, but not selfadjoint there, so that it turns out that this pair of A and B does neither satisfy Rogava's condition nor correspond to the case for the pair of H_0 and V in Theorem 2.

We close this section with the following two comments.

One concerns whether the convergence of (2.3) and (2.6) in Theorems 2 and 4 and of (2.7) and (2.8) in the abstract results just mentioned above is "uniform" or "locally uniform" in $t \geq 0$. In fact, Theorems 2 and 4 hold "uniformly" in $t \geq 0$, if $V(x)$ is strictly positive, i.e. $V(x) \geq \delta$ on $\mathbf{R}^d$ for some $\delta > 0$, and the same will be true for those abstract results mentioned above, if one of A and B is strictly positive.

The other concerns how about Theorem 1 for $K(t; V, H_0)$, $G(t; V, H_0)$ and $G(t; H_0, V)$ in place of $K(t; H_0, V)$. In this case we could only prove an estimate worse than (2.2). The same is true for Theorem 3. However, the Lie–Trotter–Kato product formula in Theorems 2 and 4 as well as the abstract version described in Section 2.2 holds, of course, for them as well as for $K(t; A, B)$ and $G(t; A, B)$.

3. Sketch of proof of Theorem 1

We only sketch the idea of proof of Theorem 1 by probabilistic methods. The proof of Theorem 3 is similar. Putting $Q(t) = K(t; H_0, V) - e^{-tH}$, we can write by the Feynman–Kac formula (1.5) as

$$(3.1)\quad (Q(t)f)(x) = E_x\Big[\big(e^{-\frac{t}{2}(V(x)+V(X(t)))} - e^{-\int_0^t V(X(s))ds}\big)f(X(t))\Big],$$

for $f \in C_0^\infty(\mathbf{R}^d)$ with $\|f\|_p = 1$, where E_x is the expectation or integral with respect to the Wiener measure μ_x. We need to prove the L^p norm of $Q(t)f$ has the estimate as on the right-hand side of (2.2).

Let $p(t,x) = (2\pi t)^{-d/2}e^{-x^2/2t}$ be the integral kernel of e^{-tH_0}, heat kernel. We use the conditional expectation $E_x[\,\cdot \mid X(t) = y]$ to rewrite (3.1) as

$$(3.2)\quad \begin{aligned} (Q(t)f)(x) &= \int f(y)p(t, x-y)E_x[v(t,x,y) \mid X(t) = y]dy, \\ v(t,x,y) &= e^{-\frac{t}{2}(V(x)+V(y))} - e^{-\int_0^t V(X(s))ds}. \end{aligned}$$

We have by Taylor's theorem,

$$\begin{aligned} v(t,x,y) =& -w(t,x,y)e^{-\frac{t}{2}(V(x)+V(y))} - \sum_{j=2}^{m} \tfrac{w(t,x,y)^j}{j!} e^{-\frac{t}{2}(V(x)+V(y))} \\ &- \tfrac{w(t,x,y)^{m+1}}{m!} \int_0^1 d\theta (1-\theta)^m e^{-(1-\theta)\frac{t}{2}(V(x)+V(y)) - \theta \int_0^t V(X(s))ds} \\ \equiv& \sum_{i=1}^{3} v_i(t,x,y), \end{aligned}$$

where

$$\begin{aligned} w(t,x,y) &= \tfrac{t}{2}(V(x)+V(y)) - \int_0^t V(X(s))ds \\ &= -\int_0^{t/2} \big(V(X(s)) - V(x)\big)ds - \int_{t/2}^t \big(V(X(s)) - V(y)\big)ds. \end{aligned}$$

