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THE HEISENBERG INEQUALITY ON ABSTRACT WIENER SPACES

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ABSTRACT. The Heisenberg inequality associated with the uncertainty principle is extended to an infinite dimensional abstract Wiener space (H, B) with an abstract Wiener measure p_1. For $\varphi \in L^2(p_1)$ and $T \in \mathcal{L}(B, H)$, it is shown that

$$\left[\int_B |Tx|_H^2 |\varphi(x)|^2 p_1(dx)\right]\left[\int_B |Tx|_H^2 |\mathcal{F}\varphi(x)|^2 p_1(dx)\right] \geq \| T|_H \|_{HS}^4 \|\varphi\|_2^4,$$

where $\mathcal{F}\varphi$ is the Fourier-Wiener transform of φ. The conditions when the equality holds also discussed.

1. Introduction

The well-known Heisenberg uncertainty principle [11] asserts that for any function $f \in L^2(\mathbf{R}^n)$ with $|f|_2 = 1$,

$$\int_{\mathbf{R}^n} |xf(x)|^2 dx \cdot \int_{\mathbf{R}^n} |\lambda \hat{f}(\lambda)|^2 d\lambda \geq \frac{n^2}{4(2\pi)^{n-1}}, \tag{1}$$

where $\hat{f}$ is the Fourier transform of f. The first infinite dimensional version of the Heisenberg inequality has been established on a white noise space by Lee and Stan [9] in the following way:

Take a basic Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$; e.g. $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$, where $\mathcal{S}(\mathbf{R})$ is the Schwartz space of rapidly decreasing functions on $\mathbf{R}$. Let $|\cdot|_0$ denote the norm on E. Replace the space $\mathbf{R}^n$ by $\mathcal{E}'$ and the Lebesgue measure on $\mathbf{R}^n$ by the standard Gaussian measure μ on $\mathcal{E}'$. Then the infinite dimensional analogue of the inequality in (1) takes the

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form

$$(2)\quad \left[\int_{\mathcal{E}'} |(x,\eta)\varphi(x)|^2\mu(dx)\right]\left[\int_{\mathcal{E}'} |(x,\eta)\mathcal{F}\varphi(x)|^2\mu(dx)\right] \geq |\eta|_0^4\|\varphi\|_0^4$$

for any $\eta \in \mathcal{E}$ and $\varphi \in L^2(\mu)$, where $(\cdot,\cdot)$ denotes the $\mathcal{E}' - \mathcal{E}$ pairing and $\mathcal{F}$ is the Fourier-Wiener transform [5, 7], i.e.

$$\mathcal{F}\varphi(x) = \int_{\mathcal{E}'} \varphi(\sqrt{2}y + ix)\mu(dy).$$

Along the line of the proof of inequality (2) in [9] it is easy to see that the inequality (2) remains true when the Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$ is replaced by an abstract Wiener space $B^* \subset H \subset B$ and the Gaussian measure μ is replaced by an abstract Wiener measure p_t with variance parameter $t = 1$, where H is a real separable Hilbert space with norm $|\cdot|_H = \sqrt{\langle\cdot,\cdot\rangle}$, B is the completion of H with respect to a measurable norm $\|\cdot\|$ and B^* is the dual space of B (see Gross [1]).

Comparing (2) with (1), one observes that (2) essentially generalizes (1) only for the case $n = 1$ since (x,η) corresponds merely to the one dimensional coordinate function. We find that a more appropriate generalization of (1) to infinite dimensions should take the following form

$$(3)\quad \begin{aligned} &\left[\int_B |Tx|_H^2|\varphi(x)|^2p_1(dx)\right]\left[\int_B |Tx|_H^2|\mathcal{F}\varphi(x)|^2p_1(dx)\right] \\ &\geq \|\widetilde{T}\|_{HS}^4\|\varphi\|_2^4, \end{aligned}$$

where $T \in \mathcal{L}(B,H)$, the space of bounded linear operators from B into H and $\widetilde{T}$ is its restriction to H which is a Hilbert-Schmidt operator with norm given by $\|\widetilde{T}\|_{HS}$ (see [4]).

