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ITERATION PROCESSES OF ASYMPTOTICALLY PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. Some convergence theorems of modified Ishikawa and Mann iteration processes with errors for asymptotically pseudo-contractive and asymptotically nonexpansive mappings in Banach spaces are obtained. The results presented in this paper improve and extend the corresponding results in Liu [7] and Schu [10].

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, and $\langle\cdot,\cdot\rangle$ denotes the pairing of E and E^*. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

is called the normalized duality mapping.

Definition 1.1. Let $T : D \subset E \to E$ be a mapping.

(1) T is said to be asymptotically nonexpansive if there exists a subsequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n\|x - y\|$$

for all $x, y \in D$ and $n = 1, 2, \cdots$.

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(2) T is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and for any $x, y \in D$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2$$

for all $n = 1, 2, \cdots$.

The concepts of asymptotically nonexpansive mapping and asymptotically pseudo-contractive mapping were introduced by Goebel and Kirk [3], and Sch [10], respectively. An early fundamental result due to Goebel and Kirk [3] showed that if E is a uniformly convex Banach space, D is a nonempty bounded closed convex subset of E and $T : D \to D$ is an asymptotically nonexpansive mapping, then T has a fixed point in D. This result is a generalization of the corresponding results in Browder [1] and Kirk [5].

The following proposition follows from Definition 1.1 immediately.

PROPOSITION 1.1. (1) *If $T : D \subset E \to E$ is a nonexpansive mapping, then T is an asymptotically nonexpansive mapping with a constant sequence $\{1\}$.*

(2) *If $T : D \subset E \to E$ is an asymptotically nonexpansive mapping, then T is an asymptotically pseudo-contractive mapping.*

We first recall the following iteration process due to Liu [6].

(I) The Ishikawa iteration process with errors is defined as follows: For a nonempty subset D of a Banach space E and a mapping $T : D \subset E \to E$, the sequence $\{x_n\}$ in D is defined by

$$\begin{aligned} &x_0 \in D, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ &y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \quad n \ge 0, \end{aligned}$$

where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in E, i.e., $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying certain restrictions.

Note that the Mann [8] and Ishikawa [4] iteration processes are all special cases of the Ishikawa iteration process with errors. Inspired by

[7, 10], we introduce the following concept of the modified Ishikawa iteration process with errors.

(II) The modified Ishikawa iteration process with errors is defined as follows: For a nonempty subset D of a Banach space E and a mapping $T : D \subset E \to E$, the sequence $\{x_n\}$ in D is defined by

$$\begin{aligned} &x_0 \in D, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, \\ &y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \geq 0, \end{aligned}$$

where $\{u_n\}$ and $\{v_n\}$ are two summable sequences in E, i.e., $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying certain restriction.

In particular, if $\beta_n = v_n = 0$ for all $n \geq 0$, then $y_n = x_n$. The sequence $\{x_n\}$ defined by

$$\begin{aligned} &x_0 \in D, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n, \quad n \geq 0, \end{aligned}$$

is called the modified Mann iteration process with errors.

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping, and asymptotically pseudo-contractive mapping were studied extensively by Browder [1], Liu [7], Rhoades [9], and Schu [10] in the setting of Hilbert spaces.

The purpose of this paper is to study the iterative approximation problems of fixed points for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings by the modified Mann and Ishikawa iteration processes with errors in uniformly smooth Banach spaces. Our results presented in this paper improve and extend some results in [7, 10].

The following Lemmas play an important role in this paper.

LEMMA 1.1 ([2]). *Let E be a real Banach space, $J : E \to 2^{E^*}$ be a normalized duality mapping. Then for all $x, y \in E$*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

LEMMA 1.2 ([11]). *Let E be a uniformly convex real Banach space, D be a nonempty closed convex subset of E, and $T : D \to D$ be a nonexpansive mapping. If the set $F(T)$ of fixed points of T is nonempty, then for any $x \in D$, $q \in F(T)$, and any $j(x-q) \in J(x-q)$, the inequality*

$$\langle Tx - q, j(x-y)\rangle - \|x-q\|^2 = 0$$

holds if and only if $x = q$.

