

**SOME CURVATURE CONDITIONS OF  
 $n$ -DIMENSIONAL  $CR$ -SUBMANIFOLDS OF  $(n - 1)$   
 $CR$ -DIMENSION IN A COMPLEX PROJECTIVE SPACE**

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ABSTRACT. The purpose of this paper is to study  $n$ -dimensional  $CR$ -submanifolds of  $(n - 1)$   $CR$ -dimension immersed in a complex projective space  $CP^{(n+p)/2}$ , and especially to determine such submanifolds under some curvature conditions.

**1. Introduction**

Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension isometrically immersed in a complex space form  $M^{(n+p)/2}(c)$ . Denoting by  $(J, \bar{g})$  the Kählerian structure of  $M^{(n+p)/2}(c)$ , it follows by definition (cf. [1, 3, 5, 6, 9, 12, 16]) that the maximal  $J$ -invariant subspace

$$\mathcal{D}_x := T_x M \cap JT_x M$$

of the tangent space  $T_x M$  of  $M$  at each point  $x$  in  $M$  has constant dimension  $(n - 1)$ . So there exists a unit vector field  $U_1$  tangent to  $M$  such that

$$\mathcal{D}_x^\perp = \text{Span}\{U_1\}, \quad \forall x \in M,$$

where  $\mathcal{D}_x^\perp$  denotes the subspace of  $T_x M$  complementary orthogonal to  $\mathcal{D}_x$ . Moreover, the vector field  $N_1$  defined by

$$(1.1) \quad N_1 := JU_1$$

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is normal to  $M$  and satisfies

$$JTM \subset TM \oplus \text{Span}\{N_1\}.$$

Hence we have, for any tangent vector field  $X$  and for a local orthonormal basis  $\{N_1, N_\alpha\}_{\alpha=2, \dots, p}$  of normal vectors to  $M$ , the following decomposition in tangential and normal components :

$$(1.2) \quad JX = FX + u^1(X)N_1,$$

$$(1.3) \quad JN_\alpha = -U_\alpha + PN_\alpha, \quad \alpha = 1, \dots, p.$$

Since the structure  $(J, \bar{g})$  is Hermitian and  $J^2 = -I$ , we can easily see from (1.2) and (1.3) that  $F$  and  $P$  are skew-symmetric linear endomorphisms acting on  $T_xM$  and  $T_xM^\perp$ , respectively and that

$$(1.4) \quad g(FU_\alpha, X) = -u^1(X)\bar{g}(N_1, PN_\alpha),$$

$$(1.5) \quad g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \bar{g}(PN_\alpha, PN_\beta),$$

where  $T_xM^\perp$  denotes the normal space of  $M$  at  $x$  and  $g$  the metric on  $M$  induced from  $\bar{g}$ . Furthermore we also have

$$(1.6) \quad g(U_\alpha, X) = u^1(X)\delta_{1\alpha}$$

and consequently

$$(1.7) \quad g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \dots, p.$$

Next, applying  $J$  to (1.2) and using (1.3) and (1.7), we have

$$(1.8) \quad F^2X = -X + u^1(X)U_1, \quad u^1(X)PN_1 = -u^1(FX)N_1,$$

from which, taking account of the skew-symmetry of  $P$  and (1.4),

$$(1.9) \quad u^1(FX) = 0, \quad FU_1 = 0, \quad PN_1 = 0.$$

Thus (1.3) may be written in the form

$$(1.10) \quad JN_1 = -U_1, \quad JN_\alpha = PN_\alpha, \quad \alpha = 2, \dots, p.$$

Moreover we may put

$$(1.11) \quad PN_\alpha = \sum_{\beta=2}^p P_{\alpha\beta}N_\beta, \quad \alpha = 2, \dots, p,$$

where  $(P_{\alpha\beta})$  is a skew-symmetric matrix which satisfies

$$(1.12) \quad \sum_{\gamma=2}^p P_{\alpha\gamma}P_{\gamma\beta} = -\delta_{\alpha\beta}, \quad \alpha, \beta = 2, \dots, p.$$

These results tell us that  $(F, g, U_1, u^1)$  defines an almost contact metric structure on  $M$  (cf. [3, 5, 6, 12]). Recently Okumura and Vanhecke [12] studied the normal almost contact metric case when  $M^{(n+p)/2}(c)$  is a complex projective space  $CP^{(n+p)/2}$  and proved