In section 2 we will provide a brief background concerning the abstract Wiener space, there the concept of weak Gross differentiation is introduced and an integration by parts formula for weak Gross differentiable functions is given. In section 3 we first reprove inequality (2) for an abstract Wiener space (H,B) for the sake of clarity and then prove inequality (3). The equality of (3) is also discussed there.

2. Calculus on abstract Wiener space

Abstract Wiener spaces

Let H be a real separable Hilbert space with norm $|\cdot|_H = \sqrt{\langle\cdot,\cdot\rangle}$, and B the completion of H with respect to a measurable norm $\|\cdot\|$. Then

(H, B) is called an abstract Wiener space (AWS). As H is identified as a dense subspace of B, we identify the dual space B^* as a dense subspace of $H^* \equiv H \subset B$, in such a way that for $x \in H$ and $y \in B^*$, $\langle x, y\rangle = (x, y)$, where $(\cdot, \cdot)$ denotes the $B - B^*$ pairing.

L. Gross [1] proved that B carries a probability measure p_t, known as the abstract Wiener measure with variance parameter $t > 0$, which is extended from the Gauss cylinder set measure μ_t with variance t on H. The abstract Wiener measure p_t may be characterized as the unique Borel measure such that

$$\int_B e^{i(x,y)}\, p_t(dx) = e^{-\frac{t}{2}|y|_H^2},$$

for all $y \in B^*$.

For each $y \in B^*$, denote the function $x \to (x, y)$ by $\tilde{y}(x)$ i.e. $\tilde{y} = (\cdot, y)$. Then $\tilde{y}$ is a random variable which is normal distributed with mean 0 and variance $t|y|_H^2$ with respect to p_t. For $h \in H$, take a sequence $\{y_n\} \subset B^*$ s.t. $y_n \to h$ in H. Then $\{\tilde{y}_n\}$ is a Cauchy sequence in $L^2(p_t)$. Define $\tilde{h} = L^2(p_t) - \lim_{n\to\infty} \tilde{y}_n$. Then $\tilde{h}$ is defined a.e. as a random variable on B with mean zero and variance $t|h|_H^2$. Customarily, we also denote $\tilde{h}(x) = \langle x, h\rangle$.

Gross differentiation

To discuss the differential calculus on an abstract Wiener space (H, B) we need the concept of B^*-differentiation and H-differentiation introduced by Gross [2]. Let V be a subspace of B which is itself a Banach space under some norm $\|\cdot\|_V$. Let f be a scalar-valued function defined on B. f is said to be *Gross-differentiable at a point $x \in B$ in the direction of V*, or *V-differentiable*, if the function $g(v) = f(x + v)$, regarded as a function defined in a neighborhood of the origin of V, is Fréchet differentiable at the origin of V. Then the V-derivative $Df(x)$ of f at x is defined by $Df(x) = g'(0)$, the Fréchet derivative of g at the origin of V. Higher order V-derivatives are defined similarly. For notional convenience, we shall use the notation $Df(x)$ to denote both Fréchet-derivative (or B-derivative) and V-derivative if no confusion is concerned.

Weak Gross differentiation

For let L^1_{exp} denote the collection of functions $g \in L^1(p_1)$ such that

$$\int_B |g(x)| e^{m\|x\|} p_1(dx) < \infty$$

for all $m > 0$. Clearly $L^\alpha(p_1) \subset L^1_{exp}$ for $\alpha > 1$.

For any $f \in L^1_{exp}$ and for any $\xi \in B^*$, $p_1 * f(\xi)$ exists. Define $Sf(\xi) = p_1 * f(\xi)$ for $\xi \in B^*$. S is called the *Segal-Bargmann transform* or simply *S-transform.*

Let $\mathcal{E}(B)$ denote the class of functions g defined on B satisfying (1) g has an analytic extention $\mathbf{g}$ to the complexification B_c of B and (2) there exist constants c, c', depending only on g, such that $|\mathbf{g}(z)| \leq c \cdot exp(c'\|z\|_B)$, where $\|z\| = \sqrt{\|x\|^2 + \|y\|^2}$ for $z = x + iy$. Clearly, $\mathcal{E}(B) \subset L^\alpha(p_1)$ for all $\alpha > 1$.

PROPOSITION 2.1.