2. Main Results

THEOREM 2.1. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, $T : D \to D$ be an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, and $F(T) \neq \phi$. Let $\{u_n\}$ and $\{v_n\}$ be two summable sequences in E and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$ satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0$,

(ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

Then for any given $x_0 \in D$, the modified Ishikawa iteration process with errors $\{x_n\} \subset D$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \geq 0, \end{aligned}$$

strongly converges to $q \in F(T)$ if and only if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ such that

$$\langle T^n y_n - q, J(y_n - q)\rangle \leq k_n\|y_n - q\|^2 - \phi(\|y_n - q\|) \tag{2.1}$$

for all $n \geq 0$.

Proof. NECESSITY. Let $x_n \to q \in F(T)$. Since D is a bounded set, $\{T^n x_n\}$ and $\{T^n y_n\}$ are both bounded. Therefore we have

$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v_n \to q \quad (\text{as} \quad n \to \infty),$$

because $\beta_n \to 0$ and $\{v_n\}$ is a summable sequence in E. Define $K = \sup_{n\geq 0}\{\|y_n - q\|\} < \infty$. If $K = 0$, then $y_n = q$ for all $n \geq 0$. Hence

(2.1) is true for all $n \geq 0$. If $K > 0$, define

$$G_t = \{n \in N : \|y_n - q\| \geq t\}, \quad t \in (0, K),$$
$$G_K = \{n \in N : \|y_n - q\| = K\},$$

where N is the set of all nonnegative integers. Since $y_n \to q$, for any $t \in (0, K]$, there exists $n_0 \in N$ such that for any $n \geq n_0$,

$$\|y_n - q\| < t.$$

This implies that for each $t \in (0, K)$

(a) G_t is a nonempty finite subset of N;

(b) $G_{t_1} \subset G_{t_2}$ if $t_1 \geq t_2$, $t_1, t_2 \in (0, K)$;

(c) $G_K = \cap_{t \in (0,K)} G_t$.

Since $T : D \to D$ is asymptotically pseudo-contractive, for given $q \in F(T)$ and for any y_n, we have

$$\langle T^n y_n - q, J(y_n - q)\rangle \leq k_n \|y_n - q\|^2, \quad n \geq 0. \tag{2.2}$$

By virtue of (2.2), we define a function

$$g(t) = \min_{n \in G_t} \{k_n \|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q)\rangle\}, \quad t \in (0, K).$$

From (2.2) and property (b), we know that

(d) $g(t) \geq 0$ for all $t \in (0, K)$;

(e) $g(t)$ is nondecreasing in $t \in (0, K)$.

Next we define a function

$$\phi(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \frac{t}{1+t} g(t) & \text{if} \quad t \in (0, K), \\ \lim\limits_{s \to K^-} \frac{t}{1+t} g(s) & \text{if} \quad t \in [K, \infty). \end{cases}$$

Hence $\phi : [0, \infty) \to [0, \infty)$ is increasing and $\phi(0) = 0$. For any $n \geq 0$, let $t_n = \|y_n - q\|$. If $t_n = 0$, then $y_n = q$. Hence $\phi(\|y_n - q\|) = 0$, and

$$\begin{aligned} \langle T^n y_n - q, J(y_n - q)\rangle &\leq k_n \|y_n - q\|^2 \\ &= k_n \|y_n - q\|^2 - \phi(\|y_n - q\|). \end{aligned}$$

If $t_n \in (0, K)$, then $n \in G_{t_n}$ and so

$$\begin{aligned}\phi(\|y_n - q\|) &= \phi(t_n)\\ &\le \min_{m \in G_{t_n}} \{k_m \|y_m - q\|^2 - \langle T^m y_m - q, J(y_m - q)\rangle\}\\ &\le k_n \|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q)\rangle.\end{aligned}$$

If $t_n = K$, then $n \in G_K = \cap_{s \in (0,K)} G_s$ and so

$$\begin{aligned}\phi(\|y_n - q\|) &= \lim_{s \to k^-} \frac{K}{1+K} g(s)\\ &\le k_n \|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q)\rangle.\end{aligned}$$

