A FEYNMAN FUNCTIONAL FOR
THE GLOBAL POSITIONING SYSTEM

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ABSTRACT. A Feynman functional formulation is given for the Global Positioning System, GPS. Both the sequential and analytic Feynman functionals are presented for the classical, exterior, gravity problems which include rigid body rotations, special relativity and some general relativity corrections. A mathematically rigorous approach is introduced whose solutions exist, are unique and which depend continuously on the initial data. This formulation is convergent and has the finite approximation property. It is emphasized that all of the problems studied are classical (not quantum) evolution systems.

1. Introduction

A *Feynman integral* [1, 2] has proven to be an interesting class of mathematical structures [3-25] and has provided useful ideas to theoretical physics [26-38]. These objects are integrals only in some general, conditionally convergent sense and can be described better as *Feynman functionals* or as the quantities in some *Feynman operational calculus*. Feynman integrals provide ways of formulating new classes of problems, i.e. the path sum (or sum over histories) can be generalized to include sums over homotopically inequivalent paths or to give new ways to think about existing problems such as polymers. They provide a bridge between deterministic and stochastic approaches, give new tools for asymptotic analysis and very general numerical approaches to physical systems. Feynman integrals have much to teach us, hence, their mathematical structures must be soundly based on theorems and proofs. In

this paper the above point of view will be directed to the Global Positioning System, GPS. The Global Positioning System, is a modern technological marvel which can measure locations on the earth's ellipsoid to within a few millimeters. To date GPS [39-41] represents the ultimate extension of geodesy and navigation so it is a fitting tribute to C. F. Gauss [42], the creator and pioneer of geodesy. A number of practical fields such as navigation and surveying have already been revolutionized by GPS, and major advances in the knowledge of such areas as general relativity, astrophysics, geophysics and many others have been provided. The GPS was used to obtain the experimental proof of the continental drift, and is presently accumulating enormous databases on earthquakes prone areas.

The GPS consists of over 24 satellites each costing about $60,000,000 orbiting the earth in six planes with three major control stations connected to over one hundred monitor stations on the ground. About thirty satellites occupy six planar orbits about four earth radii above the center of the earth, and broadcast radio frequency signals to each ground station. Each satellite and ground station has a very accurate clock. At least four different signals are required to determine x, y, z, and t of a GPS transponder (actually 6-8 are usually within the line of sight of a ground station). A Kalman filter tracks each satellite and differences of two signals reduce several errors. By weighting measured averages by their variances and by collecting enormous data sets, the errors become quite small.

This accuracy needs to be understood and it needs to be improved. In this paper, a Feynman functional based upon the sequential and analytic Feynman functionals is presented and shown to provide new perturbation solutions of some of the evolution equations for the GPS. A recent paper by B. DeFacio, G.W. Johnson and M. L. Lapidus [16] is used.

The framework used here was originally created by Cameron and Storvich, and has been extended by G. W. Johnson, M. L. Lapidus, D. Skoug, G. Kallianpur and many others. A new book by G. W. Johnson and M. L. Lapidus by Oxford University Press [15] presents much of this material in great detail. It is fitting and proper that this beautiful mathematics helps to understand and to improve the Global Positioning System. A schematic picture of the GPS constellation of satellites is shown in Fig. 1. There are three principal ideas behind the GPS system:

1. Work with two (or more) receivers which is called differential GPS, abbreviated as DGPS. By using differences of positions, several of the larger random errors almost cancel.
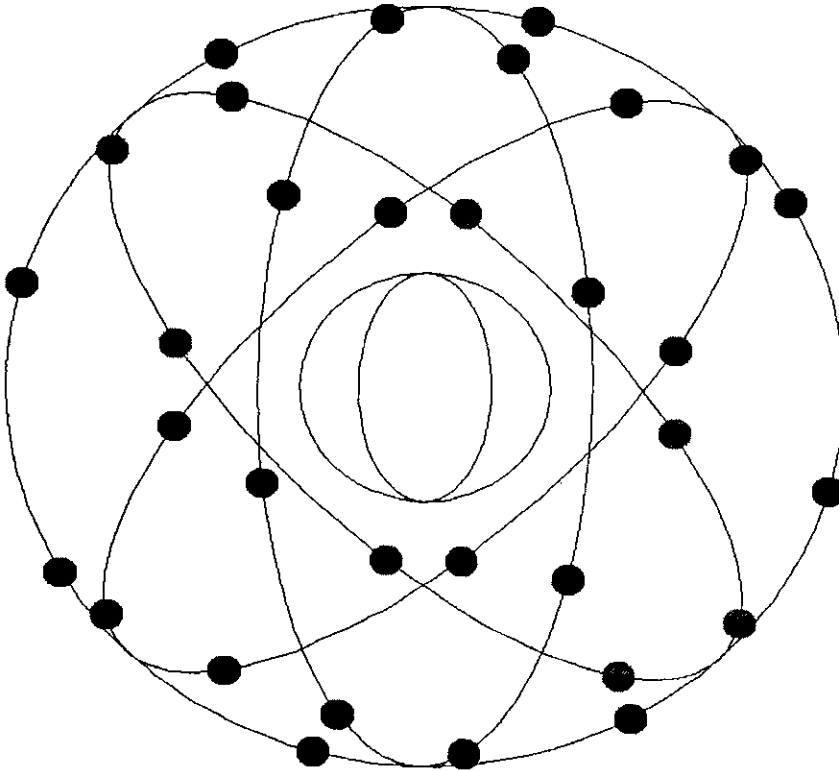
FIGURE 1. Schematic picture of the GPS constellation:
6 planes with at least 4 satellites in each. Each satellite
is about 15 minutes less than 2 orbits/day.

2. Repeat the measurements and thereby reduce the variance con-
siderably. Use updating techniques together with the weighting
of measurements by their variances in the data processing. This
saves computational time since large matrices do not need to be
diagonalized repeatedly. It also keeps inaccurate data from desta-
bilizing the calculated averages. A suitable Kalman estimator, or
filter, updates the changes in satellite and receiver positions due
to their motions.

3. Estimate and model each known source of error. The major sources of error in DGPS and the bounds on their approximate magnitudes are: Ephermeris (satellite orbital data) $\approx 2m$; satellite clock $\approx 2m$; ionosphere $\approx 1m$ (50 - 1000 km above the earth's surface); troposphere $\approx 1m$ (9 - 16 km above the earth's surface); and multipath interference of signals $\approx 0-2m$. These are each complicated problems. The reader is referred to the references, esp. [39-41], for further discussions. This paper will concentrate on the corrections to the ephermeris (orbit) data. This would be a simple problem if the accuracy demanded of GPS were not so high. But all special relativity and some of the linearized general relativity corrections are required to achieve and improve the present accuracy.

The goal of this study is to formulate a simple model to help understand the GPS and to contribute to the ongoing problem in improving its accuracy and precision. The model must be mathematically rigorous and must be generalizable to all relativity effects and must include noise. Some Feynman functionals have been applied to simpler satellite and orbit problems [34] and to a phase space Feynman functional for general relativity [29]. Uniqueness can be proven for an adequate class of these structures [16]. The sequential functionals carry the finite approximation property for insight and also converge. A more general model, applicable to the GPS system, is presented here.

In the next section, the evolution equations required will be presented and both the idealized model and a more realistic model will be discussed. In section 3, path integral perturbations for each model will be discussed. In section 4, the conclusions and outlook for further study will be presented.

2. The time evolution of a GPS satellite position

The basis of this work is most directly that of Cameron and Storvick [3,5], Johnson [6, 7, 15, 17], Lapidus [8-11, 14,21], Kallianpur, Kannan, and Karandikar [18] and DeFacio, Johnson and Lapidus [16]. Although the quantum disentanglement of [16] is not used in the present work, it is used to carry out the nonlinear coordinate transformations among different sets of spatial axes required among the different astronomy, geophysics and space physics coordinate systems. Since the SO(3) matrices which implement these coordinate changes do not commute, a classical disentanglement is required. Also the measured values used are

corrupted by noise, this classical mechanics [42-45] problem needs realistic noise for input into the Kalman estimator (filter). The accuracy of the GPS requires all special relativity [46] and some general relativity [46-50] corrections. The orbiting satellite interacts with the earth, the sun, the space environment, the moon and nearby satellites, gravitational radiation, and fluctuations occur in the atomic clocks which are used. Thus, the smooth elliptic path of a Keplerian motion is jerked one way and another by these perturbations. The "realistic noise" of a satellite path is the totality of these (random) interactions. The path integral gives a convergent perturbation expansion for evaluating these effects. From this approach some of the errors from these effects can be removed by improving the model.