(a) *For $f \in L^\alpha, \alpha > 1$, $Sf(\xi)$ exist for $\xi \in H$.*
(b) *For $f \in \mathcal{E}(B)$, $Sf(z)$ exist for all $z \in B_c$.*
(c) *For $f \in L^1_{exp}$, Sf has an analytic extention* **Sf** *to the complexification B^*_c of B given by*

$$\mathbf{Sf}(z) = \int_B f(x) e^{-\frac{1}{2}z^2 + (x,z)} p_1(dx),$$

where $z^2 = \langle z, \bar{z} \rangle$ for $z \in B^*_c$.
(d) *For $f \in L^1_{exp}$, if $Sf(\xi) = 0$ for all $\xi \in B^*$, then $f = 0$* a.e.(p_1).

Proof. (a), (b) and (c) are clear. (d) follows from (c). □

In the remaining of this paper, we shall identify Sf with **Sf** for notational convenience.

DEFINITION 2.2. Let $f \in L^1_{exp}$ and $\eta \in V$. If there exists a function $\partial_\eta f \in L^1_{exp}$ such that

$$D_\eta(Sf)(\xi) = S(\partial_\eta f)(\xi)$$

for all $\xi \in B^*$, we say that f is weakly Gross-differentiable in V-direction or, simply, weakly V-differentiable. $\partial_\eta f$ is called the weak V-derivative of f in the direction η. Define

$$\partial^*_\eta f = \tilde{\eta} f - \partial_\eta f,$$

if it exists (we shall see that ∂^*_η is in fact the adjoint of ∂_η).

EXAMPLE 2.3. Let $\varphi(x) = \langle x, h\rangle^n$, $h \in H$. φ is, in general, not H-differentiable but it is weakly H-differentiable and

$$\partial_\eta \varphi(x) = n\langle \eta, h\rangle \langle x, h\rangle^{n-1}.$$

Also we have

$$\partial_\eta^* \varphi = \tilde{\eta}\varphi - n\langle \eta, h\rangle\langle x, h\rangle^{n-1}.$$

In particular when $\eta = h$ with $|\eta| = |h| = 1$, then we have

$$\partial_\eta(\tilde{\eta}^n) = n\tilde{\eta}^{n-1} \text{ and } \partial_\eta^*(\tilde{\eta}^n) = \tilde{\eta}^{n+1} - n\tilde{\eta}^{n-1}.$$

PROPOSITION 2.4. *Let $f \in L^\alpha(p_1), \alpha > 1$. Then, for $\xi \in B^*$ and $h \in H$, we have*

$$S(\partial_h^* f)(\xi) = (h, \xi) Sf(\xi).$$

Proof.

$$\begin{aligned}
&S(\partial_h^* f)(\xi) \\
&= S(\tilde{h}f)(\xi) - S(\partial_h f)(\xi) \\
&= \int_B \langle x+\xi, h\rangle f(x+\xi)p_1(dx) - D_h(Sf)(\xi) \\
&= \int_B \langle x, h\rangle f(x+\xi)p_1(dx) + \langle h, \xi\rangle \int_B \langle x, h\rangle f(x+\xi) p_(dx) - D_h(Sf)(\xi) \\
&= (h, \xi) \int_B \langle x, h\rangle f(x+\xi)p_(dx) \\
&= (h, \xi) Sf(\xi).
\end{aligned}$$

The last equality follows from the identity $D_h(Sf)(\xi) = \int_B \langle x, h\rangle f(x + \xi)p_1(dx)$ (see [6]). □

DEFINITION 2.5. The unique function $\mathcal{F}f \in L^1_{exp}$ determined by the relation

$$S(\mathcal{F}f)(\xi) = Sf(i\xi)$$

is called the weak Fourier-Wiener transform of f.

REMARK 2.6.[5, 7] For a function $\varphi \in \mathcal{E}$, the Fourier-Wiener transform $\mathcal{F}\varphi$ is defined by

$$\mathcal{F}\varphi(y) = \int_B \varphi(\sqrt{2}x + iy)p_1(dx).$$

Then we have $\|\mathcal{F}\varphi\|_2 = \|\varphi\|_2$. For $f \in L^2(p_1)$, $\mathcal{F}f$ is defined as the $L^2(p_1)$-limit of a sequence $\{\mathcal{F}\varphi_n\}$, where $\varphi_n \to f$ in $L^2(p_1)$. In this way, $\mathcal{F}$ is extended to $L^2(p_1)$ as a unitary operator and we also have

$$S(\mathcal{F}f)(\xi) = \lim_{n\to\infty} S(\mathcal{F}\varphi_n)(\xi) = \lim_{n\to\infty} Sf_n(i\xi) = Sf(i\xi).$$

The Definition 2.3 is motivated from the above identity.