**THEOREM O-V.** *Let  $M$  be a  $CR$ -submanifold of  $(n-1)$   $CR$ -dimension isometrically immersed in  $CP^{(n+p)/2}$  and let the normal field  $N_1$  be parallel with respect to the normal connection. If the almost contact metric structure  $(F, g, U_1, u^1)$  induced in  $M$  is normal, which is equivalent to the condition  $A_1F = FA_1$  on  $M$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$  where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres and  $A_1$  denotes the shape operator corresponding to  $N_1$  ( $\pi$  is the Hopf fibration  $S^{n+p+1}(1) \rightarrow CP^{(n+p)/2}$ ).*

On the other hand, when  $\pi^{-1}(M)$  is (1) an Einstein space or (2) a locally symmetric space, it is well known (cf. [2, 7, 8, 10, 11]) that  $\pi^{-1}(M)$  has parallel second fundamental form. Projecting the quantities on  $\pi^{-1}(M)$  onto  $M$  in  $CP^{(n+p)/2}$ , we can consider  $CR$ -submanifolds of  $(n-1)$   $CR$ -dimension with the conditions corresponding to (1) or (2). In this paper we shall study such  $CR$ -submanifolds of  $(n-1)$   $CR$ -dimension isometrically immersed in  $CP^{(n+p)/2}$  by using Theorem O-V.

## 2. Fundamental equations for the $CR$ -submanifold

We first let  $M$  be as in section 1 and use the same notations as shown in that section. We denote by  $\bar{\nabla}$  and  $\nabla$  the Levi-Civita connection on  $M^{(n+p)/2}(c)$  and  $M$ , respectively. Then the Gauss and Weingarten equations are given by

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \bar{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^\perp N_\alpha, \quad \alpha = 1, \dots, p$$

for any tangent vector fields  $X, Y$  to  $M$ . Here  $\nabla^\perp$  denotes the normal connection induced from  $\bar{\nabla}$  in the normal bundle  $TM^\perp$  of  $M$ , and  $h$  and  $A_\alpha$  the second fundamental form and the shape operator corresponding to  $N_\alpha$ , respectively. It is clear that  $h$  and  $A_\alpha$  are related by

$$h(X, Y) = \sum_{\alpha=1}^p g(A_\alpha X, Y)N_\alpha.$$

Especially we put

$$(2.3) \quad \nabla_X^\perp N_\alpha = \sum_{\beta=1}^p s_{\alpha\beta}(X)N_\beta.$$

Then  $(s_{\alpha\beta})$  is the skew-symmetric matrix of connection forms of  $\nabla^\perp$ . Now, by using (2.1)-(2.3) and taking account of the Kähler condition  $\bar{\nabla}J = 0$ , we differentiate (1.2) and (1.3) covariantly and compare the tangential and normal parts. Then we can easily find that

$$(2.4) \quad (\nabla_X F)Y = u^1(Y)A_1X - g(A_1Y, X)U_1,$$

$$(2.5) \quad (\nabla_X u^1)(Y) = g(F A_1 X, Y),$$

$$(2.6) \quad \nabla_X U_1 = F A_1 X,$$

$$(2.7) \quad g(A_\alpha U_1, X) = - \sum_{\beta=2}^p s_{1\beta}(X)P_{\beta\alpha}, \quad \alpha = 2, \dots, p$$

for any  $X, Y$  tangent to  $M$ .

In the rest of this paper we suppose that the normal field  $N_1$  is parallel with respect to the normal connection  $\nabla^\perp$ . Hence (2.3) gives

$$(2.8) \quad s_{1\alpha} = 0, \quad \alpha = 2, \dots, p,$$

which together with (2.7) yields

$$(2.9) \quad A_\alpha U_1 = 0, \quad \alpha = 2, \dots, p.$$

On the other hand the ambient manifold  $M^{(n+p)/2}(c)$  is of constant holomorphic sectional curvature  $c$  and consequently its Riemannian curvature tensor  $\bar{R}$  satisfies