Next, the interaction potential energy for the exterior problem must be given. If R_1, R_2 are the radii of two approximately spherical bodies with masses M_1, M_2 whose centers are at $\vec{r_1}$ and $\vec{r_2}$ at time t, then it is common to treat the basic interaction as Newtonian gravity between two point particles in an inertial frame as

$$(1) \qquad V_1(r) = \frac{GM_1M_2}{r}$$

where G is the universal gravitational constant and $r = |\vec{r_1} - \vec{r_2}|$ is the length of seperation. This is a singular potential which can only be treated by a modified Feynman integral [11, 13, 16] using imaginary resolvent arguments. But the blowup singularity at $r \to 0$ is physically impossible for non zero radius bodies and a more realistic modified basic interaction is given by the exterior problem

$$(2) \qquad V_2(r) = \begin{cases} \frac{GM_1M_2}{r}, & R_1 + R_2 < r \\ a_1 r + i a_2, & 0 < R_0 < r < R_1 + R_2 \end{cases}$$

where a_1, a_2 are real valued constants. Clearly a satellite or observer in the more realistic situation of the potential in Eq. (2) above, has stability since this potential is bounded below. One advantage of the present formulation is that the model is stable [7, 8, 16] whereas the point particle model in Eq. (1) has only been proven to be stable in a few special cases such as the 2-D restricted 3-body problem. Another is that a solution is known to exist and to be unique in the linear case. The nonlinear cases are far more complicated with many examples and special cases known, but no general results known to me. The potential in Eq. (2) is in L^∞ and a variety of formulations ranging from the Fresnel integrable functions [4, 12, 13] to the Cameron-Storvick sequential path integral [3, 5, 19] or the analytic Feynman integrals [5, 6, 19],

the Lie-Trotter formula [9, 10] and the various Banach spaces [5, 6, 16]. The GPS problems addressed here are all described by the exponential evolution systems of [16]. It is not surprising that solutions to and insights into the (almost) random perturbations of the GPS are simply and naturally carried by a Feynman path integral. After all, some of its first rigorous formulations [5] were based on the Borel functions of a pair (v, x) consisting of the scale-invariant almost everywhere subspace of paths $x, x(0) = 0$, in a Wiener space $C_0[0, t]$ and $v \in L^2(\mathcal{R}^n)$, and were linear, stochastic integrals.

The ideal problem which generalizes Eq. (1) for the linearized general relativistic corrections for the GPS satellites is very difficult. Interesting new effects occur in the paramatized post-Newtonian effective potentials for the perturbations to Eq. (2), which can be written as in [46, 50] for $r > R$ (earth),

$$V_{G1}(r) \approx \quad + \quad \frac{GI}{2r^3}\left[-\delta_{ij} + \frac{3}{r^2}w^i w^j\right]$$

$$+ \quad 0\left(\epsilon\frac{1}{r^5}\right)$$

$$(3) \qquad V_{G2}(r) \approx G\epsilon^{ijk} \quad w_E^i \quad I^{kj}\frac{w^i}{r^3} + 0\left(\frac{\epsilon 1}{r^4}\right).$$