LEMMA 2.7. [7, 10] *Let $\mu_{\frac{1}{2}}$ denote the standard Gaussian cylinder set measure on H with variance parameter $\frac{1}{2}$ and $\{P_n\}$ a sequence of finite rank orthogonal projections on H such that $P_n(H) \subset B^*$ and which tends to the identity I_H of H. Then, for $f, g \in L^2(p_1)$, we have*

$$\begin{aligned} & \int_B f(x)\bar{g}(x)p_1(dx) \\ (4) \qquad = {} & \lim_{n\to\infty} \int_H \int_H Sf[P_n(x+iy)]S\bar{g}[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\ = {} & \sum_{j=1}^{\infty} \frac{1}{n!} \langle\langle D^n Sf(0), D^n S\bar{g}(0)\rangle\rangle_{HS}, \end{aligned}$$

where $\langle\langle\cdot,\cdot\rangle\rangle_{HS}$ denotes inner product of Hilbert-Schmidt multi-linear operators and the limit of in (4) is independent of the choices of finite rank orthogonal projections $\{P_n\}$ which tends to I_H.

As an application of lemma 2.7 we have

LEMMA 2.8. *For $h, k \in H$ and for real-valued functions f, g such that $f, g, \partial_h f, \partial_k f, \partial_h^* f$ and $\partial_k^* f \in L^2(p_1)$, we have*

$$(5) \qquad \int_B \partial_h^* f(x)\partial_k^* f(x)p_1(dx) = \langle h, k\rangle \int_B f(x)g(x)p_1(dx) + \int_B \partial_h f(x)\partial_k f(x)p_1(dx).$$

Proof. Apply lemma 2.5, we have

$$\int_B |\partial_h^* f(x)|^2 p_1(dx)$$
$$= \lim_{n\to\infty}\int_H\int_H |S(\partial_h^* f)[P_n(x+iy)]|^2 \mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$= \lim_{n\to\infty}\int_H\int_H |\langle P_n(x+iy),h\rangle|^2 |Sf[P_n(x+iy)]|^2 \mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$= \lim_{n\to\infty}\int_H\int_H \langle P_n x,h\rangle^2 Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\int_H\int_H \langle P_n y,h\rangle^2 Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$= \lim_{n\to\infty}\frac{1}{2}\int_H\int_H |P_n h|^2 Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\frac{1}{2}\int_H\int_H \langle P_n x,h\rangle D_h Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\frac{1}{2}\int_H\int_H \langle P_n x,h\rangle Sf[P_n(x+iy)]D_h Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\frac{1}{2}\int_H\int_H |P_n h|^2 Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dy)\mu_{\frac{1}{2}}(dx)$$
$$+\lim_{n\to\infty}\frac{i}{2}\int_H\int_H \langle P_n y,h\rangle D_h Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dy)\mu_{\frac{1}{2}}(dx)$$
$$-\lim_{n\to\infty}\frac{i}{2}\int_H\int_H \langle P_n y,h\rangle Sf[P_n(x+iy)]D_h Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dy)\mu_{\frac{1}{2}}(dx)$$
$$= \lim_{n\to\infty}\int_H\int_H |P_n h|^2 |Sf[P_n(x+iy)]|^2 \mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\frac{1}{2}\int_H\int_H D_h^2 Sf[P_n(\dot{x}+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\frac{1}{2}\int_H\int_H |D_h Sf[P_n(x+iy)]|^2 \mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$-\lim_{n\to\infty}\frac{1}{2}\int_H\int_H D_h^2 Sf[P_n(x+iy)]Sf[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$+\lim_{n\to\infty}\frac{1}{2}\int_H\int_H |D_h Sf[P_n(x+iy)]|^2 \mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$= \lim_{n\to\infty}|h|^2\int_H\int_H \left\{Sf[P_n(x+iy)]|^2 + |D_h Sf[P_n(x+iy)|^2\right\}\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$
$$= \lim_{n\to\infty}|h|^2\int_H\int_H \left\{Sf[P_n(x+iy)]|^2 + |S(\partial_h f)[P_n(x+iy)|^2\right\}\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy)$$