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4} \{ & \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} \\ & - \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} - 2\bar{g}(J\bar{X}, \bar{Y})J\bar{Z} \} \end{aligned}$$

for any  $\bar{X}, \bar{Y}, \bar{Z}$  tangent to  $M^{(n+p)/2}(c)$  (cf. [15, 16]). So, the equations of Gauss, Codazzi and Ricci imply

$$(2.10) \quad R(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY - 2g(FX, Y)FZ\} + \sum_{\alpha} \{g(A_{\alpha}Y, Z)A_{\alpha}X - g(A_{\alpha}X, Z)A_{\alpha}Y\},$$

$$(2.11) \quad (\nabla_X A_1)Y - (\nabla_Y A_1)X = \frac{c}{4} \{g(X, U_1)FY - g(Y, U_1)FX - 2g(FX, Y)U_1\},$$

$$(2.12) \quad [A_1, A_{\alpha}] = 0, \quad \alpha = 2, \dots, p$$

for any  $X, Y, Z$  tangent to  $M$  with the aid of (2.8), where  $R$  denotes the Riemannian curvature tensor of  $M$ .

### 3. Fibrations and immersions

In this section  $n$ -dimensional  $CR$ -submanifolds of  $(n - 1)$   $CR$ -dimension isometrically immersed in  $CP^{(n+p)/2}$  only will be considered. Moreover we shall use the assumption and the notations as in section 2.

Let  $S^{n+p+1}(a)$  be the hypersphere of radius  $a (> 0)$  in  $C^{(n+p+2)/2}$  the complex space of complex dimension  $(n + p + 2)/2$ , which is identified with the Euclidean  $(n + p + 2)$ -space  $\mathbb{R}^{n+p+2}$ . The unit sphere  $S^{n+p+1}(1)$  will be briefly denoted by  $S^{n+p+1}$ . Let  $\tilde{\pi} : S^{n+p+1} \rightarrow CP^{(n+p)/2}$  be the natural projection of  $S^{n+p+1}$  onto  $CP^{(n+p)/2}$  defined by the Hopf-fibration  $S^1 \rightarrow S^{n+p+1} \rightarrow CP^{(n+p)/2}$ . As is well known (cf. [3, 5, 12, 14, 15]),  $S^{n+p+1}$  admits a Sasakian structure  $\tilde{\xi}$  and each fibre  $\tilde{\pi}^{-1}(x)$  of  $x$  in  $CP^{(n+p)/2}$  is a maximal integral submanifold of the distribution spanned by  $\tilde{\xi}$ . Thus the base space  $CP^{(n+p)/2}$  admits the induced Kähler structure of constant holomorphic sectional curvature 4 (cf. [3, 5, 12, 14, 15]). Moreover we have a fibration  $\pi : \pi^{-1}(M) \rightarrow M$  which is compatible with the Hopf-fibration  $\tilde{\pi}$ . More precisely speaking  $\pi : \pi^{-1}(M) \rightarrow M$  is a fibration with totally geodesic fibers such that the following diagram is commutative :

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{i'} & S^{n+p+1} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{i} & CP^{(n+p)/4} \end{array}$$

where  $i' : \pi^{-1}(M) \rightarrow S^{n+p+1}$  and  $i : M \rightarrow CP^{(n+p)/2}$  are isometric immersions.

Now, let  $\xi$  be the unit vector fields tangent to the fibers of  $\pi^{-1}(M)$  such that  $i'_*\xi = \tilde{\xi}$ . (In what follows we shall delete the  $i'$  and  $i'_*$  in our notation.) Furthermore we denote by  $X^*$  the horizontal lift of a vector field  $X$  tangent to  $M$ . Then the horizontal lifts  $N_\alpha^*$  ( $\alpha = 1, \dots, p$ ) of the normal vectors  $N_\alpha$  to  $M$  form an orthonormal basis of normal vectors to  $\pi^{-1}(M)$  in  $S^{n+p+1}$ . Let  $A'_\alpha$  and  $s'_{\alpha\beta}$  be the corresponding shape operators and normal connection forms, respectively. Then, as shown in [3, 4, 5, 12, 13, 14, 15], the fundamental equations for the submersion  $\pi$  are given by

$$(3.1) \quad {}'\nabla_{X^*} Y^* = (\nabla_X Y)^* + g'((FX)^*, Y^*)\xi,$$

$$(3.2) \quad {}'\nabla_{X^*} \xi = {}'\nabla_\xi X^* = -(FX)^*,$$

where  $g'$  denotes the Riemannian metric of  $\pi^{-1}(M)$  induced from  $\tilde{g}$  that of  $S^{n+p+1}$  and  $'\nabla$  the Levi-Civita connection with respect to  $g'$ . The similar equations are valid for the submersion  $\tilde{\pi}$  by replacing  $F$  (resp.  $\xi$ ) with  $J$  (resp.  $\tilde{\xi}$ ) respectively. We denote by  $\tilde{\nabla}$  and  $'\nabla^\perp$  the Levi-Civita connection for  $\tilde{g}$  and the normal connection of  $\pi^{-1}(M)$  induced from  $\tilde{\nabla}$ , respectively. Since the diagram is commutative,  $\tilde{\nabla}_{X^*} N_\alpha^*$  implies