<u>SUFFICIENCY.</u> From Lemma 1.1 we have

$$\begin{aligned}&\|x_{n+1} - q\|^2\\ &\le \|(1-\alpha_n)(x_n - q) + \alpha_n (T^n y_n - q)\|^2\\ &\quad + 2\langle u_n, J(x_{n+1} - q)\rangle\\ &\le (1-\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle T^n y_n - q, J(x_{n+1} - q - u_n)\rangle\\ &\quad + 2\langle u_n, J(x_{n+1} - q)\rangle\\ &\le (1-\alpha_n)^2 \|x_n - q\|^2\\ &\quad + 2\alpha_n \langle T^n y_n - q, J(x_{n+1} - q - u_n) - J(y_n - q)\rangle\\ &\quad + 2\alpha_n \langle T^n y_n - q, J(y_n - q)\rangle + 2\langle u_n, J(x_{n+1} - q)\rangle.\end{aligned} \tag{2.3}$$

Now we consider the second term on the right side of (2.3). Since $\{T^n y_n - y_n\}$, $\{x_n - T^n x_n\}$ both are bounded and

$$\begin{aligned}&x_{n+1} - q - u_n - (y_n - q)\\ &\quad = (1-\alpha_n)(x_n - y_n) + \alpha_n (T^n y_n - y_n)\\ &\quad = (1-\alpha_n)[\beta_n (x_n - T^n x_n) - v_n] + \alpha_n (T^n y_n - y_n).\end{aligned}$$

Therefore

$$x_{n+1} - q - u_n - (y_n - q) \to 0 \quad (\text{as} \quad n \to \infty).$$

By the uniform continuity of J and the boundedness of $\{T^n y_n - q\}$,

$$\begin{aligned}p_n &= \langle T^n y_n - q, J(x_{n+1} - q - u_n) - J(y_n - q)\rangle\\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{2.4}$$

Substituting (2.4) and (2.1) into (2.3), we have

$$\begin{aligned} \|x_{n+1}-q\|^2 &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n p_n \\ &\quad + 2\alpha_n\{k_n\|y_n-q\|^2 - \phi(\|y_n-q\|)\} + 2\|u_n\|M_1, \end{aligned} \tag{2.5}$$

where $M_1 = \sup_{n\ge 0}\{\|x_n-q\|\} < \infty$.

Next we make an estimation for $\|y_n-q\|^2$.

$$\begin{aligned} \|y_n-q\|^2 &\le \|(1-\beta_n)(x_n-q)+\beta_n(T^n x_n - q)\|^2 + 2\langle v_n, J(y_n-q)\rangle \\ &\le (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n\langle T^n x_n - q, J(y_n-q-v_n)\rangle \\ &\quad + 2\langle v_n, J(y_n-q)\rangle \\ &\le (1-\beta_n)^2\|x_n-q\|^2 + 2\beta_n M_2 + 2\|v_n\|M_3, \end{aligned} \tag{2.6}$$

where

$$M_2 = \sup_{n\ge 0}\{\|T^n x_n - q\|\|y_n-q-v_n\|\} < \infty$$

and

$$M_3 = \sup_{n\ge 0}\{\|y_n-q\|\} < \infty.$$

Substituting (2.6) into (2.5) and using $M_4 = \sup_{n\ge 0}\ \{\|x_n-q\|^2\}$ to simplify we get

$$\begin{aligned} &\|x_{n+1}-q\|^2 \\ &\le [(1-\alpha_n)^2 + 2\alpha_n k_n(1-\beta_n)^2]\|x_n-q\|^2 \\ &\quad + 2\alpha_n(p_n + 2\beta_n k_n M_2 + 2\|v_n\|k_n M_3) \\ &\quad - 2\alpha_n\phi(\|y_n-q\|) + 2\|u_n\|M_1 \\ &\le \|x_n-q\|^2 - \alpha_n\phi(\|y_n-q\|) \\ &\quad - \alpha_n\{\phi(\|y_n-q\|) - [-2+\alpha_n+2k_n(1-\beta_n)^2]M_4 \\ &\quad - 2(p_n + 2\beta_n k_n M_2 + 2\|v_n\|k_n M_3)\} + 2\|u_n\|M_1. \end{aligned} \tag{2.7}$$

Let $\sigma = \inf_{n\ge 0}\{\|y_n-q\|\}$. Next we prove that $\sigma = 0$.