$$= \quad |h|^2 \int_B |f(x)|^2 p_1(dx) + \int_B |\partial_h f(x)|^2 p_1(dx)$$

We have proved that the identity (5) holds when $h = k$. The case $h \neq k$ now follows immediately from the linearity of the mapping $h \to \partial_h$ and the mapping $h \to \partial_h^*$ and the following identity:

$$\begin{aligned} &\int_B |\partial_{h+k}^* f(x)|^2 p_1(dx) \\ =&|h+k|^2\rangle \int_B |f(x)|^2 p_1(dx) + \int_B |\partial_{h+k} f(x)|^2 p_1(dx). \end{aligned}$$ □

Next we show that the integration by parts formula given Kuo [3] for Gross differentiable square integrable functions may be generalized to weak Gross differential function.

PROPOSITION 2.9. (Integration by parts formula) *Let $h \in H$. Assume that $f, g, \partial_h f$ and $\partial_h g$ and $\tilde{h} f$ are real-valued functions in $L^2(p_1)$. Then we have*

$$\int_B \langle x, h\rangle f(x) g(x) p_1(dx) = \int_B \Big[\partial_h f(x) g(x) + f(x) \partial_h g(x)\Big] p_1(dx).$$

Proof. Without loss of generality we may assume that f and g are real-valued functions. Let $\{P_n\}$ be the projection as given in Lemma 2.7. Then it follows from Lemma 2.7 that we have

$$\begin{aligned} &\int_B \langle x, h\rangle f(x) g(x) p_1(dx) \\ &= \lim_{n\to\infty} \int_H \int_H S(\tilde{h} f)[P_n(x+iy)] Sg[P_n(x-iy))] \mu_{\frac{1}{2}}(dx) \mu_{\frac{1}{2}}(dy) \\ &= \lim_{n\to\infty} \int_H \int_H \Big\{ S(\partial_h f)[P_n(x+iy)] + S(\partial_h^* f)[P_n(x+iy)] \Big\} \\ &\qquad\qquad Sg[P_n(x-iy)] \mu_{\frac{1}{2}}(dx) \mu_{\frac{1}{2}}(dy) \\ &= \lim_{n\to\infty} \int_H \int_H S(\partial_h f)[P_n(x+iy)] Sg[P_n(x-iy)] \mu_{\frac{1}{2}}(dx) \mu_{\frac{1}{2}}(dy) \end{aligned}$$

$$
\begin{aligned}
&+\lim_{n\to\infty}\int_H\int_H \langle P_n(x+iy),h\rangle Sf[P_n(x+iy)]Sg[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
= &\lim_{n\to\infty}\int_H\int_H S(\partial_h f)[P_n(x+iy)]Sg[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
&\quad +\frac{1}{2}\lim_{n\to\infty}\int_H\int_H D_hSf[P_n(x+iy)]Sg[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
&\quad +\frac{1}{2}\lim_{n\to\infty}\int_H\int_H Sf[P_n(x+iy)]D_hSg[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
&\quad -\frac{1}{2}\lim_{n\to\infty}\int_H\int_H D_hSf[P_n(x+iy)]Sg[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
&\quad +\frac{1}{2}\lim_{n\to\infty}\int_H\int_H Sf[P_n(x+iy)]D_hSg[P_n(x-iy)]\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
= &\lim_{n\to\infty}\int_H\int_H \Big\{S(\partial_h f)[P_n(x+iy)]Sg[P_n(x-iy)] \\
&\qquad +Sf[P_n(x+iy)]S(\partial_h g)[P_n(x-iy)]\Big\}\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
= &\int_B\Big[\partial_h(x)g(x)+f(x)\partial_h g(x)\Big]p_1(dx),
\end{aligned}
$$

where the last equality follows from the integration by parts formula on finite dimensional Gaussian space. □

COROLLARY 2.10. *Suppose that f and g satisfy the same conditions as given in Theorem 2.9. We have*

$$\int_B \partial_h^* f(x)g(x)p_1(dx) = \int_B f(x)\partial_h g(x)p_1(dx).$$

This implies that ∂_h^* is the adjoint of ∂_h.