$$\begin{aligned} {}'\nabla_{X^*}^\perp N_\alpha^* - A'_\alpha X^* &= (\tilde{\nabla}_X N_\alpha)^* + \tilde{g}((FX)^*, N_\alpha^*)\tilde{\xi} \\ &= -(A_\alpha X)^* + g(U_\alpha, X)^*\xi + (\nabla_X^\perp N_\alpha)^* \end{aligned}$$

because of (2.2), (2.5), and (3.1), from which, comparing the tangential and normal parts, we have

$$(3.3) \quad A'_\alpha X^* = (A_\alpha X)^* - g(U_\alpha, X)^*\xi,$$

$$(3.4) \quad {}'\nabla_{X^*}^\perp N_\alpha^* = (\nabla_X^\perp N_\alpha)^*.$$

Next, calculating  $\tilde{\nabla}_\xi N_\alpha^*$  and using (2.2), (2.5), and (3.2), we have

$${}'\nabla_\xi^\perp N_\alpha^* - A'_\alpha \xi = -(FN_\alpha)^* = U_\alpha^* - (PN_\alpha)^*,$$

which yields

$$(3.5) \quad A'_\alpha \xi = -U_\alpha^*,$$

$$(3.6) \quad {}'\nabla_\xi^\perp N_\alpha^* = -(PN_\alpha)^*.$$

Hence (3.3) and (3.5) with  $\alpha = 1$  and (3.6) imply

$$(3.7) \quad A'_1 X^* = (A_1 X)^* - g(U, X)^* \xi,$$

$$(3.8) \quad A'_1 \xi = -U^*,$$

$$(3.9) \quad s'_{\alpha\beta}(X^*) = s_{\alpha\beta}(X)^*, \quad s'_{\alpha\beta}(\xi) = -P_{\alpha\beta}.$$

First of all we recall the co-Gauss equations for the submersion  $\pi$  (cf. [4, 13, 14, 15]). Taking account of (3.1) and (3.2), we have

$$(3.10) \quad \begin{aligned} \nabla_{X^*} \nabla_{Y^*} Z^* &= (\nabla_X \nabla_Y Z)^* + \{g'((FX)^*, (\nabla_Y Z)^*) \\ &+ g'((F\nabla_X Y)^*, Z^*) + g'(((\nabla_X F)Y)^*, Z^*) \\ &+ g'((FY)^*, (\nabla_X Z)^*)\} \xi - g'((FY)^*, Z^*)(FX)^*, \end{aligned}$$

$$(3.11) \quad [X^*, Y^*] = [X, Y]^* + 2g'((FX)^*, Y^*)\xi, \quad [X^*, \xi] = 0.$$

Using these equations and taking account of (2.4) and (2.10) with  $c = 4$ , we can easily see that

$$(3.12) \quad \begin{aligned} 'R(X^*, Y^*)Z^* &= g(Y, Z)^* X^* - g(X, Z)^* Y^* \\ &+ \sum_{\alpha} \{g(A_{\alpha} Y, Z)^*(A_{\alpha} X)^* - g(A_{\alpha} X, Z)^*(A_{\alpha} Y)^*\} \\ &+ g'(\{u(Y)A_1 X - u(X)A_1 Y\}^*, Z^*)\xi, \end{aligned}$$

where  $'R$  denotes the Riemannian curvature tensor of  $\pi^{-1}(M)$ . Making also use of (3.1) and (3.2), we have

$$\begin{aligned} \nabla_{X^*} \nabla_{Y^*} \xi &= -\{(\nabla_X F)Y + F(\nabla_X Y)\}^* - g'((FX)^*, (FY)^*)\xi, \\ \nabla_{\xi} \nabla_{Y^*} X^* &= -(F\nabla_Y X)^* - \{g(F^2 Y, X)^* + g(FY, FX)^*\}\xi, \\ \nabla_{Y^*} \nabla_{\xi} X^* &= -\{(\nabla_Y F)X + F(\nabla_Y X)\}^* - g(FY, FX)^* \xi, \\ \nabla_{\xi} \nabla_{X^*} \xi &= \{-X + u(X)U\}^*, \end{aligned}$$