The total potential is then $V = V_2 + V_{G1} + V_{G2}$ where V_2 together with H_0 is placed in the semigroup term $\exp(-t \propto) = \exp(tH_0 - tV_2)$. Because it is much larger than the remaining terms, the semigroup is taken as the "free" or "unperturbed" term. The remaining terms of V_{G1} and V_{G2} perturb the elliptic orbit. They open the orbit by a small holonomy and produce a "rough hash" on it due to the other physical effects in these potentials and impulse interactions with other bodies. This corresponds to using the analysis in [16] in the formulation of this approach. Note the singularities $1/r^3, 1/r^5, 1/r^6$, and so on which will require the creation of *new mathematics* before the interior problem $r < R$ can be treated by a Feynman integral. In Eq. (3), I is the (spatial) moment of inertia, w^i is some spatial component of an angular momentum coordinate, "i" is a rotating coordinate frame fixed to the earth, w_E is the instantaneous angular rotation of the earth, and ϵ^{ijk} is the Levi-Civiti antisymmetric tensor. The second potential V_{G2} can be eliminated by $w_E \to 0$, but only at a terrible cost in the added complication of the other potential energy function. Even at this level, our GPS ephermeris problems are complicated. There are many other perturbations which

must also be considered. In the lowest order of perturbation theory, the paths depend only on the coordinates of the unperturbed problem. The spherical harmonics are used in a new version of the Arnowitt, Deser, Misner (hereafter ADM) expansion. The more standard tetrad expansion of the local coordinates [46] is a poor choice for the GPS problems considered here because the earth's rotation about its NS axis is an important effect, as are many relations of (most!) interesting astrophysical objects (e.g. pulsars). We do not discuss in detail the numerous interesting questions which arise in these problems for lack of space and time. We comment that our independent work verifies large portions of refs. [47-49] in detail. Several of these include:

1. The space-time coordinates are on a punctured harmonic manifold and are not restricted to Lorentz boosts, although these are included.
2. The synchronization of any three clocks A, B, and C at any three distinct points on earth, which are not the N or S pole of rotation, at successive times $t_A < t_B < t_C$ are *non-transitive*. That is if on earth, clock A is synchronized with clock B using light pulses and then clock B is synchronized with clock C using light pulses; the clock A is not synchronized with clock C. This is a holonomy effect caused by the earth's rotation.
3. Several related effects on the spatial motions follow from the potential in Eq. (3), because the perturbation to Eq. (2) change the orbit from closed to open and from periodic to almost periodic.
4. One has, and uses, an inertial frame of the sun, earth, moon, satellite system. However, measurements are made on rotating and accelerating bodies and require a careful conversion. A space-born clock measures proper time along its world line. Outside a rotating, electrically charged body a clock in a circular orbit (the eccentricity is small for GPS satellites so this is a reasonable simplification). In this case, a small gravitomagnetic effect from general relativity should occur [49]. It has never been experimentally observed but will be studied in the next generation of NASA experiments, LIGO.

There are many more open questions. The GPS may contribute to the understanding of some of these, and others may improve the GPS.

The unperturbed evolution system for this work is the system of ODE's

$$(4) \qquad \dot{Q} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \cdot D \cdot H = \begin{pmatrix} H_p \\ -H_q \end{pmatrix}$$

where $q = (q_1, q_2, \cdots q_n)^T, p = (p_1, p_2, \cdots p_n)$ is the n generalized coordinates q and their n canonical momenta p, $D = (\frac{\partial}{\partial q_1}, \cdots \frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_1}, \cdots \frac{\partial}{\partial p_n})^T$ is the differential operator with respect to phase space coordinates (q, p), $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the $2n \times 2n$ metric matrix for the symplectic group $Sp(2n)$, $H = H(p, q)$ is the system Hamiltonian and $\dot{Q} = dQ/dt$. Let ϵ_s be a small, real, positive parameter and let $g = (g_1, g_2)^T$ be a $T = \frac{2\pi}{w}$ periodic perturbation. The detailed forms of g_1, g_2 are obtained from taking suitable coordinate transformations of the gradients of the potential energy functions in Eq. (3). These lead to systems of evolution equations of the form

$$(5) \qquad \dot{Q} = JDH + \epsilon_s g$$

and a first order perturbation theory is derived. The reader is reminded that the components of g contain several orders of terms from post-Newtonian gravity and possibly noise. For example, the gravitomagnetic clock effects briefly discussed in this section first occur in the $1/r^4, 1/r^5$ terms in Eq. (2).

The solution to Eq. (5) is

$$(6) \qquad Q(t) = e^{-iHt} Q(0)$$

which is a unitary group for energy conserving systems. It can also be written as a C_0 semigroup [30] with infinitesimal generator $A = -iH$. It is clear that the system of $2n$ ODE's in Eq. (5) are unbounded linear operators, and it is useful to use the semigroup structure in these cases [15,16,30].