Suppose that $\sigma > 0$. Then $\|y_n-q\| \ge \sigma > 0$ for all $n \ge 0$. Hence $\phi(\|y_n-g\|) \ge \phi(\sigma) > 0$. From (2.7) we have

$$\begin{aligned} \|x_{n+1}-q\|^2 \le \|x_n-q\|^2 &- \alpha_n\phi(\sigma) - \alpha_n\{\phi(\sigma) \\ &- (-2+\alpha_n+2k_n)M_4 \\ &- 2(p_n + 2\beta_n k_n M_2 + 2\|v_n\|k_n M_3)\} \\ &+ 2\|u_n\|M_1. \end{aligned} \tag{2.8}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$, $p_n \to 0$, $k_n \to 1$, and $v_n \to 0$, there exists n_1 such that for all $n \geq n_1$

$$\phi(\sigma) - (-2 + \alpha_n + 2k_n)M_4 - 2(p_n + 2\beta_n k_n M_2 + 2\|v_n\| k_n M_3) > 0.$$

Hence from (2.8), we get

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n \phi(\sigma) + 2\|u_n\| M_1$$

for all $n \geq n_1$. That is,

$$\alpha_n \phi(\sigma) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\|u_n\| M_1$$

for all $n \geq n_1$. Therefore for any $m \geq n_1$ we have

$$\begin{aligned} \sum_{n=n_1}^{m} \alpha_n \phi(\sigma) &\leq \|x_{n_1} - q\|^2 - \|x_{m+1} - q\|^2 + 2M_1 \sum_{n=n_1}^{m} \|u_n\| \\ &\leq \|x_{n_1} - q\|^2 + 2M_1 \sum_{n=n_1}^{m} \|u_n\|. \end{aligned}$$

Let $m \to \infty$. Then we obtain

$$\infty = \sum_{n=n_1}^{\infty} \alpha_n \phi(\sigma) \leq \|x_{n_1} - q\|^2 + 2M_1 \sum_{n=n_1}^{\infty} \|u_n\|.$$

This is a contradiction. Hence $\sigma = 0$. Therefore there exists a subsequence $\{y_{n_j}\} \subset \{y_n\}$ such that

$$y_{n_j} \to q \quad (\text{as} \quad n_j \to \infty).$$

That is,

$$y_{n_j} = (1 - \beta_{n_j}) x_{n_j} + \beta_{n_j} T^{n_j} x_{n_j} + v_{n_j} \to q \quad (\text{as} \quad n_j \to \infty).$$

Since $\alpha_{n_j} \to 0$, $\beta_{n_j} \to 0$, $u_{n_j} \to 0$, $v_{n_j} \to 0$, and $\{T^{n_j} x_{n_j}\}$ and $\{T^{n_j} y_{n_j}\}$ both are bounded, this implies that

$$x_{n_j} \to q \quad (\text{as} \quad n_j \to \infty). \tag{2.9}$$

From (2.9) we have

$$x_{n_j+1} = (1-\alpha_{n_j})x_{n_j} + \alpha_{n_j} T^{n_j} y_{n_j} + u_{n_j} \to q \quad (\text{as} \quad n_j \to \infty)$$

and so

$$y_{n_j+1} = (1-\beta_{n_j+1})x_{n_j+1} + \beta_{n_j+1} T^{n_j+1} x_{n_j+1} + v_{n_j+1} \to q \quad (\text{as} \quad n_j \to \infty).$$

By induction we can prove that

$$x_{n_j+i} \to q \quad \text{and} \quad y_{n_j+i} \to q \quad (\text{as} \quad n_j \to \infty)$$

for all $i \geq 0$. This implies that $x_n \to q$. This completes the proof of Theorem 2.1. $\square$

From Theorem 2.1 and Proposition 1.1 we can obtain the following theorem.

THEOREM 2.2. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, and $F(T) \neq \phi$. Let $\{u_n\}$ and $\{v_n\}$ be two summable sequences in E and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$ satisfying the conditions* (i) *and* (ii) *in* Theorem 2.1. *Then for any given $x_0 \in D$, the modified Ishikawa iteration process with errors $\{x_n\} \subset D$ defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \geq 0, \end{aligned}$$

strongly converges to $q \in F(T)$ if and only if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n y_n - q, J(y_n - q)\rangle \leq k_n \|y_n - q\|^2 - \phi(\|y_n - q\|)$$

for all $n \geq 0$.