3. Heisenberg uncertainty principle

In this section we establish the infinite dimensional Heisenberg inequality associated with the uncertainty principle. The functions which satisfy the equality of the Heisenberg inequality are completely characterized.

PROPOSITION 3.1.[9] *Let $\eta \in H$ and suppose that $f, \partial_\eta f$ and $\partial_\eta^* f$ are functions in $L^2(p_1)$. Then we have*

$$\mathcal{F}\{\langle\cdot,\eta\rangle f - 2\partial_\eta f\}(y) = i\langle y,\eta\rangle \mathcal{F}f(y). \tag{6}$$

Proof. First observe that, under the assumption, $\langle\cdot,\eta\rangle f - 2\partial_\eta f \in L^2(p_1)$. Then apply the S-transform to both sides of (6), we obtain

$$\begin{aligned} &S\mathcal{F}\{\langle\cdot,\eta\rangle f - 2\partial_\eta f\}(\xi) \\ &= S\{\langle\cdot,\eta\rangle f - 2\partial_\eta f\}(i\xi) \\ &= i(\eta,\xi)Sf(i\xi) - D_\eta Sf(i\xi) \end{aligned} \tag{7}$$

On the other hand

$$\begin{aligned} &S\{\langle\cdot,\eta\rangle \mathcal{F}f\}(\xi) \\ &= S\{\partial_\eta \mathcal{F}f + \partial_\eta^* \mathcal{F}f\}(\xi) \\ &= D_\eta(S\mathcal{F}f)(\xi) + (\eta,\xi)S\mathcal{F}f(\xi) \\ &= iD_\eta Sf(i\xi) + (\eta,\xi)Sf(\xi) \end{aligned} \tag{8}$$

Formula (6) now follows from (7) and (8). □

PROPOSITION 3.2. *Suppose that $\eta, \zeta \in H$ and $\langle\cdot,h\rangle f, \partial_h f \in L^2(p_1)$. for $h = \eta, \zeta$. Then we have*

$$\left[\int_B |\langle x,\eta\rangle f(x)|^2 p_1(dx)\right]\left[\int_B |\langle x,\zeta\rangle \mathcal{F}f(x)|^2 p_1(dx)\right] \geq \langle\eta,\zeta\rangle^2 \|f\|_2^4. \tag{9}$$

In particular, for $h = \eta = \zeta$, we have

$$\left[\int_B |\langle x,h\rangle f(x)|^2 p_1(dx)\right]\left[\int_B |\langle x,h\rangle \mathcal{F}f(x)|^2 p_1(dx)\right] \geq |h|_H^4 \|f\|_2^4. \tag{10}$$

Proof. It is sufficient to prove (9) for real valued functions f. Also by Lemma 2.8 we may assume that $f \in L^2(p_1)$. Now apply lemma 2.8 and Corollary 2.10, we obtain

$$\begin{aligned} &\int_B \langle x,\eta\rangle f(x)\Big[\langle x,\zeta\rangle f(x) - 2\partial_\zeta f(x)\Big]p_1(dx) \\ &= \int_B \Big[\partial_\eta^* f(x) + \partial_\eta f(x)\Big]\Big[\partial_\zeta^* f(x) - \partial_\zeta f(x)\Big]p_1(dx) \\ &= \langle\eta,\zeta\rangle^2 \int_B |f(x)|^2 p_1(dx). \end{aligned} \tag{11}$$

The inequality (9) now follows immediately from Proposition 3.1 and the Schwarz inequality. The inequality (10) follows from (9). □

PROPOSITION 3.3. Let $h \in H\backslash\{0\}$ and let $u_h = h/|h|_H$. Denote by P_h the projection $P_h(x) = \langle x, u_h\rangle u_h$ and define $P_h{}^{\perp} = I - P_h$. Then the equality in (10) holds iff f is of the form

$$f(x) = f(P_h^{\perp}x)e^{\frac{\alpha}{2}\langle x,u_h\rangle^2}, \tag{12}$$

where $|\alpha| < \frac{1}{2}$.