3. A Feynman functional for the GPS

In this section, Feynman functional for modeling GPS errors and for data analysis will be formulated. The sequential approach is central because a finite approximation is desired. For a perturbation series, it is necessary to prove convergence. The paths of satellites or planets etc.

are not directly used because they necessitate repeated limits. An evaluation, as in [5,15,16,18] involving a single limit of Lebesque integrals is sought and a clear relation to the reproducing kernel Hilbert space, RKHS, see esp. [18], is needed. The RKHS structure is needed here because the GPS amasses very large data collections. The purpose of this data is to provide information about positions on earth, velocities and gravitational fields. The Kalman filter: estimators, the encoding and decoding on the carrier frequencies and the transmission of the electromagnetic signals carrying the codes bring data in a noisy environment are all important parts of this large complex system. The ground station receives massive input from these various direct physics problems. Whether geodesy, navigation, etc.is the task the ground stations must solve the noisy, generalized-inverse (ill-posed) problem to obtain the information needed from these signals. The RKHS have an invertibility structure and are closely related to Gaussian probability distribution functions. There are many problems in modern physics which do not satisfy the central limit theorem nor the Poisson probability distribution function, because they are "non-equilibrium" in some sense. The perturbation structure proposed here can also correct some of these pdfs by modeling away non-Gaussian effects, as well as the signal structure. Kallianpur, Kannan and Karandikar [18] gave an especially clear and useful discussion of these matters together with their relations to analytic Feynman integrals. Their paper along with those by Johnson [6,16] serve as cornerstones for the rest of this section.

In Eq. (6) the exponential form of the time evolution is given. Next, a specific set of assumptions from [16] which allow the proof of existence and uniqueness of solutions and their continuous dependence on initial data is listed in three hypothesis. (H1)-(H3). The notation is such that the exponential evolution operator in eq. (6) is written as

$$(7) \quad exp(-iHt) = exp\{-\alpha t + \int_0^1 \beta_1(s)\mu(ds) + \cdots + \int_0^1 \beta_n(s)\mu_n(ds)\}$$

where different choices of $(\beta_1, \mu_1), \cdots, (\beta_n, \mu_n)$ correspond to different models of perturbations to the time evolution of the satellites.

H.1   *The operator $-\alpha$ is the infintesimal generator of a C_o (contraction) semigroup of operators on a separable Hilbert space H over R^1*

H.2a *For each p=1, $\cdots$, n, μ_p is a measure defined on B([0, $+\infty$)), the Borel class of [0,∞).*

H.2b *The associated (necessarily positive) total variation measures $\mid \mu_p$
$\mid$ satisfy*
$\mid \mu_p \mid ([0,\ T]) < \infty$ *for every* $T \geq 0$ *and p=1, 2, $\cdots$, n.*

H.2c *The μ_p's are continuous measures, that is $\mu_p(\{t\})=0$, for every t
≥ 0.*

H.3a *For each p=1, $\cdots$, n, $\beta_p:[0, \infty) \to L(H)$ is assumed to have the
property that $\beta_p(E)^{-1}$ is in the Borel class (or Borel σ-algebra)
$B([0, \infty))$ of $[0, \infty)$ for every strong operator open subset E of
L(H).*

H.3b *We require for each p=1, $\cdots$, n that $\int_0^T ||\beta_p(s)|| |\mu_p|(ds) < \infty$ for
every $T > 0$.*

H.3c *For each p it is assumed that the family of operators $\{\beta_p(s) : 0 \leq
s < \infty\}$ is a commuting family in L(H).*

* *Note that it is NOT assumed that for $p \neq q$ that any β_p com-
mutes with β_q nor that any β_p commutes with the semigroup $T(t)$
$= \exp(-\alpha t)$.*

The fact that the potential energies (V_{G1}, V_{G2}) in Eq. (3) are in
$L^\infty(\mathcal{R}^3)$ provides great simplification — help that is needed to com-
plete this work. This allows one to rigorously prove the existence of
the integrals and to justify the interchange of limit processes. Lapidus
[10] and Johnson and Lapidus [15] did this for a 1-D example and then
we mention some of the complications which are required by the GPS
geometry. There is a nice twist to this problem, and that is that the
Feynman functional is guiding us in the establishment of a mathemati-
cally rigorous, algorithm to numerically calculate the values of physical
effects which model the residuals of real data. This will then allow us
to correct for the failure of the random parts of the system to attain the
applicability of the central limit theorem! The simple proof [10,16] allow
us to point out the exact connection of the deterministic ODE system
to the statistical aspects.