Then we estimate $d_i(t,x,y) \equiv E_x[v_i(t,x,y) \mid X(t) = y]$, $i = 1,2,3$, by a sum of powers of $|x-y|$ and t. For instance, if $m \geq 2$, we can show

$$\begin{aligned} |d_1(t,x,y)| \leq& \sum_{k=2}^{m} (|x-y|^k O(t^{k\delta}) + O(t^{(1+2\delta)k/2})) \\ &+ |x-y|^{m+\kappa} O(t) + O(t^{1+(m+\kappa)/2}), \\ |d_2(t,x,y)| \leq& \sum_{j=2}^{m} \Big\{ \sum_{k=1}^{m} (|x-y|^{jk} O(t^{jk\delta}) + O(t^{(1+2\delta)jk/2})) \\ &+ |x-y|^{j(m+\kappa)} O(t^j) + O(t^{(m+2+\kappa)j/2}) \Big\}, \\ |d_3(t,x,y)| \leq& \sum_{k=1}^{m} (|x-y|^{(m+1)k} O(t^{(m+1)k\delta}) + O(t^{(1+2\delta)(m+1)k/2})) \\ &+ |x-y|^{(m+1)(m+\kappa)} O(t^{m+1}) + O(t^{(m+1)(1+(m+\kappa)/2)}). \end{aligned}$$

Hence we have, for small $t > 0$,

$$\|Q(t)f\|_p \leq \sum_{i=1}^{3} \| \int f(y) p(t, \bullet - y) d_i(t, \bullet, y) dy \|_p \leq O(t^{1+2\delta}).$$

4. On approximation to the fundamental solution of the heat equation

Let $b = 0$ or $V(x) \geq 0$. In this section we write the Kac operator $K(t; H_0, V)$ as simply $K(t)$. Let $K(t)(x, y)$ be the integral kernel of $K(t)$, and $e^{-tH}(x, y)$ the integral kernel of e^{-tH} or the fundamental solution of the heat equation (1.1). Then a careful check of our proof of Theorem 2.1 in [10, 11], which is sketched in Section 3, will show

$$K(t)(x,y) - e^{-tH}(x,y) = O(t^{1+a})(2\pi t)^{-d/2}e^{-(x-y)^2/4t}$$

$$= O(t^{1+a})p(2t, x-y), \tag{4.1}$$

where a is the respective constant $(m+\kappa)/2$ or 2δ on the right-hand side of (2.2) depending on the regularity of $V(x)$. Recall that the free heat kernel is given by $e^{-tH_0}(x, y) = p(t, x-y) = (2\pi t)^{-d/2}e^{-(x-y)^2/2t}$. Then the integral kernel $K(t/n)^n(x, y)$ of $K(t/n)^n$ can approximate $e^{-tH}(x, y)$ in such a way as

$$K(t/n)^n(x,y) - e^{-tH}(x,y) = n^{-a}O(t^{1+a})(2\pi t)^{-d/2}e^{-(x-y)^2/4t}, \tag{4.2}$$

as $n \to \infty$.

Indeed, we have by telescoping with $x_0 = x$ and $x_n = y$

$$\begin{aligned}
&|K(t/n)^n(x,y) - e^{-tH}(x,y)| \\
&\leq \int_{\mathbf{R}^d} |K(t/n)(x_0,x_1) - e^{-tH/n}(x_0,x_1)| e^{-(n-1)tH/n}(x_1,x_n)dx_1 \\
&+ \sum_{j=2}^{n-1} \iint_{\mathbf{R}^d\times\mathbf{R}^d} K(t/n)^{j-1}(x_0,x_{j-1}) \\
&\times |K(t/n)(x_{j-1},x_j) - e^{-tH/n}(x_{j-1},x_j)| e^{-(n-j)tH/n}(x_j,x_n)dx_{j-1}dx_j \\
&+ \int_{\mathbf{R}^d} K(t/n)^{n-1}(x_0,x_{n-1})|K(t/n)(x_{n-1},x_n) - e^{-tH/n}(x_{n-1},x_n)|dx_{n-1}.
\end{aligned}$$