Proof. The Proposition follows exactly in the same way as given in [9]. □

THEOREM 3.4. *Let $T_1, T_2 \in \mathcal{L}(B,H)$ and $\widetilde{T}_j = T_j|_H$, the restriction of T_j to H. Then we have*

$$\Big[\int_B |T_1x|_H^2 |f(x)|^2 p_1(dx)\Big]\ \Big[\int_B |T_2x|_H^2 |\mathcal{F}f(x)|^2 p_1(dx)\Big] \geq \langle\langle \widetilde{T}_1, \widetilde{T}_2\rangle\rangle_{HS}^2 \|f\|_2^2. \tag{13}$$

In particular, if $T = T_1 = T_2$ we have

$$\Big[\int_B |Tx|_H^2 |f(x)|^2 p_1(dx)\Big]\Big[\int_B |Tx|_H^2 |\mathcal{F}f(x)|^2 p_1(dx)\Big] \geq \|\widetilde{T}\|_{HS}^2 \|f\|_2^2. \tag{14}$$

Proof. Clearly (13) implies (14). We need only to prove (13). Let $\{e_j\}$ be a complete orthonormal set of H consisting of elements of B^*. First observe that for any $A \in \mathcal{L}(B,H)$, $\langle Ax, e_j\rangle = \langle x, (A|_H)^*e_j\rangle$ and then apply the identity (11) in the proof of Proposition 3.2, we obtain

$$\begin{aligned}
&\int_B \langle f(x)T_1x, f(x)Sx - 2T_2\partial f(x)\rangle p_1(dx) \\
&= \sum_{j=1}^{\infty} \int_B \langle x, \widetilde{T}_1^* e_j\rangle f(x)\Big[\langle x, \widetilde{T}_2^* e_j\rangle f(x) - 2\partial_{\widetilde{T}_2^* e_j} f(x)\Big] p_1(dx) \\
&= \Big\{\sum_{j=1}^{\infty} \langle \widetilde{T}_1^* e_j, \widetilde{T}_2 e_j\rangle^2\Big\}\Big\{\int_B |f(x)|^2 p_1(dx)\Big\} \\
&= \langle\langle \widetilde{T}_1^*, \widetilde{T}_2^*\rangle\rangle_{HS} \|f\|_2^2 \\
&= \langle\langle \widetilde{T}_1, \widetilde{T}_2\rangle\rangle_{HS} \|f\|_2^2.
\end{aligned}$$

□

THEOREM 3.5. *Let $T \in \mathcal{L}(B,H)$ and write*

$$T^*Tx = \widetilde{T}^*Tx = \sum_{j=1}^{r} \lambda_j{}^2 (x, e_j) e_j,$$

where r is the rank of T and $\{e_j : j = 1, \ldots, r\} \subset B^*$ is the orthonormal set consisting of eigenvalues of T^*T. Denote by P_T the projection of B onto the closure of the subspace of H spanned by $\{e_j : j = 1, \ldots, r\}$ and $P_T^\perp = I - P_T$. Then the equality (13) holds iff $r < \infty$ and f is of the form

$$f(x) \; = \; f(P_T^\perp x) \exp\Big\{ \frac{\alpha}{2} |P_T x|^2 \Big\}, \tag{15}$$

where $|\alpha| < \frac{1}{2}$.

Proof. Observe that $|\widetilde{T}| = \sqrt{\widetilde{T}^*\widetilde{T}}$ extends to a operator in $\mathcal{L}(B,H)$ and $|Tx|_H^2 = ||\widetilde{T}|x|_H^2$. Without loss of generality we may assume that $\widetilde{T}$ is self-adjoint in H. From the proof of Theorem 3.5 and the well-known criterion in real analysis for the equality in the Schwarz inequality we see that the equality of the inequality (14) holds if and only if there exist real constants $A \geq 0$, $B \geq 0$, not both 0, such that, for almost all $x(p_1)$,

$$A|f(x)|^2 |\langle x, \widetilde{T}e_j \rangle|^2 = B|f(x)\langle x, \widetilde{T}e_j\rangle - \partial_{\widetilde{T}e_j} f(x)|^2. \tag{16}$$

Since $Te_j = \lambda_j e_j$ for $j = 1, \ldots, r$ and $\lambda_j > 0$, the condition (16) is obviously equivalent to the existence of two real constants K_1 and K_2, not both zero, such that

$$K_1 f(x) \langle x, e_j \rangle = K_2 \partial_{e_j} f(x). \tag{17}$$

Now we solve completely the equation (17).