THEOREM 2.3. *Let E be a uniformly convex and uniformly smooth real Banach space, D be a nonempty bounded closed convex subset of E, and $T : D \to D$ be a nonexpansive mapping. Let $\{u_n\}$ and $\{v_n\}$ be two summable sequences in E, and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0, 1]$ satisfying the conditions* (i) *and* (ii) *in* Theorem 2.1. *Then for any given $x_0 \in D$, the modified Ishikawa iteration process with errors $\{x_n\} \subset D$ defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \geq 0,$$

strongly converges to $q \in F(T)$ if and only if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n y_n - q, J(y_n - q)\rangle \geq \|y_n - q\|^2 - \phi(\|y_n - q\|) \tag{2.10}$$

for all $n \geq 0$.

Proof. Since $T : D \to D$ is a nonexpansive mapping, by Proposition 1.1, T is an asymptotically nonexpansive mapping with a constant sequence $\{1\}$, and so it is also an asymptotically pseudo-contractive mapping with the constant sequence $\{1\}$. By Goebel and Kirk [3], $F(T) \neq \phi$. Therefore the sufficiency of Theorem 2.3 follows from Theorem 2.1 immediately.

Next we prove the necessity of Theorem 2.3. Since E is a uniformly smooth Banach space, the normalized duality mapping $J : E \to E^*$ is single valued. Let $x_n \to q \in F(T)$. Since $\{T^n x_n\}$ is bounded, we know that

$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n + v_n \to q \quad (\text{as} \quad n \to \infty).$$

By the same way as given in proving necessity of Theorem 2.1, let us define

$$K = \sup_{n \geq 0}\{\|y_n - q\|\} < \infty,$$
$$G_t = \{n \in N : \|y_n - q\| \geq t\}, \quad t \in (0, K),$$
$$G_K = \{n \in N : \|y_n - q\| = K\},$$
$$g(t) = \min_{n \in G_t}\{\|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q)\rangle\}, \quad t \in (0, K).$$

In Theorem 2.1 we have proved that $g(t)$ is nondecreasing and $g(t) \geq 0$ for all $t \in (0, K)$.

Next we prove that $g(t) > 0$ for all $t \in (0, K)$. Suppose there exists $t_0 \in (0, K)$ such that $g(t_0) = 0$. Since G_{t_0} is a finite set, there exists an $n_0 \in G_{t_0}$ such that

$$0 = g(t_0) = \|y_{n_0} - q\|^2 - \langle T^{n_0} y_{n_0} - q, J(y_{n_0} - q)\rangle. \tag{2.11}$$

From Lemma 1.2 and (2.11), it follows that $y_{n_0} = q$. Hence $\|y_{n_0} - q\| = 0$. However, since $n_0 \in G_{t_0}$, by the definition of G_{t_0}, we have $\|y_{n_0} - q\| \geq t_0 > 0$. This is a contradiction. Therefore $g(t) > 0$ for all $t \in (0, K)$.

Now we define a function

$$\phi(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \frac{t}{1+t} g(t) & \text{if} \quad t \in (0, K), \\ \frac{t}{1+t} \lim\limits_{s \to k_-} g(s) & \text{if} \quad t \in [K, \infty). \end{cases}$$

Since g is nondecreasing and $g(t) > 0$ for all $t \in (0, K)$, $\phi(t) : [0, \infty) \to [0, \infty)$ is strictly increasing, $\phi(0) = 0$. By the same way as given in the proof of Theorem 2.1, we can prove that ϕ satisfies condition (2.10). This completes the proof of Theorem 2.3. □

THEOREM 2.4. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, $T : D \to D$ be an asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, and $F(T) \neq \phi$. Let $\{u_n\}$ be a summable sequence in E and $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

Then for any given $x_0 \in D$, the modified Mann iteration process with errors $\{x_n\} \subset D$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n + u_n, \quad n \geq 0,$$

strongly converges to $q \in F(T)$ if and only if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x_n - q, J(x_n - q)\rangle \leq k_n \|x_n - q\|^2 - \phi(\|x_n - q\|)$$

for all $n \geq 0$.

Proof. Taking $\beta_n = v_n = 0$ for all $n \geq 0$ in Theorem 2.1, then we have $y_n = x_n$ for all $n \geq 0$. Therefore the conclusion of Theorem 2.4 follows from Theorem 2.1 immediately. □

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