Let $H = H_0 + V$, $V \in L^\infty(\mathcal{R}^3)$ and consider the Feynman-Kac for-
mula for any $\varphi \epsilon L^2(\mathcal{R}^3)$ and ξ Lebesque measurable a.e. in $\mathcal{R}^3$,

$$(e^{-t(H_0+V)}\varphi)(\xi)$$

$$= \int_{C[0,t]} \exp\left\{-\int_0^t V[x(s)+\xi]ds\right\} \varphi[x(t)+\xi]dm_w(x)$$

$$(8) \quad = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{C[0,t]} \left\{\int_0^t V[x(s)+\xi]ds\right\}^m \varphi[x(t)+\xi]dm_w(x),$$

where dm_w is the Wiener measure on the Wiener space $C[0,t] = \{x = x(s) : [0,t] \to \mathcal{R}^1, x(0) = 0\}$, and where x is a continuous path in the Wiener space. Observe that e^{-tH_0} is hidden inside the Wiener measure $dm_w(x)$ on the right hand side of the two equality signs. Furthermore, the replacement of the polygonal paths in Cameron-Storvick [5] by the Fourier coefficients in Kallianpur, et al [18] is a very favorable development for the inverse problems of linear operators in the GPS. They showed that the RKHS is closely related to the spectrum of the operator H. Stated another way, with complete, noise free data, the potential energy function V can be uniquely reconstructed from the spectrum, $\sigma(H)$.

In Eq. (8), the $m = 0$ term is the free system, $V(r) \equiv 0$,

$$
\begin{aligned}
I_0 \quad &:= \{[\exp(-tH_0)]\varphi\}(\xi) \\
&= \int_{C[0,t]} \varphi[x(t) + \xi]dm_w(x).
\end{aligned}
$$
(9)

The first order perturbation, $m = 1$ in Eq. (7) is

$$
\begin{aligned}
I_1 : \quad &= \int_{C[0,t]} [-\int_0^t V(x(s_1) + \xi)ds_1]\varphi(x(t) + \xi)dm_w(x) \\
&= -\int_0^t \left\{ \int_{C[0,t]} V(x(s_1) + \xi)\varphi(x(t) + \xi)dm_w(x) \right\} ds_1,
\end{aligned}
$$
(10)

where the interchange of integration in the last equality of Eq. (10) is allowed by the Fubini theorem. The time ordering of the orbit paths and radio signal propagation occurs here because of causality, and is easy at first order. This is because the interval $0 \le s \le t$ has zero Lebesque measure when $s \in [0,t]$ for $s = 0$ or $s = t$ for the continuous measure, and it is only necessary to consider $0 < s < t$. Using the change of variables $v_0 = u_0 + \xi$ and $v_1 = u_1 + \xi$, I_1 becomes, using the exp $(-tH_0)$ semigroup properties,

$$I_1 \quad = -\int_0^t \left\{ \left(\frac{1}{(2\pi)^2 s_1(t-s_1)} \right)^{1/2} \times \right.$$

$$\times \int_{\mathcal{R}^2} V(u_1+\xi)\varphi(u_0+\xi)\exp\left[-\frac{u_1^2}{2s_1^2} - \frac{(u_0-u_1)^2}{2(t-s_1)} \right] du_0 du_1 \left. \right\} ds_1$$

$$= -\int_0^t \left\{ \left(\frac{1}{(2\pi)^2 s_1(t-s_1)} \right)^{1/2} \times \right.$$

$$(11) \quad \times \int_{\mathcal{R}^2} V(v_1)\varphi(v_0)\exp\left[-\frac{(\xi-v_1)^2}{2s_1} - \frac{(v_1-v_0)^2}{2(t-s_1)} \right] dv_0 dv_1 \left. \right\} ds_1.$$