from which, using (1.8), (2.4), (3.11) and the fibre being totally geodesic, we can easily obtain

$$(3.13) \quad 'R(X^*, Y^*)\xi = -\{u(Y)A_1 X - u(X)A_1 Y\}^*,$$

$$(3.14) \quad \begin{aligned} 'R(Y^*, \xi)X^* &= -\{u(X)A_1 Y\}^* \\ &- \{g(Y, X)^* - u(Y)^* g(U, X)^*\}\xi, \end{aligned}$$

$$(3.15) \quad 'R(\xi, X^*)\xi = \{-X + u(X)U\}^*.$$

Putting  $Z = U$  in (3.12) and using (2.9), we have

$$(3.16) \quad \begin{aligned} {}'R(X^*, Y^*)U^* &= u(Y)^*X^* - u(X)^*Y^* \\ &+ g(A_1Y, U)^*(A_1X)^* - g(A_1X, U)^*(A_1Y)^* \\ &+ g(u(Y)A_1X - u(X)A_1Y, U)^*\xi, \end{aligned}$$

from which, differentiating covariantly in the vertical direction  $\xi$  and using (3.2) and (3.16), we find

$$(3.17) \quad \begin{aligned} ({}'\nabla_\xi {}'R)(X^*, Y^*)U^* &= g(A_1Y, U)^*(A_1FX - FA_1X)^* \\ &- g(A_1X, U)^*(A_1FY - FA_1Y)^* \\ &- g(A_1FX, U)^*(A_1Y)^* + g(A_1FY, U)^*(A_1X)^* \\ &+ \{u(Y)^*g(A_1FX, U)^* - u(X)^*g(A_1FY, U)^*\}\xi. \end{aligned}$$

Thus we have

**THEOREM 1.** *Let  $M$  be an  $n$ -dimensional CR-submanifold of  $(n - 1)$  CR-dimension isometrically immersed in a complex projective space  $CP^{(n+p)/2}$  and let the normal field  $N_1$  be parallel with respect to the normal connection. If  $({}'\nabla_\xi {}'R)(X^*, Y^*)U^* = 0$  and if  $u(A_1U) \neq 0$  at least one point of  $M$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$  where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres.*

*Proof.*  $({}'\nabla_\xi {}'R)(X^*, Y^*)U^* = 0$  and (3.17) imply

$$(3.18) \quad \begin{aligned} &g(A_1Y, U)^*(A_1FX - FA_1X)^* \\ &- g(A_1X, U)^*(A_1FY - FA_1Y)^* \\ &- g(A_1FX, U)^*(A_1Y)^* \\ &+ g(A_1FY, U)^*(A_1X)^* = 0, \end{aligned}$$

$$(3.19) \quad u(Y)^*g(A_1FX, U)^* - u(X)^*g(A_1FY, U)^* = 0.$$

Putting  $Y = U$  in (3.19) and using (1.9), we have

$$(3.20) \quad FA_1U = 0.$$

Substituting  $Y = U$  in (3.18) and using (3.20), we get

$$(3.21) \quad \lambda(A_1FX - FA_1X) = 0,$$



where we put  $\lambda = u(A_1U)$ . On the other hand, using the first equation of (1.8) and (3.20), we can easily see that

$$(3.22) \quad A_1U = \lambda U,$$

from which, differentiating covariantly along  $M$  and using (2.6),

$$g((\nabla_X A_1)Y, U) + g(FA_1X, A_1Y) = (X\lambda)u(Y) + \lambda g(FA_1X, Y).$$

Taking the skew-symmetric part of the above equation and using (1.6), (1.9), (2.11) with  $c = 4$  and (3.21), we obtain

$$(3.23) \quad \begin{aligned} & -2g(FX, Y) + 2g(FA_1X, A_1Y) \\ & = (X\lambda)u(Y) - (Y\lambda)u(X) + 2\lambda g(FA_1X, Y). \end{aligned}$$