It is clear that if $f = 0$ we have equality of (14), so we may assume that $f \neq 0$. It is easy to check that the function of the form (15) satisfies the condition (17), hence the equality of (14) holds.

Now suppose that f a function in $L^2(p_1)$ which satisfies the equality of (14). Then f must satisfies (17) for any $j = 1, \ldots, r$. Since $f \neq 0$, the constant $K_2 \neq 0$. Apply the S-transform to both sides of condition (17). Then Sf satisfies the following equations:

$$\alpha(\xi, e_j) Sf(\xi) = (1-\alpha) D_{e_j} Sf(\xi), \; j = 1, 2, 3, \ldots, r, \tag{18}$$

where $\alpha = \frac{K_1}{K_2}$ and $\xi \in B^*$.
The case $\alpha = 1$ implies that $Sf(\xi) = 0$ which, in turn, implies that $f = 0$. Thus we only have to consider the case $\alpha \neq 1$.

Employ the same argument as given in the proof of Theorem 5 in [9] we obtain

$$\begin{aligned} Sf(\xi) &= Sf(P_T^{\perp}\xi)\exp\Big\{\frac{\alpha}{2(1-\alpha)}\sum_{j=1}^{r}\langle\xi,e_j\rangle^2\Big\} \\ &= Sf(P_r^{\perp}\xi)\exp\Big\{\frac{\alpha}{2(1-\alpha)}|P_T\xi|^2\Big\}. \end{aligned}$$

If $r<\infty$, it is not hard to see that $S(\exp\{\frac{\alpha}{2}|P_Tx|^2\})(\xi)=\exp\{\frac{\alpha}{(1-\alpha)}|P_T\xi|^2\}$ and there is a function $\psi(\xi) = \phi(P_T^{\perp}\xi$ such that $S\psi(\xi) = Sf(P_r^{\perp}\xi)$. Consequently we obtain

$$f(x) = \phi(P_T^{\perp}x)\exp\{\frac{\alpha}{2}|P_Tx|^2\}. \tag{19}$$

Since $f\in L^2(p_1)$ we must have $\alpha<\frac{1}{2}$. Next replace x by $P_T^{\perp}x$ in the equality (19), we find that $\phi(P_T^{\perp}x) = f(P_r^{\perp}x)$. This verifies the formula (1).

If $r=\infty$ and $\alpha<\frac{1}{2}$, we find that

$$\begin{aligned} &\lim_{n\to\infty}\int_H\int_H\Big|\exp\Big\{\frac{\alpha}{2(1-\alpha)}\Big[P_n(\xi+i\eta)\Big]^2\Big\}\Big|^2\mu_{\frac{1}{2}}(d\xi)\mu_{\frac{1}{2}}(d\eta) \\ &= \lim_{n\to\infty}\int_H\int_H\exp\Big\{\frac{\alpha}{(1-\alpha)}|P_n\xi|^2+|P_n\eta|^2\Big\}\mu_{\frac{1}{2}}(d\xi)\mu_{\frac{1}{2}}(d\eta) \\ &= \lim_{n\to\infty}\Big\{\sqrt{(1-\alpha)/(1-2\alpha)}\Big\}^n=\infty, \end{aligned}$$

where the projections $\{P_n\}$ are defined by $P_n\xi = \sum_{j=1}^n(\xi,e_j)e_j$. It follows from Lemma 2.5 that there exist no functions $g\in L^2(p_1)$ such that $Sg(\xi)=\exp\big\{\frac{\alpha}{2(1-\alpha)}|P_r\xi|^2\big\}$ which , in turn, implies that there exist no functions $f\in L^2(p_1)$ such that $Sf(\xi)=Sf(P_r^{\perp}\xi)\exp\{\frac{\alpha}{2(1-\alpha)}|P_r\xi|^2\}$. This completes the proof of the theorem. □

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