Writing the inside integral as an iterated integral yields

$$I_1 \quad = \int_0^t \left\{ \left(\frac{1}{2\pi s_1} \right)^{1/2} \int_{\mathcal{R}^1} e^{-(\xi-v_1)^2/2s_1} \left(-V(v_1) \right) \times \right.$$

$$\times \left[\left(\frac{1}{2\pi(t-s_1)} \right)^{1/2} \int_{\mathcal{R}_1} e^{-(v_1-v_0)^2/2(t-s_1)} \varphi(v_0)dv_0 \right] dv_1 \left. \right\} ds_1$$

$$(12) \quad = \int_0^t \{\exp(-s_1 H_0)(-V)\exp[-(t-s_1)H_0](\varphi)(\xi)\} ds_1.$$

The $m=2$ order term is exhibited although it is not needed in our work at this time. The time ordering is nontrivial in this order

$$I_2 \quad := \frac{1}{2!} \int_{C[0,t]} \left[\int_0^t V_1'[x(s)+\xi]ds \right]^2 \varphi(x(t)+\xi)dm_w(x)$$

$$= \frac{1}{2!} \int_{C[0,t]} \left[\int_0^t \int_0^{s_2} V_1'(x(s_1)+\xi)V(x(s_2)+\xi)ds_1 ds_2 \right.$$

$$(13) \quad + \int_0^t \int_0^{s_2} V_1'(x(s_1+\xi)V(x(s_2)+\xi)ds_2 ds_1 \right] \varphi(x(t)+\xi)dm_w(x),$$

where the two integrals over $ds_1 ds_2$ and $ds_2 ds_1$ are due to the time ordering on $[0,t]^2$. They are also equal if one is relabeled. Changing variables to $v_0 = u_0 + \xi$, $v_1 = u_1 + \xi$ and writing an iterated integral over $\mathcal{R}^3$ gives

$$I_2 = \int_0^t \int_0^{s_2} \{\exp(-s_1 H_0)(-V_1')\exp(-(s_2-s_1)H_0)(-V_1')$$

$$(14) \qquad\qquad \exp(-(t-s_2)H_0)(\varphi)(\xi)\} ds_1 ds_2.$$

The evaluation of Eqs. (12), (13) with $V_1' = V_{G1} + V_{G2}$ is not easy. If a satellite $m_{s'}$, the moon m_0, the earth m_c, and the sun m_s are considered, then Eq. (3) includes six pairs of masses $m_i m_j$ and six separation distances $r_{ij} = |\vec{r_i} - \vec{r_j}|$ four angular three-vector velocities, $\vec{w_\alpha}(\alpha = s_1', 0, e, s)$. The direction of $\vec{w_E}$ is along the normal to the satellite plane (there are four such planes) and $\vec{w_0}$, $\vec{w_s}$ are the angular velocities of the moon and sun. None of these vectors are collinear nor orthogonal. The origin is at the enter of mass of these bodies. Special relativistic transformations add velocity terms of order $\vec{V_\alpha}/c(\alpha = s', 0, e, s)$ to many of these terms and general relativity adds more. The spatial integrations are only over non-H_0, say $\vec{r_j}$ coordinates and the solutions are known to be almost-periodic [43-45].

In this framework, radiation reaction gravitational radiation can be studied by adding suitable terms to $V_{G1} + V_{G2}$. Since the hypotheses are valid even for the paramaterized post Newtonian interaction assumed in Eq. (3), $V_1' = V_{G1} + V_{G2} = g$ in the equations of motion. Then the perturbation vector g is $T = 2\pi/w$ periodic and the perturbed orbits are almost-periodic.

4. Conclusions

A class of models for the Global Positioning System including earth and its satellites in the gravitational field of the sun has been presented. Firstly, the perturbation series converges and the sum of these series is unique. Secondly, the formulation based upon the Feynman functional is versatile in the sense that next order of effects such as gravitational waves and radiation reactions can be calculated in this approximation. This provides a way to model out more of the errors from the GPS and to thereby understand it better.

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