Putting  $Y=U$  in (3.23) and using (1.9) and (3.22), we have

$$(3.24) \quad X\lambda = \beta u(X),$$

where we put  $\beta = U\lambda$ . (3.23) and (3.24) imply

$$(3.25) \quad -g(FX, Y) + g(FA_1X, A_1Y) = \lambda g(FA_1X, Y).$$

Differentiating (3.24) covariantly along  $M$  and using (2.5), we get

$$YX\lambda = (Y\beta)u(X) + \beta\{g(FA_1Y, X) + u(\nabla_Y X)\}.$$

Taking the skew-symmetric part of the above equation and using (3.24), we find

$$(Y\beta)u(X) - (X\beta)u(Y) + \beta g(FA_1Y + A_1FY, X),$$

from which, putting  $Y = U$  and using (1.6), (1.9), and (3.22),

$$X\beta = (U\beta)u(X)$$

and consequently

$$(3.26) \quad \beta(FA_1X + A_1FX) = 0.$$

Therefore  $FA_1X + A_1FX = 0$  on the open set  $S := \{x \in M \mid \beta(x) \neq 0\}$ . Taking account of (3.21), we have  $\lambda FA_1X = 0$  on  $S$ , which and (3.25) give

$$\lambda g(FX, Y) = 0.$$

Hence  $\lambda = 0$  and consequently  $\beta = 0$  on  $S$ , which is a contradiction. Thus we have  $\beta = 0$  identically on  $M$ , which and (3.24) yield that  $\lambda$  is constant. Therefore, if  $u(A_1U) \neq 0$  at least one point of  $M$ , then (3.21) gives

$$A_1FX - FA_1X = 0,$$

which together with Theorem O-V yields our result.  $\square$

Differentiating (3.13) covariantly in the direction of  $\xi$  and using (3.2) and (3.13), we have

$$(3.27) \quad \begin{aligned} & (' \nabla_{\xi}' R)(X^*, Y^*)\xi \\ &= \{u(Y)(FA_1 - A_1F)X\}^* - \{u(X)(FA_1 - A_1F)Y\}^*. \end{aligned}$$

Thus we have

**THEOREM 2.** *Let  $M$  be an  $n$ -dimensional CR-submanifold of  $(n - 1)$  CR-dimension isometrically immersed in a complex projective space  $CP^{(n+p)/2}$  and let the normal field  $N_1$  be parallel with respect to the normal connection. If  $(' \nabla_{\xi}' R)(X^*, Y^*)\xi = 0$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$  where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres.*

*Proof.*  $(' \nabla_{\xi}' R)(X^*, Y^*)U^* = 0$  and (3.27) imply

$$u(Y)(FA_1 - A_1F)X - u(X)(FA_1 - A_1F)Y = 0,$$

from which, putting  $Y = U$ , we obtain

$$(3.28) \quad (FA_1 - A_1F)X = u(X)FA_1U.$$

Taking the inner product of (3.28) with  $U$  and using (1.9), we find  $FA_1U = 0$ , which together with (3.28) yields

$$FA_1 - A_1F = 0.$$

Combining this equation with Theorem O-V gives our result.  $\square$

COROLLARY. Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension isometrically immersed in a complex projective space  $CP^{(n+p)/2}$  and let the normal field  $N_1$  be parallel with respect to the normal connection. If  $'\nabla_\xi'R = 0$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$  where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres.

Finally we differentiate (3.14) covariantly in the direction of  $\xi$ . By means of (3.2) and (3.14) we can easily find

$$(3.29) \quad ('\nabla_\xi'R)(Y^*, \xi)X^* = \{u(X)(FA_1 - A_1F)Y\}^*.$$

If  $('\nabla_\xi'R)(Y^*, \xi)X^* = 0$ , then (3.29) gives

$$u(X)(FA_1 - A_1F)Y,$$

from which, putting  $X = U$  and using (1.6), we have

$$FA_1 - A_1F = 0.$$

Thus we have

THEOREM 3. Let  $M$  be an  $n$ -dimensional  $CR$ -submanifold of  $(n - 1)$   $CR$ -dimension isometrically immersed in a complex projective space  $CP^{(n+p)/2}$  and let the normal field  $N_1$  be parallel with respect to the normal connection. If  $('\nabla_\xi'R)(Y^*, \xi)X^* = 0$ , then  $\pi^{-1}(M)$  is locally a product of  $M_1 \times M_2$  where  $M_1$  and  $M_2$  belong to some odd-dimensional spheres.

REMARK. Under our assumptions it can be easily verified from (3.15) that  $('\nabla_\xi'R)(\xi, X^*)\xi = 0$  identically on  $\pi^{-1}(M)$ .

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