The right-hand side is seen by (4.1) to be bounded by

$$\begin{aligned}
&\int O((\tfrac{t}{n})^{1+a})p(2t/n, x_0 - x_1)p((n-1)t/n, x_1 - x_n)dx_1 \\
&\quad + \sum_{j=2}^{n-1} \iint p((j-1)t/n, x_0 - x_{j-1})O((\tfrac{t}{n})^{1+a}) \\
&\qquad \times p(2t/n, x_{j-1} - x_j)p((n-j)t/n, x_j - x_n)dx_{j-1}dx_j \\
&\quad + \int p((n-1)t/n, x_0 - x_{n-1})O((\tfrac{t}{n})^{1+a})p(2t/n, x_{n-1} - x_n)dx_{n-1},
\end{aligned}$$

which is less than or equal to

$$O((\tfrac{t}{n})^{1+a})2^d\Big(\int p(2t/n, x_0 - x_1)p(2(n-1)t/n, x_1 - x_n)dx_1$$
$$+\sum_{j=2}^{n-1}\iint p(2(j-1)t/n, x_0 - x_{j-1})$$
$$\times p(2t/n, x_{j-1} - x_j)p(2(n-j)t/n, x_j - x_n)dx_{j-1}dx_j$$
$$+\int p(2(n-1)t/n, x_0 - x_{n-1})p(2t/n, x_{n-1} - x_n)dx_{n-1}\Big).$$

Thus we obtain

$$|K(t/n)^n(x,y) - e^{-tH}(x,y)| = n^{-a}O(t^{1+a})2^d p(2t, x_0 - x_n),$$

concluding (4.2). In fact, this issue was also studied by a little itricate probabilistic methods in [26].

Though it may be a new observation in the present note, by a further careful check of our proof of Theorem 2.1 in [10, 11] we can see that if $V(x)$ is a smooth, namely, C^∞–function, then (4.1) holds, together with *all the* x, y*–derivatives of the left-hand side*, so that the same is true for (4.2).

In [3] Fujiwara constructed the fundamental solution $e^{-itH}(x,y)$ of the Schrödinger equation, i.e. the integral kernel of e^{-itH} based on the idea of the Feynman path integral. He proved the following result.

Assume that $V(x)$ is smooth and satisfies

$$|\partial^\alpha V(x)| \le C_\alpha(1+x^2)^{(2-|\alpha|)_+/2}$$

for every multi-index α with constant C_α, though $V(x)$ need not be bounded below. Put

$$(E(t)f)(x) = (2\pi i t)^{-d/2}\int_{\mathbf{R}^d} e^{iS(t,x,y)}f(y)dy$$

for $f \in C_0^\infty(\mathbf{R}^d)$, with action $S(t,x,y) = \int_0^t[\frac{1}{2}(dX(s)/ds)^2 - V(X(s))]ds$, where $X(s)$ is the classical trajectory starting at $X(0) = y$ and ending at $X(t) = x$. Then for $t > 0$ sufficiently small, one has

$$E(t/n)^n(x,y) - e^{-itH}(x,y) = n^{-1}O(t^2)(2\pi t)^{-d/2}, \tag{4.3}$$

as $n \to \infty$, uniformly in x, y, together with all the x, y–derivatives of the left-hand side.

The potential $V(x)$ in this result satisfies (2.4) or condition (2.1ab) with $(\delta, m, \kappa) = (\frac{1}{2}, 2, 0)$, so that it turns out that $a = 1$. What we have observed above is that the integral kernel of the $K(t/n)^n$ can just yield an analogous approximation to the fundamental solution of the heat equation.

Added in Proof on Feb. 5, 2001: In a forthcoming paper *"The norm convergence of the Trotter-Kato product formula with error bound"* by T. Ichinose and Hideo Tamura, to appear in Commun. Math. Phys., it has been shown that the Lie-Trotter-Kato product formula holds in norm for the semigroup generated by the operator sum $H = A + B$ of two nonnegative selfadjoint operators A and B which is selfadjoint.

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